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BOUNDEDNESS OF THE BERGMAN TYPE OPERATORS ON MIXED NORM SPACES

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ABSTRACT. Conditions sufficient for boundedness of the Bergman type operators on certain mixed norm spaces $L_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$ of functions on the unit ball of C^n are given, and this is used to solve Gleason's problem for the mixed norm spaces $H_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$.

1. INTRODUCTION

For $z = (z_1, z_2, \cdots, z_n)$, $w = (w_1, w_2, \cdots, w_n) \in C^n$, we shall denote the inner product of z, w by $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm of z by $\|z\| = \sqrt{\langle z, z\rangle}$. Let $B = B_n = \{z \in C^n : \|z\| < 1\}$ be the open unit ball in C^n. The space of holomorphic functions on B will be denoted by $H(B)$. A positive continuous function φ on [0,1) is normal (see [1]) if there exist $0 < a < b, 0 \leq r_0 < 1$ such that:

(1) $\frac{\varphi(r)}{(1-r)^a}$ is nonincreasing for $r_0 \leq r < 1$ and $\lim\limits_{r\to 1} \frac{\varphi(r)}{(1-r)^a} = 0$;

(2) $\frac{\varphi(r)}{(1-r)^b}$ is nondecreasing for $r_0 \leq r < 1$ and $\lim\limits_{r\to 1} \frac{\varphi(r)}{(1-r)^b} = \infty$.

For $0 < p < \infty$, $0 < q < \infty$ and a normal function φ, let $L_{p,q}(\varphi)$ denote the spaces of measurable functions on B with

$$\|f\|_{p,q,\varphi} = \left\{\int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r)M_q^p(r,f)dr\right\}^{1/p} < \infty,$$

where

$$M_q(r,f) = \left\{\int_{\partial B} |f(r\zeta)|^q d\sigma(\zeta)\right\}^{1/q}.$$

Let $1 < q < \infty$. Equipped with the above norm, $L_{p,q}(\varphi)$ is a Banach space for $p \geq 1$. When $0 < p < 1$, $\|\cdot\|_{p,q,\varphi}^p$ is a quasinorm on $L_{p,q}(\varphi)$. Thus, $L_{p,q}(\varphi)$ is a metric space if supplied with the distance $d(f,g) = \|f-g\|_{p,q,\varphi}^p$ and the vector space operations are continuous in this metric. That this d is complete is proved in the same way as in the familiar case $p \geq 1$. There is nothing to guarantee that $\|\alpha f\|_{p,q,\varphi}^p = |\alpha|\|f\|_{p,q,\varphi}^p$, however, and so $\|\cdot\|_{p,q,\varphi}^p$ may not be a norm. So $L_{p,q}(\varphi)$ is a Frechet space but not a Banach space. The reader is referred to [2] for the

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basic theory of Frechet and Banach spaces. For $s \in R, t > 0$, the operator $P_{s,t}$ on $L_{p,q}(\varphi)$ is given by

$$P_{s,t}f(z) = c_{n,t}(1 - \|z\|^2)^s \int_B \frac{(1 - \|w\|^2)^{t-1} f(w)}{(1 - \langle z, w\rangle)^{n+t+s}} dV(w), \quad \forall f \in L_{p,q}(\varphi),$$

where the complex power is understood to be principal branches,

$$c_{n,t} = C^n_{n+t-1} = \frac{\Gamma(n+t)}{\Gamma(t)\Gamma(n+1)}.$$

G.B. Ren and J.H. Shi in [1] show that: If $t > b > a > -s$, then $P_{s,t}$ is a bounded operator of $L_{p,q}(\varphi)$ into $L_{p,q}(\varphi)$ $(1 \le p < \infty, 1 \le q < \infty)$, but the problem which is still unsolved is the case $0 < p < 1$. We know that the main tools for proving the above results are Hölder's inequality and the results due to Forelli and Rudin (e.g. see [3], Proposition 2.7 or Lemma 3.1, etc.). Because Hölder's inequality can be used only in the case $1 < p < \infty$, the method in [5], [6], [1],[7] and [8] cannot deal with the case $0 < p < 1$. In order to overcome this difficulty, we will exploit the Hardy-Littlewood analytical technique in [9]. Here we discuss the boundedness of the operator $P_{s,t}$ on $L_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$ and obtain a sufficient condition. One of the main points of the paper is to extend some results for a Banach space setting to a Frechet space setting. Our main result is:

Theorem A. *Let $0 < p < 1, 1 < q < \infty$. If $t > b > a > -s$ and φ is a normal function, then $P_{s,t} : L_{p,q}(\varphi) \to L_{p,q}(\varphi)$ is a bounded operator.*

Throughout this paper, the letter C stands for a positive different constant.

