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FIVE DEGREES OF SEPARATION

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ABSTRACT. If A is an infinite Abelian group, $S \subseteq A \times A$, then S can be transformed in five steps of type $(x, y) \mapsto (x, y+f(x))$ or $(x, y) \mapsto (x+f(y), y)$ into a predetermined subset of the diagonal (depending on $\min(|S|, |(A \times A) - S|)$).

Let A be an infinite Abelian group. Consider the "slides" of subsets of $A \times A$, i.e., the operations of the form $B \mapsto B^f$, $B \mapsto B_f$ where $f : A \to A$ is a function, $B \subseteq A \times A$, and

$$B^f = \{(x, y + f(x)) : (x, y) \in B\}, \quad B_f = \{(x + f(y), y) : (x, y) \in B\}.$$

In [2] Miklós Abért and Tamás Keleti prove that for the group $\mathbf{R}$ of reals, if $B, C \subseteq \mathbf{R} \times \mathbf{R}$, $|B| = |C|$, $|(\mathbf{R} \times \mathbf{R}) \setminus B| = |(\mathbf{R} \times \mathbf{R}) \setminus C|$, then C can be obtained from B with a bounded number of slides. Notice that the hypotheses about the cardinalities are obviously necessary. From this they proved the rather interesting corollary that every permutation of $\mathbf{R} \times \mathbf{R}$ is the product of at most N slides for some natural number N.

In this note we give an alternative proof of their first result extending it to every infinite Abelian group A.

We notice that the method of Abért and Keleti again gives the corollary above for arbitrary Abelian groups (with the same natural number N). This latter result implies that every infinite symmetric group S_κ is the set product of N Abelian subgroups (simply consider the symmetric group of all permutations of $A \times A$ where A is any Abelian group of cardinality κ; see [1]).

Rather than working with sets we work with functions, so the results will be stated for functions of the type $F : A \times A \to \{0, 1\}$. Also, instead of using slides on both coordinates we just transform in the direction of the y axis and use the change of the coordinates. All arguments should actually be done in either coordinate; the reason the author uses only transformations in the y axis is that he is scared that he would otherwise hopelessly mix up the formulas.

Notation. If $B \subseteq A$, $a \in A$, then $B + a = \{x + a : x \in B\}$. Every cardinal is, as usual, identified with its initial ordinal.

Let A be an infinite Abelian group, $\kappa = |A|$. Fix an enumeration of A as $A = \{a_\alpha : \alpha < \kappa\}$.

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If $F : A \times A \to \{0,1\}$, then set $F^*(x,y) = F(y,x)$. If, moreover $f : A \to A$, then define F^f as follows: $F^f\,(x, y + f(x)) = F(x,y)$.

We set $F \sim_0 F^*$, $F \sim_1 F^f$, and in general $F \sim_n G$ if we can reach G from F with n steps where going from F to F^* counts as 0 steps and passing from F to F^f counts as 1 step.

$F \in \mathcal{A}$ iff $|\{(x,y) : F(x,y) = \varepsilon\}| = \kappa$ holds $\varepsilon = 0, 1$. $I : A^2 \to \{0,1\}$ is the diagonal function, i.e., $I(x,y) = 1$ iff $x = y$. (Notice that $I^* = I$.)

Theorem 1. *If $F \in \mathcal{A}$, then $F \sim_5 I$. Therefore, if $F, F' \in \mathcal{A}$, then $F \sim_{10} F'$.*

Proof. In what follows we will consider various classes of functions $F : A \times A \to \{0,1\}$ and will denote them by $\mathcal{B}, \mathcal{C}, \ldots$. In these cases we set $\mathcal{B}^+ = \mathcal{B} \cup \{F^* : F \in \mathcal{B}\}$, etc.

First define the class $\mathcal{B}$ as follows. $F \in \mathcal{B}$ iff the set

$$\{x : \exists y_0 F(x,y_0) = 0, \exists y_1 F(x,y_1) = 1\}$$

has cardinality κ.

Lemma 1. *If $F \in \mathcal{A}$, then $F \in \mathcal{B}^+$.*

Proof. Assume that the statement fails for some $F \in \mathcal{A}$.

If there exist κ many x that $F(x,y) = 0$ for every y, and there exist κ many x that $F(x,y) = 1$ for every y, then clearly $F^* \in \mathcal{B}$.

