

A SHARP ESTIMATE ON THE NORM OF THE CONTINUOUS SQUARE FUNCTION

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(Communicated by Christopher D. Sogge)

ABSTRACT. In this paper, we prove that the norm of the continuous square function in $L^2(w)$ is bounded linearly in the A_2 norm of the weight w .

1. INTRODUCTION

The boundedness of singular integral operators in $L^2(w)$ for $w \in A_2$ has been known for a long time, by the Hunt-Muckenhaupt-Wheeden Theorem. Recently, some of the sharp bounds of these operators in terms of the A_2 norm of w have been discovered. S. Buckley [Buc93] proved that the Hardy-Littlewood maximal operator norm is bounded by $\|w\|_{A_2}$ and the square function operator bound is no larger than $\|w\|_{A_2}^{3/2}$. S. Pott and S. Petermichl [PP] showed that the Hilbert transform norm is bounded by no more than $\|w\|_{A_2}^{3/2}$. The author proved that the bounds of both the dyadic square function, and the dyadic analog of the Hilbert transform, the martingale transform, are bounded linearly in $\|w\|_{A_2}$ [Wit00]. It is easily seen that each of these operators cannot have a better bound than linear in $\|w\|_{A_2}$ (just look at power weights). In this paper, we give a proof that the continuous square function is bounded linearly in $\|w\|_{A_2}$.

2. NOTATION

In what follows, h_I will denote the normalized Haar function for the dyadic interval I , i.e. $h_I = \frac{\chi_{I_l} - \chi_{I_r}}{\sqrt{|I|}}$, where I_l, I_r denote the left and right children of I respectively. The collection of all dyadic intervals is denoted by D . The weight w and its inverse w^{-1} will be A_2 weights. Let $(f)_I$ denote $\frac{1}{|I|} \int_I f(x) dx$ for I a (not necessarily dyadic) interval. Sometimes the parentheses are omitted when it is clear which function we are averaging. w_I^{-1} will denote $\frac{1}{|I|} \int_I \frac{1}{w(x)} dx$. Let $\mu(I) = w_I w_I^{-1}$ and $\|w\|_{A_2} = \sup_I \mu(I)$. $\langle \cdot, \cdot \rangle_w$ will denote the inner product in $L^2(w(x) dx)$. If the subscript is omitted, $w = 1$. Let ϕ be a function with $\int \phi = 0$. $\phi^I = \frac{1}{\sqrt{B}} \phi(\frac{x-A}{B})$, where A is the starting point of the interval I , and B is the length $|I|$. Similarly, $\phi_t(x) = \frac{1}{\sqrt{t}} \phi(\frac{x}{t})$. In this paper, c will denote a constant, not necessarily the same at each occurrence, which is independent of f and w , but may depend on ϕ .

Received by the editors September 11, 2000 and, in revised form, March 9, 2001.
2000 *Mathematics Subject Classification*. Primary 42A50.

The continuous square function will be denoted by Sf and is defined by

$$Sf(x) = \left(\int_{\Gamma(x)} |f \star \phi_t(y)|^2 \frac{dydt}{t^3} \right)^{1/2},$$

where $\Gamma(x) = \{(y, t) : |y - x| < t\}$.

The dyadic square function takes the form

$$S_d f(x) = \left(\sum_{I \in D, x \in I} \langle f, h_I \rangle^2 \frac{1}{|I|} \right)^{1/2},$$

and the wavelet square function is

$$S_{\phi, d} f(x) = \left(\sum_{I \in D, x \in I} \langle f, \phi^I \rangle^2 \frac{1}{|I|} \right)^{1/2}.$$

3. STATEMENTS OF THEOREMS

Theorem 3.1. *Let $\phi(x)$ be compactly supported in $[0, 1]$, C^1 , such that $\{\phi^I\}_I$ is an orthonormal basis of $L^2(dx)$, and $\int \phi = 0$. Then*

$$\|S_{\phi, d} f\|_{L^2(w)} \leq c \|w\|_{A_2} \|f\|_{L^2(w)}$$

for $w \in A_2$ and $f \in L^2(w)$.

