

BLOW-UP OF SEMILINEAR PDE'S AT THE CRITICAL DIMENSION. A PROBABILISTIC APPROACH

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ABSTRACT. We present a probabilistic approach which proves blow-up of solutions of the Fujita equation $\partial w/\partial t = -(-\Delta)^{\alpha/2}w + w^{1+\beta}$ in the critical dimension $d = \alpha/\beta$. By using the Feynman-Kac representation twice, we construct a subsolution which locally grows to infinity as $t \rightarrow \infty$. In this way, we cover results proved earlier by analytic methods. Our method also applies to extend a blow-up result for systems proved for the Laplacian case by Escobedo and Levine (1995) to the case of α -Laplacians with possibly different parameters α .

1. INTRODUCTION AND OVERVIEW

Consider the semilinear equation

$$(1.1) \quad \begin{aligned} \frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + \gamma w_t^{1+\beta}, \\ w_0 &= \varphi, \end{aligned}$$

in \mathbb{R}^d , where $\Delta_\alpha := -(-\Delta^{\alpha/2})$, $0 < \alpha \leq 2$, denotes the α -Laplacian, β and γ are positive numbers and the initial condition φ is a nonnegative function on \mathbb{R}^d .

In Fujita's pioneering work [4] it was shown (originally for the case $\alpha = 2$) that $d = \alpha/\beta$ is the critical dimension for blow-up of (1.1): if $d > \alpha/\beta$, then (1.1) admits a global solution for all sufficiently small initial conditions, whereas if $d < \alpha/\beta$, then for any nonvanishing initial condition the solution is infinite for suitably large t .

For the case $d = \alpha/\beta$ it was proved by Sugitani [12] by subtle analytic arguments that (1.1) blows up. Using different, partly probabilistic methods, this was also proved by Portnoy ([9, 10]) for the special case $\alpha = 2$, $\beta = 1$. Related results on systems where the space variable is restricted to a bounded domain in \mathbb{R}^d can be found in the recent paper of Wang [13] and the references therein.

In this note we give a short probabilistic proof for blow-up at the critical dimension, using the Feynman-Kac representation. Here is an outline.

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Recall that the solution w of the initial value problem on $[0, T] \times \mathbb{R}^d$

$$(1.2) \quad \begin{aligned} \frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + w_t v_t, \\ w_0 &= \varphi, \end{aligned}$$

with $v : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}_+$ locally bounded has by the Feynman-Kac formula (cf. Stroock [11], §4.3, Freidlin [3], Thm. 2.2, or Dynkin [1], Thm. 9.7) a probabilistic interpretation as the density (with respect to Lebesgue measure on \mathbb{R}^d) of the measure

$$(1.3) \quad \int \mathbb{E}_x \left[\mathbf{1}(W_t \in dy) \exp \int_0^t v_s(W_s) ds \right] \varphi(x) dx = w_t(y) dy$$

where \mathbb{E}_x denotes expectation with respect to the symmetric α -stable process (W_t) started at $W_0 = x$. This shows in particular that any solution \tilde{w} of (1.2) with v replaced by $\tilde{v} \leq v$ and $\tilde{w}_0 = w_0$ fulfills $\tilde{w} \leq w$.

Consider for $i = 0, 1, 2$ the initial value problems

$$(1.4) \quad \begin{aligned} \frac{\partial w_{t,i}}{\partial t} &= \Delta_\alpha w_{t,i} + \gamma w_{t,i} w_{t,i-1}^\beta, \\ w_{0,i} &= \varphi, \end{aligned}$$

where $w_{t,-1} = 0$. Then $f_t := w_{t,0}$, $g_t := w_{t,1}$ and $h_t := w_{t,2}$ are all subsolutions of (1.1). Since $f_t(y) = \mathbb{E}_y[\varphi(W_t)]$, where (W_t) is a symmetric α -stable process, $f_t(y)$ decays like $\text{const} \cdot t^{-d/\alpha}$ (see Section 2). Since “typically” $f_s(W_s)$ should be bounded from below by $\text{const} \cdot s^{-d/\alpha}$, and also $\mathbb{P}_x \{W_t \in dy\} \geq \text{const} \cdot t^{-d/\alpha} dy$ as long as $\|y - x\| \leq t^{1/\alpha}$, one should expect (using (1.3) with $v_s = f_s^\beta$ to express the solution of (1.4) for $i = 1$) that

$$(1.5) \quad \begin{aligned} g_t(y) &= \int \mathbb{E}_x \left[\exp \int_0^t f_s(W_s)^\beta ds \mid W_t = y \right] \varphi(x) dx \\ &\geq ct^{-d/\alpha} \exp \left(\text{const} \int_1^t s^{-d\beta/\alpha} ds \right) \\ &= ct^{-d/\alpha} \exp(\text{const} \cdot \log t) \geq ct^{-d/\alpha + \varepsilon} \end{aligned}$$

as long as $\|y\| \leq t^{1/\alpha}$. This intuition can be turned into a proof basically by applying Jensen’s inequality and scaling arguments.

