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ON THE COMPLETENESS OF FACTOR RINGS

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ABSTRACT. Let T be a complete local domain containing the integers with maximal ideal M such that $|T/M|$ is at least the cardinality of the real numbers. Let p be a nonmaximal prime ideal of T such that T_p is a regular local ring. We construct an excellent local ring A such that the completion of A is T, the generic formal fiber of A is local with maximal ideal p and if I is a nonzero ideal of A, then A/I is complete.

1. INTRODUCTION

Let A be a local ring with maximal ideal M and M-adic completion $\hat{A}$. If A is not complete, and has dimension at least two, the "usual" case is that there are many nonzero ideals I of A such that A/I is not complete. In this paper, we construct rings that do not satisfy this "usual" condition. In other words, we are interested in noncomplete rings A such that A/I is complete for every nonzero ideal I of A.

In fact, examples of such rings are known to exist. In [9], it is shown that noncomplete excellent regular local rings R_s exist satisfying

$$\dim R_s = s + 1$$

and for any nonzero ideal I of R_s, the ring R_s/I is complete. In addition, R_s satisfies the following property: If q is a nonzero prime ideal of $\hat{R}_s$, then $q \cap R_s \neq (0)$. In other words, the dimension of the generic formal fiber of R_s is zero. (For definitions and background information on the formal fibers and generic formal fiber of an integral domain, see [8].) Also, in [1], S. Abhyankar, W. Heinzer, and S. Wiegand construct a two-dimensional normal local domain S such that S is not Henselian but $\frac{S}{P}$ is Henselian for every height one prime ideal P of S. In fact, one can show (using Weierstrass Preparation) that $\frac{S}{I}$ is complete for every nonzero ideal I of S. This ring S satisfies the property that if q is a nonzero prime ideal of $\hat{S}$, then $q \cap S \neq (0)$. So, once again, for this example, the dimension of the generic formal fiber is zero. As far as the authors know, these are the only known examples of noncomplete

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local rings A of dimension at least two satisfying the property that A/I is complete for every nonzero ideal I of A. In light of this, we ask the natural question: Let A be a local integral domain such that A/I is complete for all nonzero ideals I of A. Does it follow that the dimension of the generic formal fiber of A is zero?

In this paper, we construct examples showing that the answer to this question is no. Let T be a complete local domain with maximal ideal M containing the integers and such that $|T/M|$ is at least the cardinality of the real numbers. Let p be a nonmaximal prime ideal of T such that T_p is a regular local ring. We construct an excellent local ring A such that the completion of A is T, the generic formal fiber of A is local with maximal ideal p and if I is a nonzero ideal of A, then A/I is complete. So, given $n \geq 2$ and $0 \leq t < n$, we can construct an excellent local ring A with $\dim A = n$ and A/I complete for all nonzero ideals I of A. In addition, if, for example, T is a regular local ring, p can be any nonmaximal prime ideal of T. Thus, by choosing the height of p to be t, we can ensure that the dimension of the generic formal fiber of A is t.

The fact that A is a noncomplete excellent local ring possessing a local generic formal fiber is interesting. In [6], it is shown that a certain class of complete regular local rings can be realized as the completion of a noncomplete excellent local ring possessing a local generic formal fiber. Theorem 7 in this paper (thanks to suggestions by the referee) shows that a larger class of complete local domains can be realized as the completion of a noncomplete excellent local ring possessing a local generic formal fiber.

We note here that it can be important to know which ideals I of a ring A satisfy that A/I is complete. For example, in [3], Heinzer, Rotthaus and Sally show that if "enough" prime ideas of A satisfy this condition, then there is a one-to-one correspondence between the prime ideals in $\hat{A}$ that are maximal in the generic formal fiber of A and certain birational extensions of A. (See [3] for details.)