Theorem 3.2. *Let $\phi(x)$ be compactly supported in $[0, 1]$, C^1 , such that $\{\phi^I\}_I$ is an orthonormal basis of $L^2(dx)$, and $\int \phi = 0$. Then*

$$\|Sf\|_{L^2(w)} \leq c \|w\|_{A_2} \|f\|_{L^2(w)}$$

for $w \in A_2$ and $f \in L^2(w)$.

Note. Such ϕ are called wavelets. See [Dau92] for a well written discussion of wavelets and information on how to construct C^r wavelets with compact support.

4. PROOFS

Proof (Theorem 3.1). By [FP97] we know that

$$\|S_d f\|_{L^2(w)} = \left\| \left(\sum_{I \in D, x \in I} \frac{1}{|I|} \langle f, h_I \rangle^2 \right)^{1/2} \right\|_{L^2(w)} \geq c \|w\|_{A_2}^{-1/2} \|f\|_{L^2(w)}.$$

To prove the theorem, we shall prove that $S_d(x) \leq CS_{\phi, d}(x)$, and that therefore the norm of $S_{\phi, d}$ is bounded below by $c \|w\|_{A_2}^{1/2}$. Once we know the lower bound, we can derive the upper bound using the same proof used in [Wit00].

Step 1 (see also [Wil89]): Show $S_d f(x) \leq cS_{\phi,d} f(x)$.

$f(x) = \sum_{J \in D} \langle f, \phi^J \rangle \phi^J$. Inserting this expression for f into the definition of $S_d f(x)$, we have

$$\begin{aligned} S_d^2 f(x) &= \sum_{I \in D, x \in I} \left(\left\langle \sum_J \langle f, \phi^J \rangle \phi^J, h_I \right\rangle \right)^2 \frac{1}{|I|} \\ &= \sum_{I \in D, x \in I} \left(\sum_J \langle f, \phi^J \rangle \langle \phi^J, h_I \rangle \right)^2 \frac{1}{|I|}. \end{aligned}$$

Let $\lambda_J = \langle f, \phi^J \rangle$. We need to estimate

$$\sum_{I \in D} \chi_I(x) \left(\sum_{J: I \subset J} \lambda_J \langle \phi^J, h_I \rangle \right)^2 \frac{1}{|I|},$$

where we only sum over $J : I \subset J$, since otherwise the inner product is 0, as $\int \phi = 0$. Apply Cauchy-Schwarz to get

$$S_d^2 f(x) \leq \sum_I \frac{\chi_I(x)}{|I|} \left(\sum_{J: I \subset J} \lambda_J^2 \frac{|I|^3}{|J|^2} \right) \left(\sum_{J: I \subset J} \langle \phi^J, h_I \rangle^2 \frac{|J|^2}{|I|^3} \right).$$

Let us estimate $\langle \phi^J, h_I \rangle$.

Let M satisfy $\|\phi\|_\infty + \|\phi'\|_\infty \leq M$. Then

$$\langle \phi^J, h_I \rangle = \frac{1}{|I|} \left(\int_{I_l} \phi^J - \int_{I_r} \phi^J \right) \sqrt{|I|} = (\phi^J(\xi_{I_l}) - \phi^J(\xi_{I_r})) \sqrt{|I|}/2 \leq cM \frac{|I|^{3/2}}{|J|^{3/2}},$$

since $|\phi^{J'}| \leq \frac{M}{|J|^{3/2}}$. Therefore, $S_d^2 f(x)$ is bounded by

