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BRANCHED COVERINGS AND NONZERO DEGREE MAPS BETWEEN SEIFERT MANIFOLDS

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Abstract. In this note we give a necessary and sufficient condition for the existence of a fiber preserving branched covering between two closed, orientable Seifert manifolds (for sufficiency we need the additional assumption that the genus of the base orbifold of the target manifold ≥ 1). Combining this with two theorems of Rong we get a necessary and sufficient condition for the existence of a nonzero degree map between two such manifolds.

0. Introduction

In [7] Yongwu Rong proved:

Theorem. *Let $f : M \to N$ be a nonzero degree map between closed P^2-irreducible Seifert manifolds of infinite π_1. Then $f \simeq pg\pi$, where π is a composition of finitely many vertical pinches, g is a fiber preserving branched covering, and p is a covering. Furthermore, p can be chosen to be the identity map unless N is a Euclidean manifold.*

Using this theorem Rong [8] gave a necessary and sufficient condition for the existence of a degree one map between two aspherical, closed, orientable Seifert manifolds (see Theorem 3.2 of [8]), and proved as a corollary: If M is an aspherical, closed, orientable Seifert manifold, then there are only finitely many aspherical Seifert manifolds N such that there exists a degree one map $f : M \to N$ (see Corollary 4.1 of [8]).

In this note we try to generalize the results above in [8] to the case of arbitrary degree (but for sufficiency we need the additional condition that the orbifold genus of the target manifold ≥ 1). In Section 1 we give two general facts about branched coverings of Seifert manifolds, which will not be used in the later sections. Then we give a necessary and sufficient condition that there exist a fiber preserving branched covering between two closed, orientable Seifert manifolds (for sufficiency we need the additional assumption that the orbifold genus of the target manifold ≥ 1) (see Theorem 2.2 in Section 2). One of the key ingredients in the proof is a theorem of Husemoller (see Theorem 4 of [4] and Proposition 3.3 of [2]) which guarantees the existence of the branched covering of a closed, orientable surface with nonpositive Euler characteristic with any given branch data with even total branching. As a

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consequence, we get an elementary proof of the naturality of the Euler numbers of Seifert manifolds [6]. Combining Theorem 2.2 with Rong's theorems above we get a necessary and sufficient condition that there exist a nonzero degree map between two such manifolds (see Theorem 3.0 in Section 3). As a consequence, we show that for a given aspherical, closed, oriented Seifert manifold M and a nonzero integer d, there are only finitely many such Seifert manifolds N such that there is a degree d map $f : M \to N$ (see Corollary 3.1).

1. Two general facts about branched coverings of Seifert manifolds

In this section we generalize two well-known facts about coverings of Seifert manifolds to the case of branched coverings, which will not be used in the later sections.

Proposition 1.0. *A finite sheeted branched covering space of a Seifert manifold branched over fibers admits a Seifert fibration.*

The proof is easy and omitted.

Remark. If we remove the condition "branched over fibers", then the result is obviously not true.

Conversely, we have

Proposition 1.1. *Let S be a compact, connected, irreducible, orientable 3-manifold. Suppose that some finite sheeted branched covering space S' of S admits a Seifert fibration such that the inverse image of the downstairs branch set consists of some fibers. Then S itself admits a Seifert fibration such that the downstairs branch set consists of some fibers.*

Proof. Suppose we have a finite sheeted branched covering $p : S' \to S$, where S' is a Seifert manifold and the inverse image of the downstairs branch set consists of some fibers. Let $p| : S' - p^{-1}(N(B_p)) \to S - N(B_p)$ be the corresponding unbranched covering, where B_p is the downstairs branch set and $N(B_p)$ is an open regular neighborhood of B_p such that $p^{-1}(N(B_p))$ is fibered. By [5, Theorem 6.3] and the positive solution of the Seifert fibered space conjecture by Gabai, Casson and Jungreis [1], [3], $S - N(B_p)$ admits a Seifert fibration, which can extend to $N(B_p)$ such that B_p consists of some fibers as is easily seen.

Note that for any nonzero degree map $f : M \to N$ between two aspherical, closed Seifert manifolds, we have $f \simeq f_1 p$, where $p : M \to M_1$ is a fiber preserving branched covering (in fact p is the natural projection from M to some M/Z_n, where Z_n is in the S^1 action on M), and $f_{1*}(h) = h$, where h represents regular fibers of both M, N. So we shall first consider nonzero degree map $f : M \to N$ of fiber degree 1 in the next section.

