

FRÉCHET-URYSOHN SPACES IN FREE TOPOLOGICAL GROUPS

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ABSTRACT. Let $F(X)$ and $A(X)$ be respectively the free topological group and the free Abelian topological group on a Tychonoff space X . For every natural number n we denote by $F_n(X)$ ($A_n(X)$) the subset of $F(X)$ ($A(X)$) consisting of all words of reduced length $\leq n$. It is well known that if a space X is not discrete, then neither $F(X)$ nor $A(X)$ is Fréchet-Urysohn, and hence first countable. On the other hand, it is seen that both $F_2(X)$ and $A_2(X)$ are Fréchet-Urysohn for a paracompact Fréchet-Urysohn space X . In this paper, we prove first that for a metrizable space X , $F_3(X)$ ($A_3(X)$) is Fréchet-Urysohn if and only if the set of all non-isolated points of X is compact and $F_5(X)$ is Fréchet-Urysohn if and only if X is compact or discrete. As applications, we characterize the metrizable space X such that $A_n(X)$ is Fréchet-Urysohn for each $n \geq 3$ and $F_n(X)$ is Fréchet-Urysohn for each $n \geq 3$ except for $n = 4$. In addition, however, there is a first countable, and hence Fréchet-Urysohn subspace Y of $F(X)$ ($A(X)$) which is not contained in any $F_n(X)$ ($A_n(X)$). We shall show that if such a space Y is first countable, then it has a special form in $F(X)$ ($A(X)$). On the other hand, we give an example showing that if the space Y is Fréchet-Urysohn, then it need not have the form.

1. INTRODUCTION

All spaces are assumed to be Tychonoff and we denote by \mathbb{N} the set of all natural numbers. Let $F(X)$ and $A(X)$ be respectively the free topological group and the free Abelian topological group on a Tychonoff space X in the sense of Markov [4]. For each $n \in \mathbb{N}$, $F_n(X)$ stands for a subset of $F(X)$ formed by all words whose reduced length is less than or equal to n . Then each $F_n(X)$ is closed in $F(X)$. This concept is defined for $A(X)$ in the same fashion. It is well known that if a space X is not discrete, then neither $F(X)$ nor $A(X)$ is Fréchet-Urysohn, and hence first countable (see [1]). On the other hand, $F_n(X)$ and $A_n(X)$, $n \in \mathbb{N}$, have a chance to be first countable for a non-discrete space X . In fact, the author [6] recently obtained the following results:

For a metrizable space X , the following are equivalent: (i) $A_n(X)$ is metrizable for each $n \in \mathbb{N}$; (ii) $A_n(X)$ is first countable for each $n \in \mathbb{N}$; (iii) $A_2(X)$ is metrizable; (iv) $A_2(X)$ is first countable; (v) the set of all non-isolated points of

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X is compact. In the non-Abelian case the following are equivalent: (i) $F_n(X)$ is metrizable for each $n \in \mathbb{N}$; (ii) $F_n(X)$ is first countable for each $n \in \mathbb{N}$; (iii) $F_4(X)$ is metrizable; (iv) $F_4(X)$ is first countable; (v) X is compact or discrete. Furthermore, the following are also equivalent: (i) $F_3(X)$ is metrizable; (ii) $F_3(X)$ is first countable; (iii) $F_2(X)$ is metrizable; (vi) $F_2(X)$ is first countable; (v) the set of all non-isolated points of X is compact.

In the proofs of the above results, we proved that for a metrizable space X , if $F_2(X)$ ($A_2(X)$) is first countable, then the set of all non-isolated points of X is compact. It is easy to see that both $F_2(X)$ and $A_2(X)$ are Fréchet-Urysohn for a metrizable space X (see the next section). In this paper, we shall show that for a metrizable space X if $F_n(X)$ ($A_n(X)$) is Fréchet-Urysohn for some $n \geq 3$, then the set of all non-isolated points of X is compact. Moreover we shall prove that for a metrizable space X , if $F_n(X)$ is Fréchet-Urysohn for some $n \geq 5$, then X is compact or discrete. To prove it, we need some algebraic techniques; that is, we shall construct actions of groups on spaces and a semidirect product with respect to the action.