After dealing in this way in Proposition 2.1 with the case $i = 1$, we then turn to the case $i = 2$ in (1.4). The function h_t , like g_t , also has a Feynman-Kac representation, but now with f_s^β replaced by g_s^β in the exponent. By (1.5), the integrand $g_s(W_s)^\beta$ in this exponent should “typically” remain bounded from below by $\text{const} \cdot s^{-1+\varepsilon\beta}$. Thus we expect that

$$h_t(y) \geq \text{const} \cdot t^{-d/\alpha} \exp \left(-c \int_0^t s^{-1+\varepsilon\beta} ds \right),$$

and in fact we will prove this in Proposition 2.3. In particular, h_t is a subsolution of (1.1) which locally grows to infinity. This fact suffices to show blow up, as we will recall in Section 3.

Section 4 comments briefly on the case of subcritical dimensions, and Section 5 on Portnoy’s method. In Section 6 we give some extensions. Apart from re-proving Sugitani’s result, we show that blow-up of (1.1) with a certain *time-dependent* nonlinearity, which was recently proved by Guedda and Kirane [5], arises as an easy corollary of our probabilistic approach.

In Section 7 we obtain conditions for blow-up of a class of semilinear *systems*. We are able to extend a blow-up result of Escobedo and Levine [2] and show blow-up at the critical dimensions of a system which we were able to analyze before only in the case of sub- and supercritical dimensions [7, 8].

2. CONSTRUCTING SUBSOLUTIONS BY THE FEYNMAN-KAC FORMULA

In this and the following section we consider $d = \alpha/\beta$ and prove that (1.1) blows up in this case. Furthermore assume without loss of generality that the initial condition φ of (1.1) does not vanish a.s. on the unit ball. Let $p_t(x)$ denote the transition density of the symmetric α -stable process, and write

$$(2.1) \quad f_t(y) := \int p_t(y - x)\varphi(x) dx = \mathbb{E}_y [\varphi(W_t)].$$

For all $t \geq 1$ we have the inequality

$$(2.2) \quad f_t(y) \geq c_0 t^{-d/\alpha} \mathbf{1}_{B_1}(t^{-1/\alpha}y) \int_{B_1} \varphi(x) dx$$

for some $c_0 > 0$, where B_r denotes the ball in \mathbb{R}^d with radius r centered at the origin. Indeed, let $y \in B_{t^{1/\alpha}}$. Then we have by the scaling property of W_t

$$\begin{aligned} f_t(y) &= \mathbb{E}_0 [\varphi(W_t + y)] = \mathbb{E}_0 \left[\varphi \left(t^{1/\alpha}(W_1 + t^{-1/\alpha}y) \right) \right] \\ &\geq \int_{B_1} p_1(x - t^{-1/\alpha}y)\varphi(t^{1/\alpha}x) dx \geq c_0 \int_{B_1} \varphi(t^{1/\alpha}x) dx \\ &= c_0 t^{-d/\alpha} \int_{B_{t^{1/\alpha}}} \varphi(x) dx. \end{aligned}$$

This argument also shows that, for sufficiently large t ,

$$(2.3) \quad f_t(y) \geq c'_0 t^{-d/\alpha} \mathbf{1}_{B_1}(t^{-1/\alpha}y)$$

for some $c'_0 > 0$.

2.1. The first iteration: a subsolution with a slow decay. We are going to obtain a lower bound for the solution g_t of

$$(2.4) \quad \begin{aligned} \frac{\partial g_t}{\partial t} &= \Delta_\alpha g_t + \gamma g_t f_t^\beta, \\ g_0 &= \varphi, \end{aligned}$$

where f_t is defined in (2.1). Since f_t is a subsolution of (1.1), g_t is a subsolution of (1.1) as well.