2. Fiber preserving branched coverings between Seifert manifolds

From now on, "Seifert manifold" will always mean an oriented closed connected 3-manifold admitting a fixed point free action of S^1. As is well known, such a manifold is equivariantly classified by its "Seifert invariants" [10]. We shall use unnormalized Seifert invariants (see [6]). The unnormalized Seifert invariant of a Seifert manifold M is denoted by $I(M) = \{g; (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$ or $\{g; r_1, \dots, r_m\}$ where g is the orbifold genus of M, (α_i, β_i) (α_i, β_i coprime and $\alpha_i \geq 1$) are the Seifert invariants of the maybe-singular fibers, and $r_i = \beta_i/\alpha_i$.

The unnormalized Seifert invariant of a Seifert manifold M is not unique. We can transform it while not changing the isomorphic type of the Seifert fibration of M by the following moves which we call *equivalent moves* (see [6], [8]):

(1) permute the r_i.

(2) add 0 to or delete 0 from the list $r_1, \dots, r_n$.

(3) replace r_i, r_j by $r_i + 1, r_j - 1$.

As preparation, let's see when there exists a k-fold fiber preserving branched covering of fiber degree 1 between two Seifert fibered solid tori V_1 and V_2 such that upstairs and downstairs branch sets are their central fibers.

Let the Seifert invariant of V_2 be (α_2, β_2) under a (cross section, fiber) basis (c_2, h_2), i.e. the meridian μ_2 of V_2 can be represented by $c_2^{\alpha_2} h_2^{\beta_2}$. If such a branched covering $f : V_1 \to V_2$ exists, then c_2 lifts to a cross section c_1 of the fibration of V_2 restricted to ∂V_2 for fiber degree one. Thus we have $f|_*(c_1) = c_2^k$, where $f|$ is the restriction of f to ∂V_1.

By the definition of branched covering we can write $f|_*(\mu_1) = \mu_2^s$, where s is some nonzero integer. So $f|_*(c_1^{\alpha_1} h_1^{\beta_1}) = (c_2^{\alpha_2} h_2^{\beta_2})^s$, where we set the Seifert invariant of V_1 under the (section, fiber) basis (c_1, h_1) to be (α_1, β_1). Thus $c_2^{k\alpha_1} h_2^{\beta_1} = c_2^{s\alpha_2} h_2^{s\beta_2}$, i.e. $k\alpha_1 = s\alpha_2$, $\beta_1 = s\beta_2$. So $\beta_1/\alpha_1 = k\beta_2/\alpha_2$.

Conversely, if V_i has Seifert invariant (α_i, β_i), and α_i, β_i ($i = 1, 2$), and k satisfy the relation above, then clearly there exists a branched covering $V_1 \to V_2$ satisfying the given condition (by a direct construction; cf. [10]).

Definition 2.0. Let $R = \{\beta_1/\alpha_1, \dots, \beta_s/\alpha_s\}$, $R' = \{\beta'_1/\alpha'_1, \dots, \beta'_k/\alpha'_k\}$ be two unordered sets of rational numbers (repetitions allowed). Let d be a positive integer. We say $R \geq_d R'$ if R can be split into a "disjoint" union $R = R_1 \cup \ldots \cup R_k$ such that for any i ($1 \leq i \leq k$) and any $\beta_j/\alpha_j \in R_i$, $\beta_j/\alpha_j = a_j \beta'_i/\alpha'_i$ (where a_j is some positive integer), and $\sum a_j = d$, where the summation is over all j such that $\beta_j/\alpha_j \in R_i$.

Lemma 2.1. *Let M, N be closed oriented Seifert manifolds. Suppose $I(N) = \{g'; R'\}$, where $g' \geq 1$ and $R' = \{\beta'_1/\alpha'_1, \dots, \beta'_n/\alpha'_n\}$ ($n \geq 1$). Then there is a d-fold fiber preserving branched covering $M \to N$ with fiber degree one if and only if M has Seifert invariant $\{g; R\}$, where $R \geq_d R'$ and $nd - |R| \leq 2d(1-g') - 2(1-g)$. Here $|R|$ is the number of elements in the multiset R (counted with multiplicity).*