We call a subspace Y of $F(X)$ ($A(X)$) bounded in $F(X)$ ($A(X)$) if Y is contained in $F_n(X)$ ($A_n(X)$) for some $n \in \mathbb{N}$. On the other hand, a subspace Y of $F(X)$ ($A(X)$) is called unbounded in $F(X)$ ($A(X)$) if Y is not bounded in $F(X)$ ($A(X)$). As we mentioned above, neither $F(X)$ nor $A(X)$ is Fréchet-Urysohn if a space X is not discrete. On the other hand, from the above results, we can construct concrete spaces X and Y such that Y is first countable and bounded in $F(X)$ ($A(X)$). Then the following natural question can be considered:

Are there spaces X and Y such that Y is first countable or Fréchet-Urysohn, and Y is unbounded in $F(X)$ ($A(X)$)?

However it is easy to answer the question positively. We shall show, in §3, that every unbounded subspace Y must have a special form in $F(X)$ ($A(X)$) if Y is first countable. That is, the family $\{Y \cap (F_{n+1}(X) \setminus F_n(X)) : n \in \mathbb{N}\}$ ($\{Y \cap (A_{n+1}(X) \setminus A_n(X)) : n \in \mathbb{N}\}$) has to be discrete in Y . On the other hand, we give an example of a locally compact separable metric space X and a Fréchet-Urysohn subspace Y of $F(X)$ ($A(X)$) such that Y does not have the above form. The example also gives us a first countable subspace of $F(X)$ ($A(X)$) such that the above family is not discrete in $F(X)$ ($A(X)$).

We refer to [3] for elementary properties of topological groups and to [1] for the main properties of free topological groups.

2. FRÉCHET-URYSOHN PROPERTY OF $F_n(X)$ AND $A_n(X)$

In this section we study metrizable spaces X for which $F_n(X)$ and $A_n(X)$ are Fréchet-Urysohn. Recently the author showed that if a space X is paracompact, then $F_2(X)$ is a closed image of $(X \oplus \{e\} \oplus X^{-1})^2$ and $A_2(X)$ is a closed image of $(X \oplus \{0\} \oplus -X)^2$ (see [7, Proposition 4.8]). Hence, both $F_2(X)$ and $A_2(X)$ are Fréchet-Urysohn if X is metrizable. In addition, in the same paper [6], he proved that for a metrizable space, (i) if $F_n(X)$ is first countable for some $n \geq 2$, then the set of all non-isolated points of X is compact and the same is true for $A_n(X)$, and (ii) if $F_n(X)$ is first countable for some $n \geq 4$, then X is compact or discrete. We shall improve the result (i) for $n \geq 3$ and the result (ii) for $n \geq 5$ by showing the hypothesis of $F_n(X)$ and $A_n(X)$ is enough to be Fréchet-Urysohn.

For a space X , let $\mathcal{U}_{\overline{X}}$ and \mathcal{U}_X be the universal uniformities on $\overline{X} = X \oplus \{e\} \oplus X^{-1}$ and on X , respectively. For each $n \in \mathbb{N}$, $U \in \mathcal{U}_{\overline{X}}$ and $U' \in \mathcal{U}_X$, the author defined the subsets $W_n(U)$ of $F_{2n}(X)$ in [7] and $V(U')$ of $A_{2n}(X)$ in [6], as follows:

$W_n(U)$ is a subset of $F_{2n}(X)$ which consists of the identity e and all words g satisfying the following conditions;

- (1) g can be represented as the reduced form $g = x_1 x_2 \cdots x_{2k}$, where $x_i \in \overline{X}$ for $i = 1, 2, \dots, k$ and $1 \leq k \leq n$,
- (2) there is a partition $\{1, 2, \dots, 2k\} = \{i_1, i_2, \dots, i_k\} \cup \{j_1, j_2, \dots, j_k\}$,
- (3) $i_1 < i_2 < \cdots < i_k$ and $i_s < j_s$ for $s = 1, 2, \dots, k$,
- (4) $(x_{i_s}, x_{j_s}^{-1}) \in U$ for $s = 1, 2, \dots, k$ and
- (5) $i_s < i_t < j_s \iff i_s < j_t < j_s$ for $s, t = 1, 2, \dots, k$.

$V_n(U') = \{x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k : (x_i, y_i) \in U' \text{ for } i = 1, 2, \dots, k, k \leq n\}$.

Then the following are proved.

Theorem 2.1. *Let X be a space. Then:*

- (1) ([7]) $W_n(U)$ is a neighborhood of e in $F_{2n}(X)$ for each $U \in \mathcal{U}_{\overline{X}}$, and
- (2) ([6]) $V_n(U')$ is a neighborhood of 0 in $A_{2n}(X)$ for each $U' \in \mathcal{U}_X$.