So there is a set B with $|B| < \kappa$ such that if $x \in A \setminus B$, then for every y we have $F(x,y) = 1$ (say). Similarly we get that there are $\varepsilon \in \{0,1\}$ and C with $|C| < \kappa$ such that for $y \in A \setminus C$ and every x we have $F(x,y) = \varepsilon$. If we pick $x \notin B$, $y \notin C$, then we get $\varepsilon = F(x,y) = 1$. But then $\{(x,y) : F(x,y) = 0\} \subseteq B \times C$, so $F \notin \mathcal{A}$. □

We next define class $\mathcal{C}$. F is in $\mathcal{C}$ iff for every x there are y_0, y_1 with $F(x,y_0) = 0$, $F(x,y_1) = 1$.

Lemma 2. *If $F \in \mathcal{B}^+$, then there is an $F' \in \mathcal{C}^+$ with $F \sim_1 F'$.*

Proof. We can assume that $F \in \mathcal{B}$. Set

$$B = \{x : \exists y_0 \exists y_1 F(x,y_0) = 0, F(x,y_1) = 1\}.$$

By assumption, $|B| = \kappa$. Enumerate B as $B = \{b_\alpha : \alpha < \kappa\}$. Choose c_α and d_α in such a way that

$$F(b_{2\alpha}, c_\alpha) = 0, F(b_{2\alpha+1}, d_\alpha) = 1.$$

Let $f : A \to A$ be such a function that

$$f(b_{2\alpha}) = a_\alpha - c_\alpha, f(b_{2\alpha+1}) = a_\alpha - d_\alpha$$

holds for every $\alpha < \kappa$ and f is arbitrary on $A \setminus B$. Then $(F^f)^* \in \mathcal{C}$ will hold:

$$F^f(b_{2\alpha}, a_\alpha) = 0, \quad F^f(b_{2\alpha+1}, a_\alpha) = 1.$$

□

Let P be the set of quadruples of the form (x, X, Y_0, Y_1) where $x \in A$, $X = \{a_\beta : \beta < \alpha\}$ for some $\alpha < \kappa$, $Y_0, Y_1 \subseteq A \setminus X$, $Y_0 \cap Y_1 = \emptyset$, $|Y_0|, |Y_1| \leq 2$.

Set $F \in \mathcal{D}$ iff the following holds. For every $(x, X, Y_0, Y_1) \in P$ there exists some $t = t(x, X, Y_0, Y_1) \in A$ such that

$$F\,(x, t + (X \cup Y_0)) = 0, F(x, t + Y_1) = 1.$$

Lemma 3. *If $F \in \mathcal{C}^+$, then there is some $F' \in \mathcal{D}^+$ such that $F \sim_1 F'$.*

Proof. We start with a claim.

Claim. There exist elements $t = t(x, X, Y_0, Y_1) \in A$ such that the sets

$$\{t(x, X, Y_0, Y_1) + (X \cup Y_0 \cup Y_1) : (x, X, Y_0, Y_1) \in P\}$$

are pairwise disjoint.

Proof of the Claim. One can construct a well ordering of P such that for every $\alpha < \kappa$ if $A_\alpha = \{a_\beta : \beta < \alpha\}$, then the set

$$P_\alpha = \{(x, X, Y_0, Y_1) \in P : \{x\} \cup X \cup Y_0 \cup Y_1 \subseteq A_\alpha\}$$

is bounded in P. Then, as $|P_\alpha| < \kappa$, we can choose $t(x, X, Y_0, Y_1)$ by transfinite recursion (at every step all but $< \kappa$ elements of A can be chosen). □

The conditions

$$F\left(x, t(x, X, Y_0, Y_1) + (X \cup Y_0)\right) = 0, F(x, t(x, X, Y_0, Y_1) + Y_1) = 1$$

(for $(x, X, Y_0, Y_1) \in P$) impose one requirement (at the most) per vertical line, so there is a slide F^f of F meeting it. □

F is in class $\mathcal{F}$ if it is *function-like*, i.e., for every $x \in A$ there is a unique y such that $F(x, y) = 1$.