$$\begin{aligned} &cM^2 \sum_I \frac{\chi_I(x)}{|I|} \left(\sum_{J: I \subset J} \lambda_J^2 \frac{|I|^3}{|J|^2} \right) \left(\sum_{J: I \subset J} \frac{1}{|J|} \right) \\ &\leq cM^2 \sum_I \frac{(\chi_I(x))^2}{|I|} \left(\sum_{J: I \subset J} \lambda_J^2 \frac{|I|^2}{|J|^2} \right) \\ &\leq cM^2 \sum_J \frac{\chi_J(x)}{|J|^2} \lambda_J^2 \sum_{I: I \subset J} |I| \chi_I \leq cM^2 \sum_{J: x \in J} \frac{\lambda_J^2}{|J|} = cM^2 S_{\phi,d}^2 f(x), \end{aligned}$$

where we have used the fact that $\chi_I \leq \chi_J$. Therefore, $S_d f(x) \leq cMS_{\phi,d} f(x)$.

Step 2: Derive the upper bound from the lower bound.

Let

$$f_I = \frac{\phi^I}{(w_I)^{1/2}}$$

and

$$g_I = \phi^I (w_I)^{1/2} w^{-1}.$$

Then $\langle f_I, g_J \rangle_w = 1$ if $I = J$ and 0 otherwise. Let $f \in L^2(w)$.

$$\|S_{\phi,d}\|_{L^2(w)} = \left(\sum_{I \in D} \langle f, \phi^I \rangle^2 w_I \right)^{1/2} = \left(\sum_{I \in D} \langle f, g_I \rangle_w^2 \right)^{1/2}.$$

In that notation, the estimate to prove becomes

$$\sum_{I \in D} \langle f, g_I \rangle_w^2 \leq c \|w\|_{A_2}^2 \|f\|_{L^2(w)}^2.$$

We will bound this by first showing that

$$\sum_{I \in D} \langle f, f_I \rangle_w^2 \leq c \|w\|_{A_2} \|f\|_{L^2(w)}^2,$$

from which the desired inequality can be extracted by algebraic manipulations.

By Step 1, we know the following:

$$\|f\|_{L^2(w)} \leq c \|w\|_{A_2}^{1/2} \left(\sum_{I \in D} \langle f, g_I \rangle_w^2 \right)^{1/2}.$$

This allows us to compute the norm of the sequence valued operator $(Jg)_I = \langle g, f_I \rangle_w$ from $L^2(w)$ to l^2 :

$$\begin{aligned} & \sup_{\|g\|_{L^2(w)}=1} \sup_{\|\{k_I\}\|_{l^2}=1} \sum_{I \in D} k_I \int g f_I w \\ &= \sup_{\|g\|_{L^2(w)}=1} \sup_{\|\{k_I\}\|_{l^2}=1} \int g \sum_{I \in D} k_I f_I w \end{aligned}$$

by Cauchy-Schwarz, and the above inequality,

$$\begin{aligned} & \leq \sup_{\|\{k_I\}\|_{l^2}=1} \sup_{\|g\|_{L^2(w)}=1} \left\| \sum_{I \in D} k_I f_I(x) \right\|_{L^2(w)} \|g\|_{L^2(w)} \\ & \leq c \|w\|_{A_2}^{1/2} \sup_{\|\{k_I\}\|_{l^2}=1} \left[\sum_{J \in D} \left\langle \left(\sum_{I \in D} k_I f_I \right), g_J \right\rangle_w^2 \right]^{1/2} \\ & = c \|w\|_{A_2}^{1/2} \sup_{\|\{k_I\}\|_{l^2}=1} \left(\sum_{I \in D} k_I^2 \right)^{1/2} = c \|w\|_{A_2}^{1/2}. \end{aligned}$$

This means that

$$\sum_{I \in D} \langle g, f_I \rangle_w^2 \leq c \|w\|_{A_2} \|g\|_{L^2(w)}^2$$

for every weight in A_2 . In particular, this is true for the weight w^{-1} :

$$\sum_{I \in D} \left\langle g, \frac{\phi^I}{(w_I^{-1})^{1/2}} \right\rangle_{w^{-1}}^2 \leq c \|w\|_{A_2} \|g\|_{L^2(w^{-1})}^2.$$