The above neighborhoods are used to prove the following result.

Theorem 2.2. *Let X be a metrizable space. If $F_n(X)$ for some $n \geq 3$ is Fréchet-Urysohn, then the set of all non-isolated points of X is compact. The same is true for $A_n(X)$.*

Proof. It suffices to prove the theorem for $n = 3$. Suppose that the set of all non-isolated points of X is not compact, and take sequences $\{y_i : i \in \mathbb{N} \cup \{0\}\}$, $\{x_i : i \in \mathbb{N}\}$ and $\{x_{i,j} : j \in \mathbb{N}\}$ ($i \in \mathbb{N}$) in X such that

- (1) the set $Y = \{y_i : i \in \mathbb{N} \cup \{0\}\} \cup \{x_i : i \in \mathbb{N}\} \cup \{x_{i,j} : i, j \in \mathbb{N}\}$ consists of distinct points of X ,
- (2) the sequence $\{y_i : i \in \mathbb{N}\}$ converges to y_0 ,
- (3) the sequence $\{x_{i,j} : j \in \mathbb{N}\}$ converges to x_i for each $i \in \mathbb{N}$ and
- (4) $\{\{y_i : i \in \mathbb{N} \cup \{0\}\}\} \cup \{\{x_{i,j} : j \in \mathbb{N}\} : i \in \mathbb{N}\}$ is a discrete family of closed subsets of X .

For every $i \in \mathbb{N}$ put $D_i = \{x_{i,j} x_i^{-1} y_i : j \in \mathbb{N}\}$ and $D = \bigcup_{i=1}^{\infty} D_i$. Then D is a subset of $F_3(X)$ and (3) implies that the sequences D_i converge to y_i , respectively. Hence, by (2), we have that $y_0 \in \overline{D}$. To prove that $F_3(X)$ is not Fréchet-Urysohn, we need to show that there are no sequences in D which converge to y_0 .

Let S be an arbitrary sequence in D . If $S \cap D_i$ is infinite for some i , then S cannot converge to y_0 by (1) and (3). So, we may assume that there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $S \cap D_i \subseteq \{x_{i,j} x_i^{-1} y_i : j \leq f(i)\}$ for each $i \in \mathbb{N}$. Let $V = (\{y_i : i \in \mathbb{N} \cup \{0\}\})^2 \cup \bigcup_{i=1}^{\infty} (\{x_{i,j} : j > f(i)\} \cup \{x_i\})^2$. Then V is an open neighborhood of the diagonal Δ_Y in Y^2 . Let W be an open neighborhood of the diagonal Δ_X in X^2 such that $W \cap Y^2 = V$ and put $U = W \cup \{(x^{-1}, y^{-1}) : (x, y) \in W\} \cup \{(e, e)\}$. Then $U \in \mathcal{U}_{\overline{X}}$. By Theorem 2.1(1), $W_2(U)$ is a neighborhood of e in $F_4(X)$. In addition, if we put $B_{y_0} = y_0 W_2(U) \cap F_3(X)$, then B_{y_0} is a neighborhood of y_0 in $F_3(X)$ (see [7, Lemma 3.1]). On the other hand, from the definition of V , it is easy to see that $B_{y_0} \cap S = \emptyset$. This means that the sequence S cannot converge to y_0 . Consequently, $F_3(X)$ is not Fréchet-Urysohn.

In the Abelian case, put $D_i = \{x_{i,j} - x_i + y_i : j \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. Then, applying Theorem 2.1(2), the above argument also implies that $(y_0 + V_2(W)) \cap A_3(X) \cap S = \emptyset$, and hence $A_3(X)$ is not Fréchet-Urysohn. \square

Remark 2.3. Let C be a non-trivial convergent sequence $\{x_i : i \in \mathbb{N}\}$ with its limit x and $D = \{d_i : i \in \mathbb{N} \cup \{0\}\}$ be an infinite discrete space consisting of distinct points. Then $C \times D$ is homeomorphic to the space Y which appears in the above proof. Put $D_i = \{(x_j, d_i) - (x, d_i) + (x, d_0) : j \in \mathbb{N}\} \subseteq A_3(C \times D)$ ($i \in \mathbb{N}$) and $D = \bigcup_{i=1}^{\infty} D_i$. Then the above argument yields that, in $A_3(C \times D)$, no sequences in D converge to (x, d_0) , however $(x, d_0) \in \overline{D}$. We apply the fact in the proof of the next theorem.