All rings in this paper are commutative with unity. When we say a ring is local, Noetherian is implied. For a ring with exactly one maximal ideal that is not necessarily Noetherian, we will use the term quasi-local. When we write (T, M) is a local ring, we mean that T is a local ring with maximal ideal M. Following Matsumura in [8], we will use $\alpha(A)$ to denote the dimension of the generic formal fiber of a local integral domain A. Finally, we will use c to denote the cardinality of the real numbers.

The construction of the ring A in our theorem is simply a refinement of the construction used in [6]. We make use of the result found in [5] that if $(R, M \cap R)$ is a quasi-local subring of a complete local ring (T, M), the map $R \longrightarrow T/M^2$ is onto and $IT \cap R = I$ for every finitely generated ideal I of R, then R is Noetherian and the natural homomorphism $\hat{R} \longrightarrow T$ is an isomorphism. Given a complete local domain (T, M) and a prime ideal p of T, both satisfying the conditions of our theorem, we start with the prime subring of T. We then build a chain of subrings of T (starting with the prime subring of T) with each member of the chain satisfying some "nice" properties. These rings will be called p-subrings and defined later. Our ring A will be the union of this chain of subrings. Clearly, then, we must choose our rings in the chain carefully. Specifically, we must ensure that $IT \cap A = I$ for every finitely generated ideal I of A and that $A \longrightarrow T/M^2$ is onto. These conditions will guarantee that the completion of A is T. In addition, we must have that $p \cap A = (0)$ and if q is a prime ideal of T not contained in p, then $q \cap A \neq (0)$. Finally, we must choose the chain so that A/I is complete for every nonzero ideal I of A.

2. The construction

As discussed in the previous section, the ring A of our theorem will be the union of a chain of subrings of T. We use the following proposition from [5] to guarantee that the completion of A is, in fact, T.

Proposition 1. *If $(R, M \cap R)$ is a quasi-local subring of a complete local ring (T, M), the map $R \longrightarrow T/M^2$ is onto and $IT \cap R = I$ for every finitely generated ideal I of R, then R is Noetherian and the natural homomorphism $\hat{R} \longrightarrow T$ is an isomorphism.*

The following definition is taken from [6]. In our construction, we will want to ensure that each ring in our chain satisfies the conditions needed to be a p-subring.

Definition. Let (T, M) be a complete local ring and $(R, R \cap M)$ a quasi-local subring. Let $p \neq M$ be a prime ideal of T. Suppose:

(i) $|R| \leq \sup(\aleph_0, |T/M|)$ with equality implying T/M is countable,
(ii) $R \cap p = (0)$, and
(iii) $R \cap Q = (0)$ for every $Q \in \operatorname{Ass} T$.

Then we call R a p-subring of T.

Condition (i) will be used so that R is "small" compared to T. This will allow us to choose elements in T that satisfy certain transcendental properties over R. Adjoining these elements to R is the process we will use to build our chain of subrings. Condition (ii) is needed so that p will be in the generic formal fiber of A. When T is an integral domain, condition (iii) is trivial. However, all our results except for Theorem 7 hold when T is not an integral domain and so we prove them as such. In this case, we will use condition (iii) to guarantee that A will be an integral domain.

The following lemma is Lemma 2 from [4]. Note that it is really a generalization of the prime avoidance lemma.

Lemma 2. *Let (T, M) be a local ring. Let $C \subset \operatorname{Spec} T$, let I be an ideal such that $I \not\subset P$ for every $P \in C$, and let D be a subset of T. Suppose $|C \times D| < |T/M|$. Then $I \not\subset \bigcup\{P + r \mid P \in C,\ r \in D\}$.*

Recall that to use Proposition 1, we need the map $A \longrightarrow T/M^2$ to be onto. In addition, we want A/I to be complete for every nonzero ideal I of A. Note that if T is, in fact, the completion of A where I is a nonzero ideal of A and $A \longrightarrow T/IT$ is onto, then A/I will be complete. To satisfy this condition and ensure that $A \longrightarrow T/M^2$ is onto, we will construct A so that $A \longrightarrow T/J$ is onto for every ideal J of T that is not contained in the ideal p. It turns out that this condition will also help us show that the ring A we construct is excellent. If J is an ideal of T not contained in p, and $\bar{u} \in T/J$, we use the following lemma to construct an appropriate p-subring of T that contains an element of $\bar{u}$. This lemma will be used later to guarantee that $A \longrightarrow T/J$ is onto.