Proof. Suppose there is such a branched covering $f : M \to N$. Recall how we get the Seifert invariant of N as in [6]. Let $O_1, \dots, O_n$ be a nonempty collection of fibers in N, including all singular fibers. Let $T_1, \dots, T_n$ be disjoint Seifert fibered tubular neighborhoods of $O_1, \dots, O_n$ and $N_0 = N - int(T_1 \cup \ldots \cup T_n)$. Take the section c_2 of the bundle $N_0 \to N_0/S^1$ which leads to the Seifert invariant $\{g'; R'\}$ of N. Let $M_0 = M - f^{-1}(int(T_1 \cup \ldots \cup T_n))$. Then c_2 lifts to a section c_1 of the bundle $M_0 \to M_0/S^1$ for fiber degree one. Let $\{g; R\}$ be the Seifert invariant of M under the (section, fiber) frame (c_1, h_1), where h_1 stands for a regular fiber of M. Considering T_i and its inverse image under f we get the relation $R \geq_d R'$ by the local computation before Definition 2.0. The inequality $nd - |R| \leq 2d(1 - g') - 2(1 - g)$ follows from the Riemann-Hurwitz formula since $nd - |R|$ is a part of the total branching of the induced branched covering $\overline{f} : M/S^1 \to N/S^1$.

Conversely suppose the Seifert invariants of M,N satisfy the relation in the lemma we'll go to construct a branched covering $f: M \to N$ as required. Let

$$k = 2d(1-g') - 2(1-g) - (nd - |R|).$$

Let $\mathcal{D} = \{A_1, \ldots, A_n, A_{n+1}, \ldots, A_{n+k}\}$, where A_i $(i = 1, \ldots, n)$ are the partitions of d given by the relation $R \geq_d R'$ (i.e. each A_i consists of the a_j's (depending on i) in Definition 2.0), $A_{n+j} = [2, 1, \ldots, 1]$ ($d-2$ times 1) for $j = 1, \ldots, k$. By Theorem 4 of [4] or Proposition 3.3 of [2] there is a d-fold branched covering $\overline{f}: F_g \to F_{g'}$ with branch data $\mathcal{D}$, where F_g is the closed, orientable surface of genus g. Based on this and the local constructions (as what we have done in the preparation before Definition 2.0) decided by the relation $R \geq_d R'$, a branched covering $M \to N$ as required can be constructed: We have the induced covering between surfaces with boundary $\overline{f}| : F_g - \overline{f}^{-1}(int(D_1 \cup \ldots \cup D_{n+k})) \to F_{g'} - int(D_1 \cup \ldots \cup D_{n+k})$. Then clearly

$$\overline{f}| \times id : (F_g - \overline{f}^{-1}(int(D_1 \cup \ldots \cup D_{n+k}))) \times S^1 \to (F_{g'} - int(D_1 \cup \ldots \cup D_{n+k})) \times S^1$$

extends to a d-fold fiber preserving branched covering $f: M \to N$ with fiber degree one.

Remark. In the proof of the necessity of the lemma above we do not need the condition $g' \geq 1$. But for sufficiency we do need $g' \geq 1$ as shown in the following example.

Example. Let $I(M) = \{0; R\} = \{0; 2/13, 2/13, 4/17, 4/17, 9/19, 3/19\}$, $I(N) = \{0; R'\} = \{0; 1/13, 2/17, 3/19\}$. Then $R \geq_4 R'$ and $nd - |R| = 6 = 2d(1-g') - 2(1-g)$. However there is no 4-fold fiber preserving branched covering $M \to N$ with fiber degree one, for $\{[2,2], [2,2], [3,1]\}$ cannot arise as the branch data of a connected branched covering of S^2 (see Proposition 5.3 of [2]).

In the case of arbitrary fiber degree we have the following:

Theorem 2.2. *Let M, N be closed, oriented Seifert manifolds with Seifert invariants*

$$I(M) = \{g; R\} = \{g; \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m\},$$
$$I(N) = \{g'; R'\} = \{g'; \beta'_1/\alpha'_1, \ldots, \beta'_n/\alpha'_n\},$$

where $g' \geq 1$. Let d, d_1 be positive integers and $d_1|d$. Then there is a d-fold fiber preserving branched covering $M \to N$ with fiber degree d_1 if and only if $d_1 R := \{d_1\beta_1/\alpha_1, \ldots, d_1\beta_m/\alpha_m\}$ can be equivalently moved to a set $\overline{R} \geq_{d_2} R'$ (where $d_2 = d/d_1$) and $nd_2 - |\overline{R}| \leq 2d_2(1-g') - 2(1-g)$.