Theorem 2.4. *Let X be a metrizable space. If $F_n(X)$ is Fréchet-Urysohn for some $n \geq 5$, then X is compact or discrete.*

Proof. Suppose that a metrizable space X is neither compact nor discrete, and choose sequences $\{x_i : i \in \mathbb{N}\}$ and $\{d_i : i \in \mathbb{N}\}$ consisting of distinct points in X and a point x in X such that the sequence $\{x_i : i \in \mathbb{N}\}$ converges to x and $\{\{x_i : i \in \mathbb{N}\} \cup \{x\}\} \cup \{\{d_i : i \in \mathbb{N}\}\}$ is a discrete closed family in X . For each $i \in \mathbb{N}$ put $E_i = \{d_i x_j x^{-1} d_i^{-1} x_i : j \in \mathbb{N}\}$ and $E = \bigcup_{i=1}^{\infty} E_i$. Then E is a subset of $F_5(X)$. Since each sequence E_i converges to x_i , we have that $x \in \overline{E}$. To prove that $F_5(X)$ is not Fréchet-Urysohn, we need to show that there are no sequences in E which converge to x in $F_5(X)$. Let S be an arbitrary sequence in E . Then we may assume that $S \cap E_i$ is a non-empty finite set for each $i \in \mathbb{N}$, that is, we may assume that for each $i \in \mathbb{N}$ there is a non-empty finite set $p_i \subseteq \mathbb{N}$ such that $S \cap E_i = \{g_{i,j} = d_i x_j x^{-1} d_i^{-1} x_i : j \in p_i\}$. To prove that S cannot converge to x , we need to construct some mappings and topological groups which are defined by Pestov and the author in [5].

Let $C = \{x_i : i \in \mathbb{N}\} \cup \{x\}$ and $D = \{d_i : i \in \mathbb{N}\}$. Define a mapping $\tau : F(D) \times (C \times F(D)) \rightarrow C \times F(D)$ by letting $\tau((g, (x, h))) = (x, gh)$ for each $(g, (x, h)) \in F(D) \times (C \times F(D))$. Since $F(D)$ is a discrete space, τ is a continuous action of the group $F(D)$ on the space $C \times F(D)$. For every $g \in F(D)$, the self-homeomorphism $\tau_g : C \times F(D) \rightarrow C \times F(D) : (x, h) \rightarrow (x, gh)$ can be extended to an automorphism $\overline{\tau}_g : A(C \times F(D)) \rightarrow A(C \times F(D))$. Then, put $\overline{\tau} : F(D) \times A(C \times F(D)) \rightarrow A(C \times F(D))$ as $\overline{\tau}((g, h)) = \overline{\tau}_g(h)$ for each $(g, h) \in F(D) \times A(C \times F(D))$. Since $F(D)$ is a discrete space, $\overline{\tau}$ is a continuous action of $F(D)$ on $A(C \times F(D))$.

Let $G = F(D) \rtimes_{\overline{\tau}} A(C \times F(D))$ be the semidirect product formed with respect to the action $\overline{\tau}$. In other words, as a topological space, G is the product of $F(D)$ and $A(C \times F(D))$ and the group operation is given by $(g, a) \cdot (h, b) = (gh, a + \overline{\tau}_g(b))$, where $g, h \in F(D)$ and $a, b \in A(C \times F(D))$. Since $A(C \times F(D))$ (identified with $\{e\} \times A(C \times F(D))$, where e is the unit element of $F(D)$) forms an open normal subgroup of G , let $\pi : G \rightarrow A(C \times F(D))$ be the quotient mapping.

Define a mapping $\psi : C \oplus D \rightarrow G$ by

$$\psi(t) = \begin{cases} (e, (t, e)) \in F(D) \rtimes_{\overline{\tau}} A(C \times F(D)), & \text{if } t \in C, \\ (t, 0) \in F(D) \rtimes_{\overline{\tau}} A(C \times F(D)), & \text{if } t \in D, \end{cases}$$

where 0 denotes the unit element of $A(C \times F(D))$. Since

$$\lim_{n \rightarrow \infty} \psi(x_n) = \lim_{n \rightarrow \infty} (e, (x_n, e)) = (e, (x, e)) = \psi(x),$$

the mapping ψ is continuous and therefore extends to a continuous homomorphism $\bar{\psi} : F(C \oplus D) \rightarrow G$. Let $Y = C \times (\{e\} \oplus D) \subseteq C \times F(D)$ and

$$f = (\pi \circ \bar{\psi})|_{(\pi \circ \bar{\psi})^{-1}(A(Y))} : (\pi \circ \bar{\psi})^{-1}(A(Y)) \rightarrow A(Y).$$

Then, since $A(Y)$ is a topological subgroup of $A(C \times F(D))$, f is a continuous homomorphism.