Proposition 2.1. *There exist $\varepsilon, c > 0$ such that, for all $t \geq 2$ and all $y \in \mathbb{R}^d$ obeying $\|y\| \leq t^{1/\alpha}$,*

$$(2.5) \quad g_t(y) \geq c t^{-d/\alpha+\varepsilon}.$$

Proof. By the Feynman-Kac formula, g_t arises as the density of the measure defined in (1.3) (with v_s replaced by f_s^β). We therefore have, using (2.2) and Jensen's

inequality,

$$\begin{aligned}
 g_t(y) &= \int \varphi(x) p_t(y-x) \mathbb{E}_x \left[\exp \int_0^t \gamma f_s(W_s)^\beta ds \mid W_t = y \right] dx \\
 &\geq \int \varphi(x) p_t(y-x) \mathbb{E}_x \left[\exp \int_1^{t/2} c_2 s^{-\beta d/\alpha} \mathbf{1}_{B_{s^{1/\alpha}}}(W_s) ds \mid W_t = y \right] dx \\
 &\geq \int_{B_1} \varphi(x) p_t(y-x) \exp \left(c_2 \int_1^{t/2} s^{-\beta d/\alpha} \mathbb{P}_x \{ W_s \in B_{s^{1/\alpha}} \mid W_t = y \} ds \right) dx \\
 (2.6) \quad &\geq c_3 t^{-d/\alpha} \exp \left(c_4 \int_1^{t/2} s^{-\beta d/\alpha} ds \right)
 \end{aligned}$$

where the last estimate relies on Lemma 2.2 below. (Here and below $c_i, i = 1, 2, \dots$, denote “locally defined” positive constants.) The assertion now follows from our assumption $d = \alpha/\beta$. \square

The intuition behind the following assertion is clear: conditioning on some “typical” state at time t does not much affect the behavior of (W_t) between times 0 and $t/2$.

Lemma 2.2. *There exists a $c > 0$ such that for all $t \geq 2, y \in B_{t^{1/\alpha}}, x \in B_1$ and $s \in [1, t/2]$,*

$$(2.7) \quad \mathbb{P}_x \{ W_s \in B_{s^{1/\alpha}} \mid W_t = y \} \geq c.$$

Proof. First note that (2.7) is equivalent to

$$(2.8) \quad \int_{B_{s^{1/\alpha}}} p_s(z-x) p_{t-s}(y-z) dz \geq c_5 p_t(y-x).$$

Next, let us state the following facts, which are easy consequences of the scaling property of (W_t) :

(i) For all $z \in B_{s^{1/\alpha}}$ and $r := t - s$

$$\begin{aligned}
 p_r(y-z) dz &= \mathbb{P}_0 \{ r^{1/\alpha} W_1 + y \in dz \} \geq \inf_{a \in B_{2^{1/\alpha}}} \mathbb{P}_0 \{ W_1 \in 2^{1/\alpha} t^{-1/\alpha} dz - a \} \\
 &\geq c_5 t^{-d/\alpha} dz.
 \end{aligned}$$

(ii) Similarly, for all $z \in B_{s^{1/\alpha}}, p_s(z-x) \geq c_6 s^{-d/\alpha}$.

Combining (i) and (ii) we see that the LHS of (2.8) is bounded from below by $c_7 t^{-d/\alpha}$. Since $p_t(\cdot)$ is bounded above by $\text{const} \cdot t^{-d/\alpha}$ the claim is proved. \square

2.2. The second iteration: a subsolution growing to infinity. We are now aiming at a lower estimate for the solution h_t of

$$\begin{aligned}
 (2.9) \quad \frac{\partial h_t}{\partial t} &= \Delta_\alpha h_t + h_t g_t^\beta, \\
 h_0 &= \varphi,
 \end{aligned}$$

where g_t is the subsolution of (1.1) constructed in the previous subsection. Clearly, h_t is also a subsolution of (1.1).

Proposition 2.3. *$\inf \{ h_t(y) \mid \|y\| \leq 1 \} \rightarrow \infty$ as $t \rightarrow \infty$. More specifically there exist constants $\varepsilon, c', c'' > 0$ such that*

$$h_t(y) \geq c' t^{-d/\alpha} \exp(c'' t^{\varepsilon\beta}) \mathbf{1}_{B_1}(y).$$

Proof. We proceed as in the proof of Proposition 2.1. First we note that the Feynman-Kac formula gives

$$(2.10) \quad h_t(y) = \int \varphi(x) p_t(y-x) \mathbb{E}_x \left[\exp \int_0^t \gamma g_s(W_s)^\beta ds \mid W_t = y \right] dx.$$