Lemma 3. *Let (T, M) be a complete local ring with $|T/M| \geq c$, p a nonmaximal prime ideal of T and J an ideal of T with $J \not\subset p$. Suppose p contains all associated prime ideals of T. Let R be a p-subring of T and $\bar{u} \in T/J$. Then there exists a p-subring S of T such that $R \subset S \subset T$ and $\bar{u}$ is in the image of the map $S \longrightarrow T/J$. Moreover, if $\bar{u} = \bar{0}$, then $S \cap J \neq (0)$.*

Proof. Let $C = \text{Ass}T \cup \{p\}$ and suppose $P \in C$. Define $D_{(P)}$ to be a full set of coset representatives $t + P$ that make $(u + t) + P$ algebraic over R as an element of T/P. Now, $|D_{(P)}| = \max(|R|, \aleph_0)$. As $|T/M| \geq c$, and R is a p-subring, we have $|R| < |T/M|$. So, $|D_{(P)}| < |T/M|$. Note that by hypothesis, $J \not\subset P$ for every $P \in C$. Let $D = \bigcup_{P \in C} D_{(P)}$ and note that $|C \times D| < |T/M|$. Now, use Lemma 2 to find an element $x \in J$ such that $x \notin \bigcup\{r + P | r \in D \text{ and } P \in C\}$. We claim that $S = R[x + u]$ localized at $R[x + u] \cap M$ is the desired p-subring.

Since $|R| < |T/M|$ and $|S| = \max(|R|, \aleph_0)$, we have $|S| < |T/M|$. Hence, condition (i) of p-subrings is satisfied. Now, suppose that $P \in C$. Then if $f \in R[u + x] \cap P$, we have

$$f = r_n(u + x)^n + \cdots + r_1(u + x) + r_0 \in P.$$

So, $f \equiv 0$ modulo P. By the way x was chosen, we have that $r_i \equiv 0$ modulo P for every i. So, $r_i \in P \cap R = (0)$. Hence, $S \cap P = (0)$ and it follows that S is a p-subring. Note that under the map $S \longrightarrow T/J$, $u + x$ is mapped to $u + J$, so $u + J$ is in the image of the map. Also, note that if $\bar{u} = \bar{0}$, then $u + x \in J$. Since $(u + x) + p$ is transcendental over R as an element of T/p, we have $u + x \neq 0$. It follows that $S \cap J \neq (0)$. □

Lemma 4 is Lemma 6 from [6]. It will be needed to ensure that $IT \cap A = I$ for every finitely generated ideal I of A. This property, of course, is needed to apply Proposition 1.

Lemma 4. *Let (T, M) be a complete local ring, p a prime ideal of T, and R a p-subring. Suppose I is a finitely generated ideal of R and $c \in IT \cap R$. Then there exists a p-subring S of T with $R \subset S \subset T$ and $c \in IS$.*

Definition. Let Ω be a well-ordered set and $\alpha \in \Omega$. We define $\gamma(\alpha) = \sup\{\beta \in \Omega | \beta < \alpha\}$.

In Lemma 5, we construct a p-subring that simultaneously satisfies many of the properties we desire.

Lemma 5. *Let (T, M) be a complete local ring with $|T/M| \geq c$ and p a nonmaximal prime ideal of T. Let J be an ideal of T with $J \not\subset p$ and suppose all associated prime ideals of T are contained in p. Let $\bar{u} \in T/J$. Suppose R is a p-subring of T. Then there exists a p-subring S of T such that:*

(i) *$R \subset S \subset T$,*
(ii) *if $\bar{u} = \bar{0}$, then $S \cap J \neq (0)$,*
(iii) *u is in the image of the map $S \longrightarrow T/J$,*
(iv) *for every finitely generated ideal I of S, we have $IT \cap S = I$.*

Proof. Use Lemma 3 to find a p-subring R_0 such that $R \subset R_0 \subset T$ and $\bar{u} \in \text{Image}(R_0 \longrightarrow T/J)$ and if $\bar{u} = \bar{0}$, then $R_0 \cap J \neq (0)$. The rest of the proof follows the proof of Lemma 12 in [6], but the proof is not long and so we include it here.