We return to prove that the sequence S cannot converge to x in $F_5(X)$. Since $F_5(C \oplus D)$ is a subspace of $F_5(X)$ and $S \cup \{x\} \subseteq F_5(C \oplus D)$, it suffices to show that S does not converge to x in $F_5(C \oplus D)$. Let $i \in \mathbb{N}$ and $j \in p_i$. Before we calculate $\bar{\psi}(g_{i,j})$, let us note that the inverse elements of $(d, 0)$ and $(e, (x, e))$ in G are $(d^{-1}, 0)$ and $(e, -(x, e))$ for each $d \in D$ and $x \in C$, respectively. Hence $g_{i,j}$ is mapped by $\bar{\psi}$, as follows:

$$\begin{aligned} \bar{\psi}(g_{i,j}) &= \bar{\psi}(d_i x_j x^{-1} d_i^{-1} x_i) = \bar{\psi}(d_i) \bar{\psi}(x_j) \bar{\psi}(x)^{-1} \bar{\psi}(d_i)^{-1} \bar{\psi}(x_i) \\ &= \psi(d_i) \psi(x_j) \psi(x)^{-1} \psi(d_i)^{-1} \psi(x_i) \\ &= (d_i, 0)(e, (x_j, e))(e, (x, e))^{-1} (d_i, 0)^{-1} (e, (x_i, e)) \\ &= (d_i, \overline{\tau_{d_i}}((x_j, e)))(e, -(x, e))(d_i^{-1}, 0)(e, (x_i, e)) \\ &= (d_i, (x_j, d_i))(d_i^{-1}, -(x, e) + \overline{\tau_e}(0))(e, (x_i, e)) \\ &= (d_i, (x_j, d_i))(d_i^{-1}, -(x, e))(e, (x_i, e)) \\ &= (d_i, (x_j, d_i))(d_i^{-1}, -(x, e) + \overline{\tau_{d_i^{-1}}}(x_i, e)) \\ &= (d_i, (x_j, d_i))(d_i^{-1}, -(x, e) + (x_i, d_i^{-1})) \\ &= (e, (x_j, d_i) + \overline{\tau_{d_i}}(-(x, e) + (x_i, d_i^{-1}))) \\ &= (e, (x_j, d_i) - \overline{\tau_{d_i}}(x, e) + \overline{\tau_{d_i}}(x_i, d_i^{-1})) \\ &= (e, (x_j, d_i) - (x, d_i) + (x_i, e)). \end{aligned}$$

Hence $\pi \circ \bar{\psi}(g_{i,j}) = (x_j, d_i) - (x, d_i) + (x_i, e)$. Since $(x_j, d_i), (x, d_i), (x_i, e) \in Y$, $\pi \circ \bar{\psi}(g_{i,j}) \in A(Y)$. This means that $f(g_{i,j}) = (x_j, d_i) - (x, d_i) + (x_i, e)$. Therefore, we have that $f(S) = \{(x_j, d_i) - (x, d_i) + (x_i, e) : j \in p_i, i \in \mathbb{N}\} \subseteq A_3(Y)$. On the other hand, in the same way, we can show that $f(x) = (x, e) \in A(Y)$. Hence, by Remark 2.3, the sequence $f(S)$ does not converge to $f(x)$ in $A_3(Y)$. This yields that S does not converge to x in $F_5(C \oplus D)$. Therefore, we conclude that $F_5(X)$ is not Fréchet-Urysohn. \square

From the above theorems, we obtain the following results which improve Theorem 4.5, 4.9 and 4.12 in [7].

Corollary 2.5. *For a metrizable space X , the following are equivalent:*

- (1) $A_n(X)$ is metrizable for each $n \in \mathbb{N}$;
- (2) $A_n(X)$ is first countable for each $n \in \mathbb{N}$;
- (3) $A_n(X)$ is Fréchet-Urysohn for each $n \in \mathbb{N}$;
- (4) $A_2(X)$ is metrizable;
- (5) $A_2(X)$ is first countable;
- (6) $A_3(X)$ is Fréchet-Urysohn;
- (7) $F_3(X)$ is metrizable;
- (8) $F_3(X)$ is first countable;
- (9) $F_3(X)$ is Fréchet-Urysohn;
- (10) $F_2(X)$ is metrizable;

- (11) $F_2(X)$ is first countable;
 (12) the set of all non-isolated points of X is compact.