Let $g^* = gw^{-1}$. Then this becomes

$$\sum_{I \in D} \left\langle g^*, \frac{\phi^I}{(w_I^{-1})^{1/2}} \right\rangle_{dx}^2 \leq c \|w\|_{A_2} \|g^*\|_{L^2(w)}^2.$$

The left-hand side can be rewritten as

$$\sum_{I \in D} \left\langle g^*, \phi^I (w_I)^{1/2} w w^{-1} \right\rangle_{dx}^2 \frac{1}{\mu(I)}.$$

So finally, we have the inequality

$$\sum_{I \in D} \langle g^*, g_I \rangle_w^2 \leq c \|w\|_{A_2}^2 \|g^*\|_{L^2(w)}^2,$$

which is what we needed to prove. □

Proof (Theorem 3.2). We will prove Theorem 3.2 using Theorem 3.1.

We will proceed as follows: First we will prove that the upper bound on the wavelet square function is still valid if we translate and dilate the dyadic grid. Then we write the continuous square function as an average of the translated/dilated operators over a continuum of translations and dilations.

Recall that $\phi^I = \frac{1}{\sqrt{|I|}} \phi(\frac{x-A}{|I|})$, where A is the starting point of the interval I , and B is the length $|I|$. By Theorem 3.1,

$$\left\| \left(\sum_{I \in D, x \in I} \frac{1}{|I|} \langle f, \phi^I \rangle^2 \right)^{1/2} \right\|_{L^2(w)} \leq c \|w\|_{A_2} \|f\|_{L^2(w)}.$$

For $\alpha \in \mathbb{R}$ and $1 \leq \beta < 2$, define

$$D_{\alpha, \beta} = \{I : I = [\beta k 2^i + \alpha, \beta(k+1)2^i + \alpha]; i, k \in \mathbb{Z}\}.$$

Lemma 4.1. *For fixed α and β , let*

$$S_{\phi, \alpha, \beta} f(x) = \left(\sum_{I \in D_{\alpha, \beta}, x \in I} \frac{1}{|I|} \langle f, \phi^I \rangle^2 \right)^{1/2}.$$

Then,

$$\|S_{\phi, \alpha, \beta} f(x)\|_{L^2(w)} \leq c \|w\|_{A_2} \|f\|_{L^2(w)},$$

where c is independent of α and β .

Proof.

$$\begin{aligned} & \left\| \left(\sum_{I \in D_{\alpha, \beta}, x \in I} \frac{1}{|I|} \langle f, \phi^I \rangle^2 \right)^{1/2} \right\|_{L^2(w)}^2 = \sum_{I \in D_{\alpha, \beta}} \langle f, \phi^I \rangle^2 w_I \\ & = \sum_{i, k} \left(\int f(u) \frac{\phi(\frac{u - (\beta k 2^i + \alpha)}{\beta 2^i})}{\sqrt{\beta 2^i}} du \right)^2 \frac{1}{\beta 2^i} \int_{[\beta k 2^i + \alpha, \beta(k+1)2^i + \alpha]} w(u) du. \end{aligned}$$

Change variables by letting $\tau = (u - \alpha)/\beta$.

The above becomes

$$\begin{aligned} & \sum_{i, k} \left(\int f(\beta\tau + \alpha) \phi\left(\frac{\tau - k 2^i}{2^i}\right) \frac{\beta d\tau}{\sqrt{\beta 2^i}} \right)^2 \frac{1}{2^i} \int_{[k 2^i, (k+1)2^i]} w(\beta\tau + \alpha) d\tau \\ & = \sum_{I \in D} \langle f(\beta\tau + \alpha), \phi^I(\tau) \rangle_{d\tau}^2 \beta \frac{1}{|I|} \int_I w(\beta\tau + \alpha) d\tau, \end{aligned}$$

which, by the boundedness of the operator with regular dyadic cubes, is

$$\leq \beta c \|w\|_{A_2}^2 \int f^2(\beta\tau + \alpha)w(\beta\tau + \alpha)d\tau = c \|w\|_{A_2}^2 \|f\|_{L^2(w)}^2$$

using the reverse change of variables.