We will construct S to contain R_0, so conditions (i), (ii) and (iii) will follow automatically. Let

$$\Omega = \{(I, c) \mid I \text{ a finitely generated ideal of } R_0 \text{ and } c \in IT \cap R_0\}.$$

Now, since I can be R_0, we have $|R_0| \leq |\Omega|$, and $|\Omega| \leq |R_0|$ is clear since the number of finite subsets of R_0 is $|R_0|$. So, $|\Omega| = |R_0|$. Well-order Ω so that it does not have a maximal element and let 0 denote its initial element. Now, we will define a family of p-subrings. Begin with R_0. If $\gamma(\alpha) \neq \alpha$, and $\gamma(\alpha) = (I, c)$, then choose R_α to be the p-subring gotten from Lemma 4 so that $R_{\gamma(\alpha)} \subset R_\alpha \subset T$ and $c \in IR_\alpha$. If $\gamma(\alpha) = \alpha$, choose $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$. Set $R_1 = \bigcup R_\alpha$. Now, if I is any finitely generated ideal of R_0, and $c \in IT \cap R_0$, then $(I, c) = \gamma(\alpha)$ for some $\alpha \in \Omega$. So, $c \in IR_\alpha \subset IR_1$. Thus, $IT \cap R_0 \subset IR_1$. It is easy to verify that R_1 is a p-subring.

We repeat the process to obtain a p-subring extension R_2 of R_1 such that $IT \cap R_1 \subset IR_2$ for every finitely generated ideal I of R_1. Continue to obtain an ascending chain $R_0 \subset R_1 \subset \cdots$ such that $IT \cap R_n \subset IR_{n+1}$ for every finitely generated ideal I of R_n. Then, $S = \bigcup R_i$ is a p-subring. If I is a finitely generated ideal of S, then some R_n contains a generating set for I, say $y_1, \ldots, y_k$. If $c \in IT \cap S$, then $c \in R_m$ for some $m \geq n$. So, $c \in (y_1, \ldots, y_k)T \cap R_m$; therefore, $c \in (y_1, \ldots, y_k)R_{m+1} \subset I$. Thus, $IT \cap S = I$, so condition (iv) holds. □

Finally, we can construct a ring A satisfying all conditions of our theorem except that A be excellent. Note that Lemma 6 provides nonexcellent examples of rings A with a local generic formal fiber and such that A/I is complete for every nonzero ideal I of A. In a certain sense, though, the ring A in Lemma 6 is "almost" excellent. Condition (ii) implies that at least in characteristic 0, all formal fibers except the generic formal fiber are geometrically regular. So, the only obstacle to A being excellent is that the generic formal fiber may not be geometrically regular.

Lemma 6. *Let (T, M) be a complete local ring with $|T/M| \geq c$ and such that no integer of T is a zero-divisor. Let p be a nonmaximal prime ideal of T that contains all the associated prime ideals of T. Suppose that p intersected with the prime subring of T is the zero ideal. Then there exists a local domain A such that:*

(i) $\hat{A} = T$.
(ii) *If P is a nonzero prime ideal of A, then $T \otimes_A k(P)$ is a field where $k(P) = A_p/PA_P$.*
(iii) *The generic formal fiber of A is local with p the maximal ideal.*
(iv) *If I is a nonzero ideal of A, then A/I is complete.*

Proof. Define

$$\Omega = \{u + J \in T/J \mid J \text{ is an ideal of } T \text{ with } J \not\subset p\}.$$