Proof. The equivalence of the statements (1), (2), (4), (5), (7), (8), (10), (11) and (12) are due to Theorem 4.5 and 4.12 in [7]. The implications (2) \Rightarrow (3) \Rightarrow (6) and (8) \Rightarrow (9) are trivial. Theorem 2.2 yields the implications (6) \Rightarrow (12) and (9) \Rightarrow (12). \square

Corollary 2.6. *For a metrizable space X , the following are equivalent:*

- (1) $F_n(X)$ is metrizable for each $n \in \mathbb{N}$;
- (2) $F_n(X)$ is first countable for each $n \in \mathbb{N}$;
- (3) $F_n(X)$ is Fréchet-Urysohn for each $n \in \mathbb{N}$;
- (4) i_n is a closed mapping for each $n \in \mathbb{N}$;
- (5) $F_4(X)$ is metrizable ;
- (6) $F_4(X)$ is first countable ;
- (7) $F_5(X)$ is Fréchet-Urysohn ;
- (8) i_3 is a closed mapping ;
- (9) X is compact or discrete.

Proof. The equivalence of the statements (1), (2), (4), (5), (6), (8) and (9) is due to Lemma 4.7 and Theorem 4.9 in [7], and the implications (2) \Rightarrow (3) \Rightarrow (7) are trivial. Theorem 2.4 yields the implication (7) \Rightarrow (9). \square

As we mentioned at the beginning of this section, we have already shown that the mapping i_2 is closed if and only if every neighborhood of the diagonal in X^2 is an element of the universal uniformity \mathcal{U}_X of X (see [7, Proposition 4.8]). In particular, i_2 is closed for a paracompact space. Therefore both $F_2(X)$ and $A_2(X)$ are Fréchet-Urysohn for a metrizable space X . So, the reader must note that it is not clarified that the equivalent condition of a metrizable space X for $F_4(X)$ be Fréchet-Urysohn. Unfortunately, the author does not know about it. He just conjectures that $F_4(X)$ is Fréchet-Urysohn if the set of all non-isolated points of a metrizable space X is compact, and hence we could add the statement that $F_4(X)$ is Fréchet-Urysohn on the list of equivalences in Corollary 2.5.

3. UNBOUNDED SUBSPACES OF $F(X)$ AND $A(X)$

As we mentioned in §1, however, neither $F(X)$ nor $A(X)$ is Fréchet-Urysohn for a non-discrete space X , and there are non-discrete spaces X and Y such that Y is a first countable unbounded subspace of $F(X)$ ($A(X)$). For example, let X be any non-discrete first countable space. Fix an element $x \in X$ and for each $n \in \mathbb{N}$ let $X_n = X \cdot x^n$. Then the subspace X_n of $F(X)$ is homeomorphic to X . Let f be a function from X to the additive group of integers such that $f(x) = 1$ for each $x \in X$. Then the homomorphic extension $F(f)$ of f over $F(X)$ is continuous. Since $F(f)(g) = n + 1$ for each $g \in X_n$ and $n \in \mathbb{N}$, it follows that the subspace $Y = \bigcup_{i=1}^{\infty} X_n$ is the sum of $\{X_n : n \in \mathbb{N}\}$. Hence Y is a required unbounded subspace of $F(X)$.

For each $n \in \mathbb{N}$, let $E_n(X) = F_n(X) \setminus F_{n-1}(X)$. Then, for the above subspace Y , the family $\{X_n = Y \cap E_n : n \in \mathbb{N}\}$ is discrete in Y . Generally, we can show that every first countable subspace Y of $F(X)$ ($A(X)$) has this property.

Proposition 3.1. *Let X be a Tychonoff space and Y a first countable subspace of $F(X)$. Then $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is discrete in Y . The same is true for $A(X)$.*

Proof. Suppose that there is a first countable subspace Y of $F(X)$ such that $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is not discrete at a word g in Y . Since Y is first countable, g has a countable neighborhood base in Y . So, we can choose sequences $\{k_n : n \in \mathbb{N}\}$ of natural numbers and $\{g_n : n \in \mathbb{N}\}$ in Y such that

- (1) $k_1 < k_2 < \dots$,
- (2) $g_n \in Y \cap E_{k_n}$ for each $n \in \mathbb{N}$ and
- (3) the sequence $\{g_n\}$ converges to g in Y .