Proof. First we prove necessity. If there exists such a branched covering $f: M \to N$, then f factors through $M/(Z/d_1)$ (where Z/d_1 is inside the S^1 action on M). By a fact due to Seifert (see [10] or Lemma 1.3 of [6]) $M/(Z/d_1)$ has Seifert invariant $\{g; d_1\beta_1/\alpha_1, \ldots, d_1\beta_m/\alpha_m\}$. Now the necessity clearly follows from the lemma above.

Sufficiency can be proved by a direct construction as in the lemma above.

Remark. Again in the proof of the necessity $g' \geq 1$ is not used.

As an application we have the following.

Corollary 2.3 ([6]). *Let M, N be aspherical, closed, oriented Seifert manifolds. Suppose there is a degree d ($d \neq 0$) fiber preserving map $f : M \to N$ with fiber degree d_1. Then $e(M) = (d_2/d_1)e(N)$, where $e(M)$ (resp. $e(N)$) is the Euler number of the Seifert fibration of M (resp. N), and $d_2 = d/d_1$.*

Proof. By the theorem of Rong cited at the beginning of this note, $f \simeq g\pi$, where $\pi : M \to \hat{M}$ is a composition of finitely many of vertical pinches, and $g : \hat{M} \to N$ is a fiber preserving branched covering. By Corollary 3.3 of [8] $e(M) = e(\hat{M})$. We can write $g = g_2 g_1$, where $g_1 : \hat{M} \to \overline{M} := \hat{M}/(Z/d_1)$ is the natural projection, and $g_2 : \overline{M} \to N$ is a d_2-fold fiber preserving branched covering of fiber degree one. By the fact due to Seifert cited above, $e(\hat{M}) = (1/d_1)e(\overline{M})$. By Theorem 2.2 and Definition 2.0, $e(\overline{M}) = d_2 e(N)$. So $e(M) = (d_2/d_1)e(N)$.

Remark. The proof of the fact above sketched in [6] used more algebraic topology, while our proof here is quite elementary.

Corollary 2.4 ([8]). *Let M, N be closed, oriented Seifert manifolds, where $\chi(O_N) \leq 0$. Suppose there is a d-fold fiber preserving branched covering $f : M \to N$ with orbifold degree d_2. Then $|\chi(O_M)| \geq d_2|\chi(O_N)|$, where O_M (resp. O_N) is the base orbifold of M (resp. N) and $\chi(O_M)$ (resp. $\chi(O_N)$) is its Euler characteristic. The equality holds if and only if f is a covering.*

Proof. It follows from a direct computation using the Theorem.

Let $\overline{M} = M/(Z/d_1)$, where $d_1 = d/d_2$. We have

$$\begin{aligned}
& -\chi(O_{\overline{M}}) + d_2\chi(O_N) \\
= \ & -\chi(|O_{\overline{M}}|) + \textstyle\sum_{i=1}^{|\overline{R}|}(1 - 1/\overline{\alpha}_i) + d_2\chi(|O_N|) - d_2 \sum_{j=1}^{n}(1 - 1/\alpha'_j) \\
= \ & d_2\chi(|O_N|) - \chi(|O_{\overline{M}}|) - (nd_2 - |\overline{R}|) + \textstyle\sum_{j=1}^{n}(d_2/\alpha'_j - \sum_{\overline{\beta}_i/\overline{\alpha}_i \in \overline{R}_j} 1/\overline{\alpha}_i) \\
\geq \ & d_2\chi(|O_N|) - \chi(|O_{\overline{M}}|) - (nd_2 - |\overline{R}|) \quad \text{(using the relation } \overline{R} \geq_{d_2} R') \\
\geq \ & 0.
\end{aligned}$$

The last two equalities hold if and only if $\overline{M} \to N$ is a covering.

On the other hand, $-\chi(O_M) \geq -\chi(O_{\overline{M}})$ by the fact due to Seifert cited above; the equality holds if and only if $\overline{f} : M \to \overline{M}$ is a covering.

Combining these two things the proof is complete.