Note. $c \|w(\beta\tau + \alpha)\|_{A_2} = c \|w\|_{A_2}$, since dilation and translation do not affect the A_2 norm. □

Let N be a measurable subset of the $\alpha - \beta$ plane. Then $\|S_{\phi,\alpha,\beta}f(x)\|_{L^2(N, \frac{d\alpha d\beta}{|N|})}$ is a function of x , with norm in $L^2(w)$, and furthermore,

$$\left\| \|S_{\phi,\alpha,\beta}f(x)\|_{L^2(N, \frac{d\alpha d\beta}{|N|})} \right\|_{L^2(w)} = \left\| \|S_{\phi,\alpha,\beta}f(x)\|_{L^2(w)} \right\|_{L^2(N, \frac{d\alpha d\beta}{|N|})} \leq c \|w\|_{A_2} \|f\|_{L^2(w)}$$

for any N .

Let N be $[-M, M] \times [1, 2]$ for a fixed real M . For this N ,

$$\begin{aligned} & \|S_{\phi,\alpha,\beta}f(x)\|_{L^2(N, \frac{d\alpha d\beta}{|N|})}^2 \\ &= \frac{1}{2M} \int_{-M}^M \int_1^2 \sum_{I \in D_{\alpha,\beta}, x \in I} \left(\int f(u) \frac{\phi(\frac{u-(\beta k 2^i + \alpha)}{\beta 2^i}}{\sqrt{\beta 2^i}} du \right)^2 \frac{1}{\beta 2^i} d\beta d\alpha \\ &= \sum_i \int_1^2 \frac{1}{2M} \int_{-M}^M \left(\int f(u) \frac{\phi(\frac{u-(\beta k 2^i + \alpha)}{\beta 2^i}}{\sqrt{\beta 2^i}} du \right)^2 \frac{1}{\beta 2^i} d\alpha d\beta, \end{aligned}$$

where $k = k(\alpha, \beta, i, x)$, namely the unique k such that $x \in [k2^i\beta + \alpha, (k+1)2^i\beta + \alpha)$.

Now break up the integral \int_{-M}^M into chunks of α for which k is the same, i.e. $\alpha \in (x - 2^i\beta(k+1), x - 2^i\beta k]$. Obviously, we cannot always break up the integral exactly into such chunks (and for some i , there will be no chunks at all), so we only take the whole chunks, and throw away the end pieces, since they are positive anyway. Therefore the above is bigger than

$$\sum_i \int_1^2 \frac{1}{2M} \sum_{k=A(M,\beta,i,x)}^{B(M,\beta,i,x)} \int_{x-2^i\beta(k+1)}^{x-2^i\beta k} \left(\int f(u) \frac{\phi(\frac{u-(\beta k 2^i + \alpha)}{\beta 2^i}}{\sqrt{\beta 2^i}} du \right)^2 \frac{1}{\beta 2^i} d\alpha d\beta$$

where a calculation shows that $A(M, \beta, i, x) =$ the closest integer bigger than $\frac{x-M}{2^i\beta}$ and $B(M, \beta, i, x) =$ the closest integer small than $\frac{x+M}{2^i\beta} - 1$. The sum is understood to be 0 if $B < A$.

Now change variables $y = \alpha + k2^i\beta$ and $t = 2^i\beta$. This change of variables has determinant 2^{-i} , which is proportional to $\frac{1}{t}$. (Remember that $\beta \in [1, 2]$.)