Using Jensen's inequality and (2.5), we see that the RHS of (2.10) is bounded from below by

$$(2.11) \quad \begin{aligned} & \int \varphi(x) p_t(y-x) \exp \left(\gamma \int_2^{t/2} \mathbb{E}_x [g_s(W_s)^\beta \mid W_t = y] ds \right) dx \\ & \geq \int_{B_1} \varphi(x) p_t(y-x) \\ & \quad \cdot \exp \left(\gamma \int_2^{t/2} c s^{-\beta d/\alpha + \varepsilon \beta} \mathbb{P}_x \{W_s \in B_{s^{1/\alpha}} \mid W_t = y\} ds \right) dx \end{aligned}$$

$$(2.12) \quad \geq c_8 t^{-d/\alpha} \exp(c_9 t^{\varepsilon \beta}).$$

Here, we used Lemma 2.2 and the assumption $d = \alpha/\beta$ in the last inequality. \square

3. COMPLETION OF THE PROOF OF BLOW-UP

From Proposition 2.3 we know that

$$(3.1) \quad K(t) := \inf_{x \in B_1} w_t(x) \rightarrow \infty \text{ as } t \rightarrow \infty$$

where B_1 denotes the unit ball. In fact this is enough to guarantee blow-up. Here is an easy argument which is borrowed from [6], §4, and which we include for convenience.

We are going to re-start (1.1) with the initial condition w_{t_0} , with a suitable choice of t_0 given below. Writing $u_t := w_{t_0+t}$ we first recall the integral form of (1.1):

$$(3.2) \quad u_t(x) = \int p_t(y-x) u_0(y) dy + \int_0^t \gamma ds \int p_{t-s}(y-x) u_s(y)^{1+\beta} dy.$$

Noting that $\zeta := \min_{x \in B_1} \min_{0 \leq s \leq 1} \mathbb{P}_x \{W_s \in B_1\}$ is strictly positive, we obtain for all $t \in [0, 1]$ from (3.1) the estimate

$$(3.3) \quad \min_{x \in B_1} u_t(x) \geq \zeta K(t_0) + \gamma \zeta \int_0^t \left(\min_{y \in B_1} u_s(y) \right)^{1+\beta} ds.$$

Now choose t_0 so big that the blow-up time of the equation

$$(3.4) \quad v(t) = \zeta K(t_0) + \gamma \zeta \int_0^t v(s)^{1+\beta} ds$$

is smaller than 1. Then, *a fortiori*, $\min_{x \in B_1} u_1(x) = \infty$, which shows blow-up of w .

4. SUBCRITICAL DIMENSIONS: ONE ITERATION SUFFICES

In the case $d < \alpha/\beta$, (2.6) shows that already the first subsolution g_t (constructed in Section 2.1) grows to infinity on the unit ball B_1 in the sense that $\inf\{g_t(y) \mid \|y\| \leq 1\} \rightarrow \infty$ as $t \rightarrow \infty$. Thus, in view of the previous section, for subcritical dimensions a single application of the Feynman-Kac formula suffices to show blow-up of (1.1).

5. A REMARK ON PORTNOY'S METHOD

Portnoy [9] studies the iteration scheme

$$(5.1) \quad \begin{aligned} v_{n+1}(x) &= (\Pi_1 v_n)(x) + (\Pi_1 v_n)^2(x) \\ v_0 &= \varphi \geq 0 \end{aligned}$$

where Π_1 is a transition probability on \mathbb{R}^d . He shows that under suitable assumptions on Π_1 (which include the case of a standard Brownian transition probability), (5.1) admits no bounded solution for $d = 1$ and $d = 2$ provided φ does not a.s. vanish.

A closer look on his proofs shows that he achieves this by analyzing subsolutions $v_n^{(i)}$ of (5.1) which are given by the scheme

$$(5.2) \quad \begin{aligned} v_{n+1}^{(0)} &= \Pi_1 v_n^{(0)} = \Pi_{n+1} \varphi, \\ v_{n+1}^{(i)} &= \Pi_1 v_n^{(i)} + \left(\Pi_1 v_n^{(i)} \right) \left(\Pi_1 v_n^{(i-1)} \right), \quad i = 1, 2. \end{aligned}$$

The analysis of (5.2) is carried through probabilistically in terms of random walks, which is much in the spirit of a discrete time Feynman-Kac approach.