Since T is Noetherian, if J is an ideal of T, then J is finitely generated. Hence, $|\{J \text{ ideal of } T | J \not\subset p\}| \leq |T|$. But since $|T/M| \geq c$, we have $|T| = |T/M|$. Also, $|T/M| \leq |T/J| \leq |T|$, and so $|T/J| = |T/M|$ for every ideal J of T. It follows that $|\Omega| = |T/M|$. Well-order Ω so that each element has fewer than $|\Omega|$ predecessors. Let 0 denote the first element of Ω. We define R_0' to be the prime subring of T and R_0 to be R_0' localized at $R_0' \cap M$. Note that R_0 is a p-subring.

We recursively define a family of p-subrings as follows. R_0 is already defined. Let $\lambda \in \Omega$ and assume R_β has been defined for every $\beta < \lambda$. Then $\gamma(\lambda) = u + J$ for some ideal J of T with $J \not\subset p$. If $\gamma(\lambda) < \lambda$, use Lemma 5 to obtain a p-subring R_λ so that $R_{\gamma(\lambda)} \subset R_\lambda \subset T$, $u + J \in \operatorname{Image}(R_\lambda \longrightarrow T/J)$ and for every finitely generated ideal I of R_λ we have $IT \cap R_\lambda = I$. Moreover, if $u + J = 0 + J$, we have $R_\lambda \cap J \neq (0)$. If $\gamma(\lambda) = \lambda$, define $R_\lambda = \bigcup_{\beta<\lambda} R_\beta$. Then, R_λ is a p-subring for every $\lambda \in \Omega$. We claim $A = \bigcup_{\lambda \in \Omega} R_\lambda$ is the desired domain.

Now, as each R_λ is a p-subring, we have $R_\lambda \cap p = (0)$. Hence, $A \cap p = (0)$. Also, if J is an ideal of T with $J \not\subset p$, then $0 + J \in \Omega$. So, $\gamma(\lambda) = 0 + J$ for some $\lambda \in \Omega$

with $\gamma(\lambda) < \lambda$. By construction, $R_\lambda \cap J \neq (0)$. It follows that $J \cap A \neq (0)$. Hence, the generic formal fiber of A is local with maximal ideal p.

We will now show that the completion of A is T. To do this, we make use of Proposition 1. Note that as p is not the maximal ideal of T, we have $M^2 \not\subset p$. Hence, by the construction, the map $A \longrightarrow T/M^2$ is surjective. Let I be a finitely generated ideal of A with $I = (y_1, \dots, y_k)$. Let $c \in IT \cap A$. Then $\{c, y_1 \dots, y_k\} \subset R_\lambda$ for some $\lambda \in \Omega$ with $\gamma(\lambda) < \lambda$. By construction, $(y_1, \dots, y_k)T \cap R_\lambda = (y_1, \dots, y_k)R_\lambda$. As $c \in (y_1, \dots, y_k)T \cap R_\lambda$, we have $c \in (y_1, \dots, y_k)R_\lambda \subset I$. Hence, $IT \cap A = I$. It follows by Proposition 1 that A is Noetherian and the completion of A is T.

Now suppose I is a nonzero ideal of A. Let $J = IT$. If $J \subseteq p$, then $I \subseteq J \cap A \subseteq p \cap A = (0)$, a contradiction. Hence, $J \not\subset p$. It follows by the construction that the map $A \longrightarrow T/J$ is surjective. Hence, $A/I \hookrightarrow T/J$ is surjective and hence A/I is complete.