Then the compact set $\{g_n : n \in \mathbb{N}\} \cup \{g\}$ is unbounded in $F(X)$. Since every compact subset of $F(X)$ is bounded in $F(X)$, this is a contradiction. \square

Corollary 3.2. *Let X be a Tychonoff space and Y a subspace of $F(X)$ satisfying one of the following properties: locally compactness, Čech-completeness, first countability and point-countable type. Then $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is discrete in Y . The same is true for $A(X)$.*

Proof. Recall that a space Y is of *point-countable type* iff for each point $p \in Y$, there is a compact set K and K has countable character. Then, the argument of the proof of Proposition 3.1 implies that $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is discrete in a subspace Y of $F(X)$ if Y is of point-countable type. Since the point-countable type is the weakest property among the above properties, this completes the proof. \square

In general, The family $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ of Corollary 3.2 is not necessary to be discrete in $F(X)$ ($A(X)$). Furthermore, the family $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is not necessary to be discrete in Y if Y is Fréchet-Urysohn. The following example shows these facts.

Example 3.3. There is an unbounded subspace Y of the free topological group $F(X)$ on a metrizable space X such that Y is a locally compact separable metric space and $\{Y \cap E_n(X) : n \in \mathbb{N}\}$ is not discrete at e in $F(X)$. In addition, if we put $Z = Y \cup \{e\}$, then Z is Fréchet-Urysohn and $\{Z \cap E_n(X) : n \in \mathbb{N}\}$ is not discrete at e in Z .

Proof. Let $X = \bigoplus_{n=1}^{\infty} C_n$, where each C_n is a non-trivial convergent sequence $\{x_{k,n} : k \in \mathbb{N}\}$ with its limit x_n . For each $n \in \mathbb{N}$ let $p(n) = \frac{n(n-1)}{2}$ and

$$S_n = \{x_{p(n)+1}^{-1} x_{k,p(n)+1} x_{p(n)+2}^{-1} x_{k,p(n)+2} \cdots x_{p(n)+n}^{-1} x_{k,p(n)+n} : k \in \mathbb{N}\}.$$

Since each S_n is a subset of E_{2n} , the set $Y = \bigcup_{n=1}^{\infty} S_n$ is unbounded in $F(X)$. To prove that Y is a required subspace of $F(X)$, it suffices to show that the subspace $Z = Y \cup \{e\}$ of $F(X)$ is homeomorphic to the sequential fan $S(\omega)$. It is clear that each sequence S_n converges to e and $S_i \cap S_j = \emptyset$ if $i \neq j$. Hence it suffices to show that a subset L of Z is closed in Z whenever the intersection of L with $S_n \cup \{e\}$ is closed in $S_n \cup \{e\}$ for each $n \in \mathbb{N}$. Let L be a subset of Z such that $L \cap (S_n \cup \{e\})$ is closed in $S_n \cup \{e\}$ for each $n \in \mathbb{N}$ and K be a compact subset of $F(X)$. Since K is bounded in $F(X)$, there is $n \in \mathbb{N}$ such that $L \cap K \subseteq \bigcup_{i=1}^n S_i \cup \{e\}$. It follows that $L \cap K = \bigcup_{i=1}^n (L \cap (S_i \cup \{e\})) \cap K$, and hence $L \cap K$ is closed in K . Since X is a locally compact separable metrizable space, $F(X)$ is a k -space (see [2, Theorem 2.11]). Thus the above argument yields that L is closed in $F(X)$, and hence in Z . Consequently we can prove that Z is homeomorphic to $S(\omega)$. \square

In particular, Corollary 3.2 yields that the subspace $\bigcup_{i=1}^{\infty} E_{n_i}$ does not satisfy any properties of Corollary 3.2 for each subsequence $\{n_i : i \in \mathbb{N}\}$ of natural numbers, as follows.