Remark. It is indicated in [8] that the inequality in Corollary 2.4 can be proved by a covering space argument. This in turn depends on the fact that every good, compact 2-dimensional orbifold without boundary is finitely covered by a manifold (see for example [9]). But the proof of this fact is not very easy. We also note that the inequality has independent meaning, i.e. it can be applied directly to 2-orbifold branched coverings.

We note that however the condition "$nd - |R| \leq 2d(1-g') - 2(1-g)$" in Lemma 2.1 cannot be replaced by $|\chi(O_M)| \geq d|\chi(O_N)|$ as shown in the following example.

Example. Let $I(M) = \{5; R\} = \{5; 2/5, 3/5, 2/7, 3/7, 2/11, 3/11\}$, $I(N) = \{1; R'\} = \{1; 1/5, 1/7, 1/11\}$, $d = 5$. Then $R \geq_5 R'$ and $|\chi(O_M)| \geq 5|\chi(O_N)|$. But it is easy to see that R cannot be equivalently moved to a set R_1 such that $R_1 \geq_5 R'$ and $15 - |R_1| \leq 10(1-1) - 2(1-5) = 8$.

Similarly using Theorem 4 of [4] (or Proposition 3.3 of [2]) we have the following.

Proposition 2.5. *Let N be a closed, orientable Seifert manifold with Seifert invatiant $I(N) = \{g'; (\alpha'_1, \beta'_1), \dots, (\alpha'_n, \beta'_n)\}$, where $g' \geq 1$. Suppose d, d_2 are two positive integers and $d_2|d$. Let $\mathcal{D} = \{A_1, \dots, A_n, A_{n+1}, \dots, A_{n+k}\}$ be a collection of partitions of d_2 (with repetitions allowed). Then there is a closed, oriented Seifert manifold M and a d-fold fiber preserving branched covering $M \to N$ with orbifold degree d_2 and orbifold branch data $\mathcal{D}$ (the corresponding branch set in N/S^1 is the n cone points and k regular points) if and only if the total branching $v(\mathcal{D}) := (n+k)d_2 - \sum_{l=1}^{n+k} |A_l|$ is even, where $|A_l|$ denotes the number of components of A_l (abusing notation).*

Remark. A cone point of the orbifold O_N of N may be a downstairs branch point of the induced map $\overline{f} : O_M \to O_N$ while the corresponding singular fiber is not in the downstairs branch set of $f : M \to N$, and vice versa.

3. Applications to nonzero degree maps between Seifert manifolds

Now we can generalize Theorem 3.2 of [8] to the case of arbitrary degree (but for sufficiency we need the additional assumption that the orbifold genus of the target manifold ≥ 1).

Theorem 3.0. *Let M, N be aspherical, closed, oriented Seifert manifolds with Seifert invariants $I(M) = \{g; R\}$, $I(N) = \{g'; R'\}$, where $g' \geq 1$. Let d be a positive integer. Suppose N admits no Euclidean geometry. Then there is a map $M \to N$ with degree d if and only if there is a set $\{\hat{g}; \hat{R}\}$ such that $g \geq \hat{g}$ and $R \geq \hat{R}$ (which is defined in [8]), and $\{\hat{g}; \hat{R}\}$ and $\{g'; R'\}$ satisfy the relation in Theorem 2.2 (satisfied by $\{g; R\}$ and $\{g'; R'\}$) for some integer $d_1|d$.*

Proof. First we prove necessity. If there is a map $f : M \to N$ of degree d, then by the theorem of Rong cited at the beginning of this note we have $f \simeq g\pi$, where $\pi : M \to \hat{M}$ is a composition of finitely many vertical pinches, and g is a fiber preserving branched covering. By Theorem 3.2 of [8] $\hat{M}$ has Seifert invariant $\{\hat{g}; \hat{R}\}$ which satisfies $g \geq \hat{g}$ and $R \geq \hat{R}$. Now the necessity follows from Theorem 2.2.

Sufficiency follows immediately from Theorem 3.2 of [8] and Theorem 2.2.

Now we have the following generalization of Corollary 4.1 of [8].