We now claim that if $J \not\subset p$, then $J = (A \cap J)T$. To see this, let $J \not\subset p$ and note that since $M \not\subset p$, we have $JM \not\subset p$. So, $A \longrightarrow T/JM$ is onto by our construction. Consider the $T-$modules J, M, and $(A \cap J)T$. We will show that $J = MJ + (A \cap J)T$. Note that $MJ + (A \cap J)T \subseteq J$ is clear. Now, let $x \in J$. Then since $A \longrightarrow T/JM$ is onto, there is an $a \in A$ such that $a + JM = x + JM$. So, $x = a + jm$ where $j \in J$ and $m \in M$. Also, $a = x - jm \in J$, so $a \in (A \cap J)T$. Hence, $x \in MJ + (A \cap J)T$ and it follows that $MJ + (A \cap J)T = J$. By Nakayama's Lemma, we have $J = (A \cap J)T$. So, our claim that if $J \not\subset p$, then $J = (A \cap J)T$ holds.

Now, suppose P is a nonzero prime ideal of A and q is a prime ideal of T such that $q \cap A = P$. Then $q \subseteq p$ implies that $q \cap A = p \cap A = (0)$, a contradiction. So, we must have that $q \not\subset p$. Hence, $q = (q \cap A)T = PT$ and it follows that the only prime ideal of T that lies over P is PT. Therefore, there is only one prime ideal in the ring $T \otimes_A k(P)$. Since PT is a prime ideal of T, we must have that $T \otimes_A k(P)$ is a domain. It follows that $T \otimes_A k(P)$ is a field and hence property (ii) of the theorem holds. (In fact, one can show that $T \otimes_A k(P)$ is isomorphic to $k(P)$.)

It is interesting to note that we have also shown that there is a one-to-one correspondence between nonzero prime ideals of A and prime ideals of T that are not in the generic formal fiber of A. $\square$

Theorem 7 is the desired result.

Theorem 7. *Let (T, M) be a complete local domain containing the integers and such that $|T/M| \geq c$. Let p be a nonmaximal prime ideal of T such that T_p is a regular local ring. Then there exists an excellent local domain A such that the completion of A is T, the generic formal fiber of A is local with maximal ideal p and if I is a nonzero ideal of A, then A/I is complete.*

Proof. Use Lemma 6 to construct A. The Theorem is clear except for the fact that A is excellent. Note that $T \otimes_A k((0)) \cong T_p$ is a regular local ring by assumption. By Lemma 6, the ring $T \otimes_A k(P)$ is a field. Since T contains the integers, $T \otimes_A k(P)$ is a field of characteristic zero. Hence, it follows that A is excellent. $\square$

We do not know if Theorem 7 holds when T contains a finite field.

It is interesting to note that the proof of Lemma 6 shows that for the rings we constructed in Lemma 6 and Theorem 7, there is actually a one-to-one correspondence between nonzero prime ideals of A and prime ideals of T that are not in the generic formal fiber of A.

Also, note in particular that the ring A from Lemma 6 and Theorem 7 satisfies the following property: For all prime ideals q of T such that $p \subset q$, the ring $A/(q \cap A)$ is complete. Therefore, by Theorem 2.7 in [3], there is exactly one analytically irreducible normal local Noetherian domain C that birationally dominates A satisfying $\alpha(C) = 0$ and satisfying $C/\mathbf{m}C$ is a finite A-module. (Here, $\mathbf{m}$ denotes the maximal ideal of A.)

In fact, one could modify the proofs in this paper by replacing the ideal p with a countable set of nonmaximal, incomparable prime ideals of T. Doing this, one would see that all the results in this paper hold when p is replaced with such a set. In particular, one could show that the following result holds: Let (T, M) be a complete local domain containing the integers and such that $|T/M| \geq c$. Let L be a countable set of nonmaximal incomparable prime ideals of T such that T_p is a regular local ring for all $p \in L$. Then there exists an excellent local domain A such that the completion of A is T, the maximal ideals in the generic formal fiber of A is exactly the set L and if I is a nonzero ideal of A, then A/I is complete. Then, by Theorem 2.7 in [3], there is a one-to-one correspondence between elements of L and analytically irreducible normal local Noetherian domains C that birationally dominate A, have $\alpha(C) = 0$ and satisfy that $C/\mathbf{m}C$ is a finite A-module where $\mathbf{m}$ denotes the maximal ideal of A.

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