It can be extracted from Portnoy's arguments that, for the Brownian case, say,

$$(5.3) \quad v_n^{(1)} \text{ grows to infinity for } d = 1$$

and

$$(5.4) \quad v_n^{(2)} \text{ grows to infinity for } d = 2.$$

An easy application of Jensen's inequality plus induction shows that w_n is bounded from below by v_n (where w_t is the solution of (1.1) with $\beta = 1$). Indeed,

$$\begin{aligned} w_n &= \Pi_1 w_{n-1} + \int_0^1 \Pi_s w_{n-s}^2 ds \geq \Pi_1 w_{n-1} + \left(\int_0^1 \Pi_s w_{n-s} ds \right)^2 \\ &\geq \Pi_1 w_{n-1} + \left(\int_0^1 \Pi_s \Pi_{1-s} w_{n-1} ds \right)^2 \geq \Pi_1 v_{n-1} + (\Pi_1 v_{n-1})^2 = v_n. \end{aligned}$$

Together with the argument in Section 3 above, (5.3) and (5.4) thus imply blow-up of w for $\beta = 1$ and $\alpha = 2$ in one and two dimensions. (In [10], a more complicated argument is used to show $w_n \geq v_n$ and the blow-up of w .)

6. EXTENSIONS

6.1. Sugitani's condition. Sugitani [12] considers instead of (1.1) the slightly more general equation

$$(6.1) \quad \begin{aligned} \frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + F(w_t), \\ w_0 &= \varphi, \end{aligned}$$

where $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is increasing and convex, and $F(u) \sim \gamma u^{1+\beta}$ as $u \rightarrow 0$. This requires only slight modifications in Section 2:

In (2.4) and below, $f_t(u)^\beta$ has to be replaced by $F(f_t(u))/f_t(u)$, which by assumption can be bounded from below by $c f_t(u)^\beta$.

Similarly, in (2.9) and below, $g_t(u)^\beta$ has to be replaced by $F(g_t(u))/g_t(u)$.

6.2. A time dependent nonlinearity. Recently, Guedda and Kirane [5] showed by analytic methods blow-up of the equation

$$(6.2) \quad \frac{\partial w_t}{\partial t} = \Delta_\alpha w_t + \gamma t^\sigma w_t^{1+\beta}, \quad w_0 = \varphi \ (\geq 0, \neq 0)$$

for $\sigma \geq \beta d/\alpha - 1$. This result also follows quickly from our probabilistic approach. In fact, it suffices to consider the case $\sigma = \beta d/\alpha - 1$.

Lemma 6.1. *The solution of*

$$(6.3) \quad \begin{aligned} \frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + v_t w_t^{1+\beta}, \\ w_0 &= \varphi \ (\geq 0, \neq 0) \end{aligned}$$

with $v : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}_+$, $v_t(x) \geq \text{const} \cdot t^{\beta d/\alpha - 1} \mathbf{1}_{B_1}(t^{-1/\alpha}x)$ for $t \geq 1$ blows up in finite time.

We briefly indicate the changes required in the arguments presented in Sections 2 and 3 in order to prove Lemma 6.1.

1. Concerning the subsolution g_t , all that happens is that a factor $s^\sigma \mathbf{1}_{B_{s^{1/\alpha}}}(\cdot)$ enters into the exponentials in the Feynman-Kac representation in the RHS of (2.6). Since $s^{-\beta d/\alpha}$ in the RHS of (2.6) cancels against s^σ , the lower bound (2.6) remains unchanged, and so does the estimate (2.5).

2. Concerning the subsolution h_t , again a factor s^σ enters into the exponentials in (2.10) and (2.11). Since again $(s^{-d/\alpha})^\beta$ cancels against s^σ , the lower bound (2.12) remains unchanged, and so does the assertion in Proposition 2.3.

3. Concerning the argument in Section 3, from the space-time-inhomogeneity in (6.3) a factor $(t_0 + t)^\sigma$ enters in front of the integral in (3.3). (Observe that by our assumption $v_t \geq \text{const} \cdot t^\sigma$ uniformly on B_1 for $t \geq 1$.) Still, since (2.12) guarantees a super-algebraic growth of $K(t)$, we can choose t_0 so big that the blow-up time of the equation

$$v(t) = \zeta K(t_0) + \gamma \zeta (t_0 + 1)^\sigma \int_0^t v(s)^{1+\beta} ds$$

is smaller than 1, so that the argument of Section 3 remains valid.

7. BLOW-UP OF SYSTEMS

In this section we apply our probabilistic approach to extend a blow-up result of Escobedo and Levine [2] (Theorem 7.1 and Remark 7.2). In Theorem 7.3 we show that a system which we investigated in [8] in high dimensions blows up at the critical dimension.