Corollary 3.4. *Let X be a non-discrete Tychonoff space. Then, for every sequence $\{n_i : i \in \mathbb{N}\}$ of natural numbers, $\bigcup_{i=1}^{\infty} E_{n_i}(X)$ is not of point-countable type, and hence it does not satisfy any properties of Corollary 3.2. The same is also true for $A(X)$.*

Proof. Let $Y = \bigcup_{i=1}^{\infty} E_{n_i}(X)$. By Corollary 3.2, it suffices to show that $\mathcal{E} = \{E_{n_i}(X) : i \in \mathbb{N}\}$ is not discrete in Y . To prove that, choose a subsequence $\{m_i : i \in \mathbb{N}\}$ of $\{n_i : i \in \mathbb{N}\}$ and natural numbers k_i such that $m_{i+1} - m_i = 2k_i$ for each $i \in \mathbb{N}$. Fix $g \in E_{m_1}$, a non-isolated point x in X and an open neighborhood U of g in $F(X)$. Denote the set of all elements of X taking part in the reduced form of the word g by $\text{car } g$. Then we can choose a sequence $\{U_i : i \in \mathbb{N}\}$ of an open neighborhood of e in $F(X)$ and sequences $\{a_i : i \in \mathbb{N}\}$ and $\{b_i : i \in \mathbb{N}\}$ in X satisfying the following properties:

- (1) $U_i = U_i^{-1}$ for each $i \in \mathbb{N}$,
- (2) $gU_1 \subseteq U$ and $U_{i+1}^{2p_i} \subseteq U_i$ for each $i \in \mathbb{N}$, where $p_i = \sum_{j=1}^i k_j$,
- (3) the family $\{a_i : i \in \mathbb{N}\} \cup \{b_i : i \in \mathbb{N}\} \cup \text{car } g$ consists of distinct points in X and
- (4) $a_i, b_i \in xU_{i+1}$.

For each $i \in \mathbb{N}$, since $a_i^{-1}b_i \in (xU_{i+1})^{-1}xU_{i+1} = U_{i+1}^{-1}U_{i+1} = U_{i+1}^2$, we have that $(a_i^{-1}b_i)^{p_i} \in (U_{i+1}^2)^{p_i} = U_{i+1}^{2p_i} \subseteq U_i$. For each $i \in \mathbb{N}$, put $g_i = g(a_i^{-1}b_i)^{p_i}$. Then each $g_i \in gU_i \subseteq gU_1 \subseteq U$. Furthermore, property (3) implies the length of $g_i = m_1 + 2p_i = m_1 + 2\sum_{j=1}^i k_j = m_{i+1}$. It follows that $g_i \in E_{m_{i+1}}$. Thus we have that $U \cap E_{m_{i+1}} \neq \emptyset$ for each $i \in \mathbb{N}$. This means that \mathcal{E} is not discrete at g in Y . \square

However Proposition 3.1 is not true for Fréchet-Urysohn spaces as is shown in Example 3.3; Corollary 3.4 is also true for Fréchet-Urysohn spaces.

Proposition 3.5. *Let X be a non-discrete Tychonoff space. Then, for every sequence $\{n_i : i \in \mathbb{N}\}$ of natural numbers, $\bigcup_{i=1}^{\infty} E_{n_i}(X)$ is not a Fréchet-Urysohn space.*

Proof. Let $Y = \bigcup_{i=1}^{\infty} E_{n_i}(X)$ and suppose that Y is Fréchet-Urysohn. Choose a subsequence $\{m_i : i \in \mathbb{N}\}$ of $\{n_i : i \in \mathbb{N}\}$ such that $m_{i+1} - m_i = 2k_i$ ($k_i \in \mathbb{N}$) for each $i \in \mathbb{N}$. Then, applying the proof of Corollary 3.4, we can show that for each $i \in \mathbb{N}$ and $g \in E_{m_i}(X)$, $g \in \overline{E_{m_j}(X)}^Y$ for each $j \geq i$. Choose $g \in E_{m_1}$. Since $g \in \overline{E_{m_2}}^Y \setminus E_{m_2}$ there is a non-trivial sequence $\{g_i : i \in \mathbb{N}\}$ in E_{m_2} that converges to g . For each $i \in \mathbb{N}$ since $g_i \in \overline{E_{m_{i+2}}}^Y \setminus E_{m_{i+2}}$ we can take a non-trivial sequence $\{g_{i,j} : j \in \mathbb{N}\}$ in $E_{m_{i+2}}$ that converges to g_i . We put $A = \{g_{i,j} : i, j \in \mathbb{N}\}$. Since the sequence $\{g_i\}$ converges to g , we have that $g \in \overline{A}$. So there is a sequence S in A that converges to g . Each sequence $\{g_{i,j} : j \in \mathbb{N}\}$ converges to g_i and $g_i \neq g$. It follows that the sequence S is unbounded in $F(X)$. Since $S \cup \{g\}$ is compact, this is a contradiction. Consequently Y is not Fréchet-Urysohn. \square

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