2. Background and preliminaries

The following lemmas will be needed in the proof of Theorem A.

Lemma 1 ([4], Proposition 1.4.10)**.** *For $z \in B$, c real, $t > -1$, define*

$$J_{c,t}(z) = \int_B \frac{(1 - \|w\|^2)^t}{|1 - \langle z, w\rangle|^{n+1+t+c}} dV(w).$$

Then:

(1) *when $c < 0$, $J_{c,t}$ is bounded in z;*
(2) *when $c > 0$, $J_{c,t}(z) \sim (1 - \|z\|^2)^{-c}$, where $\sim$ denotes $(1 - \|z\|^2)^c J_{c,t}(z)$ has a positive limit $(\|z\| \to 1)$.*

Lemma 2 ([1], Lemma 2.3)**.** *Let φ be a normal function. If $s + t > b > a > s$, then*

$$\int_0^1 \frac{\varphi^p(\rho) d\rho}{(1-\rho)^{ps+1}(1 - r\rho)^{pt}} \le C \frac{\varphi^p(r)}{(1-r)^{p(s+t)}} \ (0 \le r < 1, p > 0).$$

Lemma 3 ([1], Lemma 2.1)**.** *If $s + t > 0, 1 \le q < \infty$, then*

$$M_q(\rho, P_{s,t}f) \le C(1 - \rho^2)^s \int_0^1 \frac{r^{2n-1}(1 - r^2)^{t-1}}{(1 - r\rho)^{t+s}} M_q(r, f) dr.$$

This follows from Hölder's inequality and Lemma 1 (see [1]).

In order to prove our main result, we first prove the following lemma.

Lemma 4. *If $t > b > a > -s$, $0 < p < 1$ and $1 < q < \infty$, then*

$$M^p_q(\rho, P_{s,t}f) \le C(1 - \rho^2)^{ps} \int_0^1 \frac{r^{p(2n-1)}(1 - r)^{pt-1}}{(1 - r\rho)^{p(t+s)}} M^p_q(r, f) dr.$$

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Proof. Denote $s_k = 1 - 2^{-k}$. By the monotonicity of the integral means $M_q^p(r, f)$ with respect to r, Lemma 3 and the elementary inequality $(a + b)^p \leq a^p + b^p$ $(a, b \geq 0, 0 < p < 1)$, we obtain

$$\begin{aligned} M_q^p(\rho, P_{s,t}f) &\leq C(1-\rho^2)^{ps}\left(\int_0^1 \frac{r^{2n-1}(1-r)^{t-1}}{(1-r\rho)^{t+s}} M_q(r,f)dr\right)^p \\ &= C(1-\rho^2)^{ps}\left(\sum_{k=1}^{\infty}\int_{s_{k-1}}^{s_k} \frac{r^{2n-1}(1-r)^{t-1}}{(1-r\rho)^{t+s}} M_q(r,f)dr\right)^p \\ &\leq C(1-\rho^2)^{ps}\left(\sum_{k=1}^{\infty} \frac{s_k^{2n-1}(1-s_{k-1})^{t}}{(1-s_k\rho)^{t+s}} M_q(s_k,f)\right)^p \\ &\leq C(1-\rho^2)^{ps}\sum_{k=1}^{\infty} \frac{s_k^{p(2n-1)}(1-s_{k-1})^{pt}}{(1-s_k\rho)^{p(t+s)}} M_q^p(s_k,f) \\ &\leq C(1-\rho^2)^{ps}\int_0^1 \frac{r^{p(2n-1)}(1-r)^{pt-1}}{(1-r\rho)^{p(t+s)}} M_q^p(r,f)dr. \end{aligned}$$

This completes the proof of the lemma.

3. Proof of the main theorem

We can now prove the main result of this paper.