Theorem 7.1. *Assume that (u, v) solves*

$$(7.1) \quad \begin{aligned} \frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + u_t^{1+\beta_1} v_t^{\beta_2}, \\ \frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + F(u_t, v_t), \\ u_0 &= \varphi_1, \quad v_0 = \varphi_2, \end{aligned}$$

where $\alpha_1, \alpha_2 \in (0, 2]$, $\beta_1 > 0$, $\beta_2 \geq 0$, $F \geq 0$, $\varphi_1 \geq 0$, $\varphi_2 \geq 0$ and both φ_1 and φ_2 do not a.s. vanish. Then u blows up if

$$(7.2) \quad \alpha_2 \leq \alpha_1 \text{ and } d \leq \left(\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right)^{-1}.$$

Remark 7.2. For $\alpha_1 = \alpha_2 =: \alpha$, (7.2) turns into the condition $d \leq \alpha/(\beta_1 + \beta_2)$, which is also the condition for blow-up of the partial differential equation

$$\frac{\partial u}{\partial t} = \Delta_\alpha u + u^{1+\beta_1+\beta_2}.$$

For $\alpha = 2$, this specializes to one of the main results in Escobedo and Levine’s paper [2]. They investigate by analytic tools the system

$$\frac{\partial u}{\partial t} = \Delta u + u^{1+\beta_1}v^{\beta_2}, \quad \frac{\partial v}{\partial t} = \Delta v + u^{\theta_1}v^{\theta_2}$$

and prove blow-up under the condition $d \leq 2/(\beta_1 + \beta_2)$.

Proof of Theorem 7.1. Let $f_{t,j}(y) := \int \varphi_j(x)p_{t,j}(y - x) dx$, $j = 1, 2$, where $p_{t,j}$ denotes the symmetric α_j -stable transition density. Obviously, $(f_{t,1}, f_{t,2})$ is a sub-solution of (7.1), and from (2.2) we have for $t \geq 1$

$$(7.3) \quad f_{t,1}(y) \geq Ct^{-d/\alpha_1} \mathbf{1}_{B_1}(t^{-1/\alpha_1}y)$$

and

$$(7.4) \quad f_{t,2}(y) \geq Ct^{-d/\alpha_2} \mathbf{1}_{B_1}(t^{-1/\alpha_1}y),$$

where we used the assumption $\alpha_2 \leq \alpha_1$ to obtain (7.4). Consequently for $t \geq 1$ and $\|y\| \leq t^{1/\alpha_1}$

$$v_t(y)^{\beta_2} \geq C't^{-d\beta_2/\alpha_1} \geq C't^{d\beta_1/\alpha_1-1}$$

where we used the assumption (7.2) in the last inequality. Now we infer blow-up of u using Lemma 6.1. □

Theorem 7.3. Assume that (u, v) solves

$$(7.5) \quad \begin{aligned} \frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + u_t v_t, \\ \frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + u_t v_t, \\ u_0 &= \varphi_1, \quad v_0 = \varphi_2, \end{aligned}$$

where $\alpha_1, \alpha_2 \in (0, 2]$, $\varphi_1 \geq 0$, $\varphi_2 \geq 0$ and both φ_1 and φ_2 do not a.s. vanish. Then (u, v) blows up if $d \leq \min(\alpha_1, \alpha_2)$.

Remark 7.4. It was shown in [8] that (7.5) admits global solutions if $d > \min(\alpha_1, \alpha_2)$ and φ_1 and φ_2 are sufficiently small.

Before proving Theorem 7.3, we prepare with a lemma which is an easy generalization of Lemma 2.2. Here and below, $(W_t^{(i)})$ denotes the symmetric stable process with index α_i and $p_{t,i}(x)$ its transition density, $i = 1, 2$.

Lemma 7.5. Assume that $\alpha := \alpha_2 \leq \alpha_1$. There exists a $c > 0$ such that for all $t \geq 2$, $y \in B_{t^{1/\alpha}}$, $x \in B_1$ and $s \in [1, t/2]$,

$$\mathbb{P}_x \left\{ W_s^{(2)} \in B_{s^{1/\alpha_1}} \mid W_t^{(2)} = y \right\} \geq cs^{d/\alpha_1-d/\alpha_2}.$$

Proof. It suffices to show (2.8) with $cs^{d/\alpha_1-d/\alpha_2}$ instead of c_5 and $p_{t,2}$ instead of p_t .

Again we have (i) and (ii) from the proof of Lemma 2.2, now with $(W_t^{(2)})$ instead of (W_t) . Integrating the bound s^{-d/α_2} over $B_{s^{1/\alpha_1}}$ then gives the factor $\text{const} \cdot s^{d/\alpha_1-d/\alpha_2}$. □

Proof of Theorem 7.3. The proof proceeds in three steps. First we prove using the Feynman-Kac representation (see (1.3)) that (at least one component of) the solution (u, v) locally grows to ∞ . In a second step we show that (u, v) can be bounded below uniformly in $B_1 \times B_1$ similarly as in Section 3 but this time by comparison with the solution of a suitable coupled pair of ODEs. Finally, in step 3 we show that this system of ODEs blows up.