Corollary 3.1. *Let M be an aspherical, closed, oriented Seifert manifold. Then given a nonzero integer d, there are only finitely many aspherical, closed, oriented Seifert manifolds N such that there is a map $f : M \to N$ with degree d.*

Proof. We may assume $d > 0$. Suppose N is an aspherical, closed, oriented Seifert manifold such that there is a map $f : M \to N$ with degree d. Note that there are only 10 closed Euclidean 3-manifolds, so we can assume that N admits no Euclidean geometry. Then the Seifert invariants of M and N have the relation as described in Theorem 3.0. By Corollary 4.1 of [8] there are only finitely many aspherical, closed, oriented Seifert manifolds $\hat{M}$ such that there is a degree one map $M \to \hat{M}$. So we need only show that given an aspherical, closed, oriented Seifert manifold $\hat{M}$, there are only finitely many aspherical, closed, oriented Seifert manifolds N such that there is a d-fold fiber preserving branched covering $\hat{M} \to N$. By Corollary 2.4 $-\chi(O_{\hat{M}}) \geq -d_2\chi(O_N)$, where d_2 is the orbifold degree. Let $\overline{M} = \hat{M}/(Z/d_1)$, where $d_1 = d/d_2$. From Theorem 2.2 and Definition 2.0 we have $\overline{\alpha}_j \le \alpha'_i \le d_2\overline{\alpha}_j$ for any i, where $\overline{\beta}_j/\overline{\alpha}_j \in \overline{R}_i$. There are only finitely many 2-orbifolds O_N which satisfy these two conditions. Also $e(N) = (d_1/d_2)e(M)$. So there are only finitely many such N. □

Corollary 3.2. *Let M, N be aspherical, closed, oriented Seifert manifolds with given Seifert invariants, where the orbifold genus of $N \ge 1$. Then given a nonzero integer d there is an algorithm to decide whether there is a map $f : M \to N$ of degree d. Furthermore, if $e(N) \ne 0$ and $\chi(O_N) \ne 0$ the set $\{d :$ there is a map $f : M \to N$ of degree $d\}$ is finite and can be decided.*

Proof. For simplicity we assume $e(N) \ne 0$ and $\chi(O_N) \ne 0$. If there is a map $f : M \to N$ of degree d, say, then $e(M) \ne 0$ and $|d| \le |e(N)|\chi(O_M)^2/(|e(M)|\chi(O_N)^2)$ by Corollaries 2.3 and 2.4. We can find the finitely many Seifert manifolds $\hat{M}$ such that there is a degree one map $M \to \hat{M}$ by the proof of Corollary 4.1 of [8]. For each $\hat{M}$ let $\overline{M} = \hat{M}/(Z/d_1)$ (where d_1 is the divisor of d such that $e(M) = (d_2/d_1)e(N)$ and $|\chi(O_M)| \ge d_2|\chi(O_N)|$ (where $d_2 = d/d_1$)). By Lemma 2.1 and the finiteness of the number of partitions of d_2 we can decide whether there is a d_2-fold fiber preserving branched covering $\overline{M} \to N$ of fiber degree one.

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References

[1] A. Casson, D. Jungreis, Convergence groups and Seifert fibered 3-manifolds, Invent. Math. 118 (1994), 441-456. MR **96f:**57011

[2] A. L. Edmonds, R. S. Kulkarni and R. E. Stong, Realizability of branched coverings of surfaces, Trans. Amer. Math. Soc. 282 (1984), No. 2, 773-790. MR **85k:**57005

[3] D. Gabai, Convergence groups are Fuchisian groups. Ann. of Math. 136 (1992), 447-510. MR **93m:**20065

[4] D. Husemoller, Ramified coverings of Riemann surfaces, Duke Math. J. 29 (1962), 167-174. MR **25:**188

[5] W. Jaco, P. Shalen, Seifert Fibered Spaces in 3-manifolds, Memoirs of Amer. Math. Soc. 21 (1979) (no. 220). MR **81c:**57010

[6] W. D. Neumann, F. Raymond, Seifert manifolds, plumbing, μ-invariant and orientation reversing maps, Lecture Notes in Mathematics 664, 162-195. MR **80e:**57008

[7] Y. Rong, Maps between Seifert fibered spaces of infinite π_1, Pacific J. Math. 160 (1993), No. 1, 143-154. MR **94e:**55026

[8] Y. Rong, Degree one maps of Seifert manifolds and a note on Seifert volume, Topol. Appl. 64 (1995), 191-200. MR **96c:**57034

[9] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc. 15 (1983), 401-487. MR **84m:**57009

[10] H. Seifert, Topologie dreidimensionaler gefaserter Raume, Acta Math. 60 (1933), 147-238.

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