The resulting integral is

$$\begin{aligned} & \sum_i \int_{2^i}^{2^{i+1}} \frac{1}{2M} \sum_{k=A(M,t,x)}^{B(M,t,x)} \int_{x-t}^x \left(\int f(u) \frac{\phi(\frac{u-y}{t}}{\sqrt{t}} du \right)^2 \frac{1}{t^2} dy dt \\ &= \int_{\mathbb{R}^+} \frac{1}{2M} \sum_{k=A(M,t,x)}^{B(M,t,x)} \int_{x-t}^x |f \star \tilde{\phi}_t(y)|^2 \frac{1}{t^2} dy dt, \end{aligned}$$

where $\tilde{\phi}(x)$ denotes $\phi(-x)$.

Notice that k no longer appears in the integrand, so we can replace $\sum_{k=A(M,t,x)}^{B(M,t,x)}$ by $\max\{0, B(M, t, x) - A(M, t, x) + 1\}$. $B(M, t, x) - A(M, t, x) + 1 \geq \frac{M+x}{t} - \frac{x-M}{t} - 2 = \frac{2M}{t} - 2$, so if $M \geq t$ (and certainly if $M/2 \geq t$), the second term of the max is larger. Therefore the above is at least

$$\begin{aligned} & \int_{0 < t \leq \frac{M}{2}} \frac{1}{2M} \left(\frac{2M}{t} - 2\right) \int_{x-t}^x |f \star \tilde{\phi}_t(y)|^2 \frac{1}{t^2} dy dt \\ & \geq c \int_{0 < t \leq \frac{M}{2}} \left(\frac{2}{t} - \frac{1}{t}\right) \int_{x-t}^x |f \star \tilde{\phi}_t(y)|^2 \frac{1}{t^2} dy dt \\ & = c \int_{0 < t \leq \frac{M}{2}} \int_{x-t}^x |f \star \tilde{\phi}_t(y)|^2 \frac{1}{t^3} dy dt. \end{aligned}$$

To summarize, we have proven that

$$\begin{aligned} & \|S_{\phi, \alpha, \beta} f(x)\|_{L^2([-M, M] \times [1, 2], \frac{d\alpha d\beta}{2M})}^2 \\ & \geq c \int_{0 < t \leq \frac{M}{2}} \int_{x-t}^x |f \star \tilde{\phi}_t(y)|^2 \frac{1}{t^3} dy dt, \end{aligned}$$

uniformly in M . Therefore,

$$\begin{aligned} c \|w\|_{A_2} \|f\|_{L^2(w)} & \geq \left\| \|S_{\phi, \alpha, \beta} f(x)\|_{L^2([-M, M] \times [1, 2], \frac{d\alpha d\beta}{2M})} \right\|_{L^2(w)} \\ & \geq \left\| \left(c \int_{0 < t \leq \frac{M}{2}} \int_{x-t}^x |f \star \tilde{\phi}_t(y)|^2 \frac{1}{t^3} dy dt \right)^{1/2} \right\|_{L^2(w)} \end{aligned}$$

uniformly in M . Take the limit as $M \rightarrow \infty$ to get

$$\left\| \left(\int_{\mathbb{R}^+} \int_{x-t}^x |f \star \tilde{\phi}_t(y)|^2 \frac{1}{t^3} dy dt \right)^{\frac{1}{2}} \right\|_{L^2(w)} \leq c \|w\|_{A_2} \|f\|_{L^2(w)}.$$

By considering $f(-u)$ instead of $f(u)$, and following the minus sign through, it is also easily seen that

$$\left\| \left(\int_{\mathbb{R}^+} \int_x^{x+t} |f \star \tilde{\phi}_t(y)|^2 \frac{1}{t^3} dy dt \right)^{\frac{1}{2}} \right\|_{L^2(w)} \leq c \|w\|_{A_2} \|f\|_{L^2(w)},$$

which proves that

$$\|Sf(x)\|_{L^2(w)} \leq c \|w\|_{A_2} \|f\|_{L^2(w)}.$$

□

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