Proof of Theorem A. For any $f \in L_{p,q}(\varphi)$, by Lemma 3, Lemma 2, Lemma 4 and Fubini's Theorem, we have

$$\begin{aligned} \|P_{s,t}f\|_{p,q,\varphi}^p &\leq \int_0^1 (1-\rho)^{-1}\varphi^p(\rho)M_q^p(\rho, P_{s,t}f)d\rho \\ &\leq C\int_0^1\left(\int_0^1 \frac{r^{p(2n-1)}(1-r)^{pt-1}}{(1-r\rho)^{p(t+s)}} M_q^p(r,f)dr\right)(1-\rho)^{ps-1}\varphi^p(\rho)d\rho \\ &= C\int_0^1\left(\int_0^1 \frac{\varphi^p(\rho)}{(1-\rho)^{1-ps}(1-r\rho)^{p(t+s)}}d\rho\right) r^{p(2n-1)}(1-r)^{pt-1}M_q^p(r,f)dr \\ &\leq C\int_0^1 r^{p(2n-1)}(1-r)^{-1}\varphi^p(r)M_q^p(r,f)dr. \end{aligned}$$

Use the change of variables $r = \rho^{\frac{1}{p}}$ and note that $p < 1$, $\rho^{\frac{1}{p}} < \rho$. Thus

$$(1-\rho^{\frac{1}{p}})^{-1} < (1-\rho)^{-1} \text{ and } M_q^p(\rho^{\frac{1}{p}}, f) < M_q^p(\rho, f).$$

Finally, since φ is normal,

$$\varphi^p(\rho^{\frac{1}{p}}) \leq \frac{(1-\rho^{\frac{1}{p}})^{bp}}{(1-\rho)^{bp}}\varphi^p(\rho).$$

That is, $\varphi^p(\rho^{\frac{1}{p}}) \leq C\varphi^p(\rho)$. So

$$\begin{aligned} \|P_{s,t}f\|_{p,q,\varphi}^p &\leq C\int_0^1 \rho^{2n-1}(1-\rho^{\frac{1}{p}})^{-1}\varphi^p(\rho^{\frac{1}{p}})M_q^p(\rho^{\frac{1}{p}}, f)d\rho \\ &\leq C\int_0^1 \rho^{2n-1}(1-\rho)^{-1}\varphi^p(\rho)M_q^p(\rho,f)d\rho = C\|f\|_{p,q,\varphi}^p. \end{aligned}$$

Hence the proof of Theorem A follows.

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Remark. Let

$$P^{\sim}_{s,t} f(z) = c_{n,t}(1 - \|z\|^2)^s \int_B \frac{(1 - \|w\|^2)^{t-1}|f(w)|}{|1 - \langle z, w\rangle|^{n+t+s}} dV(w), \quad \forall f \in L_{p,q}(\varphi).$$

The proof of Theorem A actually gives us a little more, that is, there exists a constant C such that $\|P^{\sim}_{s,t} f\|_{p,q,\varphi} \le C\|f\|_{p,q,\varphi}$, $\forall f \in L_{p,q}(\varphi)$.

4. An application

We denote the holomorphic mixed norm space $L_{p,q}(\varphi) \cap H(B)$ by $H_{p,q}(\varphi)$, and the norm in $H_{p,q}(\varphi)$ is equivalent to

$$\|f\|_{p,q,\varphi} = \left\{ \int_0^1 (1 - r)^{-1} \varphi^p(r) M_q^p(r, f) dr \right\}^{1/p}.$$

In this section, we investigate Gleason's problem on $H_{p,q}(\varphi)$. To the best of our knowledge, this problem has been answered respectively by Zhu in [10], Ortega in [11], Choe in [5], and Ren and Shi in [1], [12], [13] in the case $1 \le p < \infty$. So a natural question comes up: How is the case $0 < p < 1$? As an application of Theorem A, we obtain the following:

Theorem B. *Gleason's problem can be solved on $H_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$. More precisely, for any integer $m > 1$, there exist bounded linear operators A_α on $H_{p,q}(\varphi)$ such that if $f \in H_{p,q}(\varphi)$, $D^\alpha f(0) = 0$ $(|\alpha| \le m - 1)$, then $f(z) = \sum_{|\alpha|=m} z^\alpha A_\alpha f(z)$ on B, with $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.*

The fundamental ideas used in arguing this theorem come from the references [10] and [13].

Proof. Assume $m = 1$. By Leibenson's technique, we get

$$f(z) = \sum_{k=1}^n z_k \int_0^1 \frac{\partial f}{\partial z_k}(rz) dr,$$

where $f \in H_{p,q}(\varphi)$ and $f(0) = 0$. Set $A_k f(z) = \int_0^1 \frac{\partial f}{\partial z_k}(rz)dr, z \in B$. A_k is obviously linear, so it remains to show that A_k is bounded on $H_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$. Given $f \in H_{p,q}(\varphi)$, let $f_r(z) = f(rz)$, $r \in (0, 1)$. We have $P_{0,t} f_r = f_r$; see [1]. Letting $r \to 1^-$, the boundedness of $P_{0,t}$ implies that $P_{0,t} f = f$. Thus

$$f(z) = c_{n,t} \int_B \frac{(1 - \|w\|^2)^{t-1} f(w)}{(1 - \langle z, w\rangle)^{n+t}} dV(w),$$

where $t > b$, $t \in \mathcal{N}$.