1. From (2.3) we have

$$(7.6) \quad u_t \geq c_1 t^{-d/\alpha_1} \mathbf{1}_{B_{t^{1/\alpha_1}}}$$

and

$$(7.7) \quad v_t \geq c_2 t^{-d/\alpha_2} \mathbf{1}_{B_{t^{1/\alpha_2}}}$$

for all $t \geq t_0$ for some sufficiently large t_0 . Let us now assume without loss of generality that $\alpha_2 \leq \alpha_1$. By the Feynman-Kac formula we have

$$u_t(y) = \int \varphi_1(x) p_{t,1}(y-x) \mathbb{E}_x \left[\exp \int_0^t v_s(W_s^{(1)}) ds \mid W_t^{(1)} = y \right] dx.$$

For $t \geq 2t_0$, by Jensen's inequality and (7.7), this can be bounded from below by

$$\int \varphi_1(x) p_{t,1}(y-x) \exp \left(\int_{t_0}^{t/2} c_2 s^{-d/\alpha_2} \mathbb{P}_x \left\{ W_s^{(1)} \in B_{s^{1/\alpha_2}} \mid W_t^{(1)} = y \right\} ds \right) dx.$$

Noting that $B_{s^{1/\alpha_2}} \supseteq B_{s^{1/\alpha_1}}$ and using Lemma 2.2, we thus arrive at the lower bound

$$(7.8) \quad c_3 t^{-d/\alpha_1} \exp \left(c_4 \int_{t_0}^{t/2} s^{-d/\alpha_2} ds \right).$$

If $d < \alpha_2$, then this lower bound grows super-algebraically from which we will infer blow-up in steps 2 and 3.

Let us now assume $d = \alpha_2$. Then (7.8) turns into the lower bound

$$(7.9) \quad u_t(y) \geq c_5 t^{-d/\alpha_1+\varepsilon}$$

(uniformly in $y \in B_{t^{1/\alpha_1}}$ for t sufficiently large). Another application of the Feynman-Kac formula gives

$$(7.10) \quad v_t(y) = \int \varphi_2(x) p_{t,2}(y-x) \mathbb{E}_x \left[\exp \int_0^t u_s(W_s^{(2)}) ds \mid W_t^{(2)} = y \right] dx.$$

Using Jensen's inequality and (7.9), we can bound this from below by

$$\int \varphi_2(x) p_{t,2}(y-x) \exp \int_{t_0}^{t/2} c_1 s^{-d/\alpha_1+\varepsilon} \mathbb{P}_x \left\{ W_s^{(2)} \in B_{s^{1/\alpha_1}} \mid W_t^{(2)} = y \right\} ds dx.$$

In view of Lemma 7.5 we thus obtain as a lower bound for $v_t(y)$ (as long as t is sufficiently large and $y \in B_{t^{1/\alpha_2}}$):

$$\begin{aligned} c_6 t^{-d/\alpha_2} \exp \int_{t_0}^{t/2} c_7 s^{-d/\alpha_1 + \varepsilon} s^{d/\alpha_1 - d/\alpha_2} ds &= c_6 t^{-d/\alpha_2} \exp \int_{t_0}^{t/2} c_7 s^{-d/\alpha_2 + \varepsilon} ds \\ &= c_6 t^{-d/\alpha_2} \exp(c_8 t^\varepsilon). \end{aligned}$$

Thus in this case v grows (super-algebraically).

2. Rewriting (7.5) in integral form we obtain for $t, t_0 \geq 0$

$$\begin{aligned} u_{t+t_0}(x) &= \int dy p_{t,1}(y-x) u_{t_0}(y) + \int_0^t ds \int dy p_{t-s,1}(y-x) u_{t_0+s}(y) v_{t_0+s}(y), \\ v_{t+t_0}(x) &= \int dy p_{t,2}(y-x) v_{t_0}(y) + \int_0^t ds \int dy p_{t-s,2}(y-x) u_{t_0+s}(y) v_{t_0+s}(y). \end{aligned}$$