Lemma 4. *If $F \in \mathcal{D}^+$, then there is an $F' \in \mathcal{F}^+$ with $F \sim_2 F'$.*

Proof. Assume first that A is countable, $A = \{a_0, a_1, \dots\}$ with $a_0 = 0$. Let F be in $\mathcal{D}$.

We are going to build a function $G = F^f$ in an inductive construction.

Set $k_0 = 0$, $X_0 = Y_0 = Z_0 = \{a_0\}$, $b_0 = 0$, $f(0) = 0$.

In the $n + 1$-st step we are going to construct $f(a_{k_n+1}), \dots, f(a_{k_{n+1}})$, so they (with the earlier choices) will determine $G(a_i, a_j)$ for $i \le k_{n+1}$, j arbitrary.

Set

$$X_{n+1} = \{a_i : 0 \le i \le k_n, F(a_i, a_{n+1}) = 1\}.$$

Choose a_r of minimal index from $A \setminus (Z_0 \cup \dots \cup Z_n)$.

Let $k_{n+1} > k_n$ be so large that if b_{n+1} is determined by $b_{n+1} + a_{k_{n+1}} = a_r$, then

$$(X_{n+1} + b_{n+1}) \cap (Z_0 \cup \dots \cup Z_n) = \emptyset$$

holds. We now determine the values $f(a_{k_n+1}), \dots, f(a_{k_{n+1}})$ in such a way that $F^f(a_i, a_j) = 0$ holds for $k_n < i \le k_{n+1}$, $0 \le j \le n+1$, except that $F^f(a_{k_{n+1}}, a_{n+1}) = 1$.

After this, set $Y_{n+1} = X_{n+1} \cup \{a_{k_{n+1}}\}$, $Z_{n+1} = Y_{n+1} + b_{n+1}$.

Notice that $Z_{n+1} \cap (Z_0 \cup \dots \cup Z_n) = \emptyset$ and $a_r \in Z_{n+1}$. We get, therefore, that $(Z_0, Z_1, \dots)$ is a partition of A into finite subsets. Moreover, if we define the function $g : A \to A$ by letting $g(a_n) = b_n$, then as the equality

$$Y_n = \{x \in A : G(x, a_n) = 1\}$$

holds for every n, we therefore have that $(G^*)^g \in \mathcal{F}^+$.

Assume now that A is uncountable and let $F^* \in \mathcal{D}$.

Let $A = \{a_\alpha : \alpha < \kappa\}$ be the well ordering from the previous lemma, but we assume that it starts with $a_{-1} = 0$. We consider the increasing, continuous decomposition $A = \bigcup\{A_\alpha : \alpha < \kappa\}$ into subgroups of smaller cardinality in such a way that $a_{2\alpha}, a_{2\alpha+1} \in A_{\alpha+1}$ and the index of A_α in $A_{\alpha+1}$ is at least 4. Also, assume that $A_0 = \{0\}$.

We first define some $F'' \sim_1 F$. Set $F''(0,0) = 1$.

By transfinite recursion on $0 \le \alpha < \kappa$ we define, for $x \in A_{\alpha+1} \setminus A_\alpha$, some values of the type $F''(x, y)$.

First, let $F''(x, a_\beta) = 0$ for $\beta < 2\alpha$.

Assume that $c_\alpha, d_\alpha, c_\alpha - d_\alpha \in A_{\alpha+1} \setminus A_\alpha$ are in distinct cosets mod A_α.

Define

(a) $F''(A_\alpha + (-c_\alpha), a_{2\alpha}) = 0$,
(b) $F''(x + (d_\alpha - c_\alpha), a_{2\alpha}) = 1 - F''(x, a_{2\alpha+1})$ for $x \in A_\alpha$,
(c) $F''(x, a_{2\alpha}) = 1$, for $x \in A_{\alpha+1} \setminus (A_\alpha \cup (A_\alpha + (-c_\alpha)) \cup (A_\alpha + (d_\alpha - c_\alpha)))$,
(d) $F''(x + (c_\alpha - d_\alpha), a_{2\alpha+1}) = 1 - F''(x, a_{2\alpha})$ for $x \in A_\alpha$,
(e) $F''(x, a_{2\alpha+1}) = 0$, for $x \in A_{\alpha+1} \setminus (A_\alpha \cup (A_\alpha + (c_\alpha - d_\alpha)))$.