Differentiating under the integral gives

$$\frac{\partial f}{\partial z_k}(z) = c_{n,t} \int_B \frac{\overline{w_k}(1 - \|w\|^2)^{t-1} f(w)}{(1 - \langle z, w\rangle)^{n+t+1}} dV(w).$$

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This implies that

$$\begin{aligned}A_k f(z) &= C\int_0^1 dr\int_B \frac{\overline{w_k}(1-\|w\|^2)^{t-1}f(w)}{(1-r\langle z,w\rangle)^{n+t+1}}dV(w)\\ &= C\int_B \overline{w_k}(1-\|w\|^2)^{t-1}f(w)dV(w)\int_0^1\frac{1}{(1-r\langle z,w\rangle)^{n+t+1}}dr\\ &= C\int_B \frac{\overline{w_k}(1-\|w\|^2)^{t-1}f(w)}{(1-\langle z,w\rangle)^{n+t}}Q(z,w)dV(w),\end{aligned}$$

where

$$Q(z,w)=\frac{1-(1-\langle z,w\rangle)^{n+t}}{\langle z,w\rangle}=\sum_{k=1}^{n+t-1}(1-\langle z,w\rangle)^k.$$

Note that $Q(z,w)$ is a polynomial in z and $\overline{w}$. Thus we can find a constant $C>0$ such that $|A_k f(z)|\le C|P_{0,t}^{\sim}f(z)|$. By the Remark after Theorem A, we see that A_k is bounded on $H_{p,q}(\varphi)$ $(0<p<1, 1<q<\infty)$. For m in the general case, it can be proved by induction. Therefore, the proof is complete.

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References

1. G.B. Ren and J.H. Shi, *Bergman type operator on mixed norm spaces with applications*, Chin. Ann. of Math. 18(B)(1997), 265–276. MR **98k:**32006
2. A. Brown and C. Pearcy, *Introduction to operator theory I: elements of functional analysis*, GTM 55, Springer-Verlag, New York, Berlin, Heidelberg. MR **58:**23463
3. F. Forelli and W. Rudin, *Projections on spaces of holomorphic functions on balls*, Indiana Univ. Math. J., 24(6)(1974): 593–602. MR **50:**10332
4. W. Rudin, *Function theory in the unit ball of C^n*, Springer-Verlag, New York, 1980. MR **82i:**32002
5. B.R. Choe, *Projection, weighted Bergman spaces, and the Bloch space,* Proc. Amer. Math. Soc., 108(1)(1990): 127–136. MR **90h:**32009
6. J. Xiao, *Compactness for Toeplitz and Hankel operators on weighted Bergman spaces in ball in C^n*, Science in China, 23A(8)(1993): 811–818. MR **95b:**47030
7. A.L. Shields and D.L. Williams, *Bounded projections, duality and multipliers in spaces of analytic functions,* Trans. Amer. Math. Soc., 162(1971): 287–302. MR **44:**790
8. S. Gadbois, *Mixed-norm generalization of Bergman space, and duality,* Proc. Amer. Math. Soc., 104(4)(1988): 1171–1180. MR **89m:**46041
9. G.H. Hardy and J.E. Littlewood, *Some property of fractional integrals II* Math Z, 34(1932): 403–439.
10. K.H. Zhu, *The Bergman spaces, the Bloch spaces and Gleason's problem*, Trans. Amer. Soc. 309(1)(1988): 253–268. MR **89j:**46025
11. J.M. Ortega, *The Gleason problem in Bergman-Sobolev spaces,* Complex Variables, 20(1992): 157–170. MR **95d:**32011
12. G.B. Ren and J.H. Shi, *Forelli-Rudin type theorem on pluriharmonic Bergman spaces with small exponent*, Science in China, 29A(10)(1999): 909–913.
13. G.B. Ren and J.H. Shi, *Gleason's problem in weighted Bergman space egg domains*, Science in China, 41A(3)(1998): 225–231. MR **99f:**32038

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