Let $\zeta := \min_{x \in B_1} \min_{0 \leq s \leq 1} \left(\mathbb{P}_x(W_s^{(1)} \in B_1) \wedge \mathbb{P}_x(W_s^{(2)} \in B_1) \right) > 0$ and $\tilde{u}(t) := \min_{x \in B_1} u_t(x)$, $\tilde{v}(t) := \min_{x \in B_1} v_t(x)$. This allows us to estimate for $t \in [0, 1]$

$$\begin{aligned} (7.11) \quad \tilde{u}(t_0 + t) &\geq \zeta \tilde{u}(t_0) + \zeta \int_0^t ds \tilde{u}(t_0 + s) \tilde{v}(t_0 + s), \\ \tilde{v}(t_0 + t) &\geq \zeta \tilde{v}(t_0) + \zeta \int_0^t ds \tilde{u}(t_0 + s) \tilde{v}(t_0 + s). \end{aligned}$$

In step 1 we saw that $(\tilde{u} \vee \tilde{v})(t_0) \rightarrow \infty$ super-algebraically while $(\tilde{u} \wedge \tilde{v})(t_0)$ decays at most algebraically. Thus, t_0 can be chosen so big that the blow-up time of

$$(7.12) \quad U(t) = \zeta \tilde{u}(t_0) + \zeta \int_0^t ds U(s) V(s), \quad V(t) = \zeta \tilde{v}(t_0) + \zeta \int_0^t ds U(s) V(s)$$

is less than 1 (see step 3). We conclude that (u, v) blows up.

3. It remains to study (7.12) which in ODE form is

$$U'(t) = \zeta U(t) V(t) = V'(t)$$

and WLOG assume that $U_0 := U(0) \geq V(0) =: V_0$. The solution is given by

$$\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = \begin{cases} \frac{U_0 - V_0}{1 - (V_0/U_0) \exp(\zeta(U_0 - V_0)t)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ V_0 - U_0 \end{pmatrix} & \text{if } U_0 > V_0, \\ \frac{1}{1/U_0 - \zeta t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } U_0 = V_0, \end{cases}$$

for $0 \leq t < \tau$ with explosion time

$$\tau = \begin{cases} \frac{\log U_0 - \log V_0}{\zeta(U_0 - V_0)} & \text{if } U_0 > V_0, \\ \frac{1}{\zeta U_0} & \text{if } U_0 = V_0. \end{cases}$$

In our scenario we have $U_0 \geq \exp(\varepsilon_1 t_0)$, $V_0 \geq t_0^{-\varepsilon_2}$ for some $\varepsilon_1, \varepsilon_2 > 0$, which allows us to choose t_0 big enough to enforce $\tau < 1$. Indeed if $V_0 \geq U_0/2$ we have $\tau \leq 2/(\zeta U_0)$, and if $1 \leq V_0 < U_0/2$ we can estimate $\tau \leq (2 \log U_0)/(\zeta U_0)$. Finally, if $V_0 < 1$ we have $\tau \leq (\log U_0)/(\zeta(U_0 - 1)) + \varepsilon_2 \log t_0/(\zeta(\exp(\varepsilon_1 t_0) - 1))$. \square

Remark 7.6. Consider instead of (7.5) the more general system

$$(7.13) \quad \begin{aligned} \frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + u_t v_t^{\beta_1}, \\ \frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + u_t^{\beta_2} v_t, \\ u_0 &= \varphi_1, \quad v_0 = \varphi_2, \end{aligned}$$

where $\alpha_1, \alpha_2, \varphi_1, \varphi_2$ are as in Theorem 7.3, and $\beta_1, \beta_2 > 0$. Assume that $\alpha_2 \leq \alpha_1$. Proceeding as in the proof of Theorem 7.3 but using the simple bound (7.6) instead of (7.9) in the Feynman-Kac representation corresponding to (7.10) one quickly obtains that (7.13) has a growing subsolution if

$$(7.14) \quad d < \max \left(\frac{\alpha_2}{\beta_1}, \left(\frac{\beta_2 - 1}{\alpha_1} + \frac{1}{\alpha_2} \right)^{-1} \right).$$

As before, from this one infers blow-up, this time by comparing with the ODE system $U'(t) = U(t)V^{\beta_1}(t)$, $V'(t) = V(t)U^{\beta_2}(t)$.

It remains an interesting question whether the RHS of (7.14) is the critical dimension for blow-up of (7.13) and whether there is blow-up at the critical dimension. We conjecture that this is the case at least for $\alpha_1 = \alpha_2 =: \alpha$, in which case the RHS of (7.14) turns into $\alpha / \min(\beta_1, \beta_2)$. Indeed, for the special case $\alpha = 2$, this was proved by Escobedo and Levine [2].

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