We can find such an $F'' \sim_1 F$, for this requires for any fixed value of $x \in A_{\alpha+1} \setminus A_\alpha$ the values $F''(x, a_\beta) = 0$ for $\beta < 2\alpha$ and either 0 or 1 at the places $a_{2\alpha}$ and $a_{2\alpha+1}$, and that is possible as $F \in \mathcal{D}$.

Set $G = (F'')^*$. Let f be the following $A \to A$ function: $f(0) = 0$, $f(a_{2\alpha}) = -c_\alpha$, $f(a_{2\alpha+1}) = -d_\alpha$ $(0 \le \alpha < \kappa)$. Let $F' = (G^f)^*$. We claim that $F' \in \mathcal{F}$, that is, F' is function-like.

First, $F'(0,0) = 1$. Moreover, $F'(0, a_{2\alpha}) = F''(-c_\alpha, a_{2\alpha}) = 0$, $F'(0, a_{2\alpha+1}) = F''(-d_\alpha, a_{2\alpha+1}) = 0$ hold for $0 \le \alpha < \kappa$.

Assume now that $x \neq 0$. Then, as $A = \bigcup\{A_\alpha : \alpha < \kappa\}$ is a continuous, increasing decomposition, there is a unique $0 \le \alpha < \kappa$ such that $x \in A_{\alpha+1} \setminus A_\alpha$.

If $\beta > \alpha$, then

$$F'(x, a_{2\beta}) = F''(x - c_\beta, a_{2\beta}) = 0$$

and

$$F'(x, a_{2\beta+1}) = F''(x - d_\beta, a_{2\beta+1}) = 0.$$

If, however, $0 \le \beta < \alpha$, then $F'(x, a_{2\beta}) = F''(x - c_\beta, a_{2\beta}) = 0$, $F'(x, a_{2\beta+1}) = F''(x - d_\beta, a_{2\beta+1}) = 0$ as $x - c_\beta, x - d_\beta \in A_{\alpha+1} \setminus A_\alpha$.

Finally, we calculate

$$F'(x, a_{2\alpha}) + F'(x, a_{2\alpha+1}) = F''(x - c_\alpha, a_{2\alpha}) + F''(x - d_\alpha, a_{2\alpha+1}).$$

If we show that this is 1 for every $x \in A_{\alpha+1} \setminus A_\alpha$ we will be done.

If $x \in A_\alpha + c_\alpha$, then this sum is 1 by (d);
if $x \in A_\alpha + d_\alpha$, then this sum is 1 by (b);
if $x \in A_{\alpha+1} \setminus ((A_\alpha + c_\alpha) \cup (A_\alpha + d_\alpha) \cup A_\alpha)$, then

$$F''(x - c_\alpha, a_{2\alpha}) + F''(x - d_\alpha, a_{2\alpha+1}) = 1 + 0 = 1$$

by (c) and (e).

□

Lemma 5. *If $F \in \mathcal{F}^+$, then $F \sim_1 I$.*

Proof. This is easy. Assume that $F \in \mathcal{F}$. Let $f : A \to A$ be the following function: $F(x, f(x)) = 1$. Then $F = I^f$. □

In order to show the remaining (easy) case of the Abért-Keleti theorem we make the following definitions. If $\mu < \kappa, F : A \times A \to \{0,1\}$ set $F \in \mathcal{A}_\mu$ iff $|\{(x, y) :$

$F(x,y) = 1\}| = \mu$. Fix a set $S \subseteq A$ of cardinal μ. $I_\mu(x,y) = 1$ for $x = y \in S$, otherwise it is 0.

Theorem 2. *If $F \in \mathcal{A}_\mu$, then $F \sim_3 I_\mu$.*

Proof. Assume that $F \in \mathcal{A}_\mu$. For $y \in A$ set $T_y = \{x \in A : F(x,y) = 1\}$. There are at most μ nonempty sets T_y, each of cardinality at most μ. We can, therefore, slide each of them, so that they become mutually disjoint. This way we get $G \sim_1 F$ such that G^* is partially function-like (every vertical line contains at most one 1), so there is some $H \sim_1 G$ which is partially function-like and the x-th vertical line contains a one iff $x \in S$. Finally, $H \sim_1 I_\mu$. $\square$

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References

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