PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 8, Pages 2487–2496
S 0002-9939(02)06354-2
Article electronically published on February 4, 2002

VARIETIES GENERATED BY COUNTABLY COMPACT ABELIAN GROUPS

DIKRAN DIKRANJAN AND MICHAEL TKACHENKO

(Communicated by Alan Dow)

ABSTRACT. We prove under the assumption of Martin's Axiom that every precompact Abelian group of size $\leq 2^{\aleph_0}$ belongs to the smallest class of groups that contains all Abelian countably compact groups and is closed under direct products, taking closed subgroups and continuous isomorphic images.

1. INTRODUCTION

The celebrated Comfort–Ross theorem states that the direct product of any family of pseudocompact topological groups is pseudocompact [CR]. However, every precompact topological group can be embedded as a closed subgroup into a pseudocompact topological group [CM, Lemma 2.1], so the class of closed subgroups of pseudocompact groups coincides with all precompact groups.

Countable compactness behaves in a completely different way. Clearly, closed subgroups of countably compact groups inherit this property. But the product of two countably compact Abelian topological groups can contain a countably infinite closed subgroup (which is neither countably compact nor pseudocompact, since every infinite pseudocompact group has cardinality $\geq 2^{\aleph_0}$). Therefore, the product of two countably compact groups can fail to be countably compact [Do1]. It is important to mention that van Douwen's construction in [Do1] makes use of Martin's Axiom (MA for short). Under the weaker assumption of $\mathrm{MA}_{countable}$, Hart and van Mill constructed in [HM] a single group H whose square was not countably compact. Later on, Tomita [To1, To2] presented several construction of this type under $\mathrm{MA}_{countable}$. At this moment, there is no single construction in ZFC of two countably compact groups whose product is not countably compact. This suggests the hypothesis that there might exist a model of ZFC where countable compactness is productive in topological groups.

Received by the editors February 4, 2000 and, in revised form, March 21, 2001.

1991 *Mathematics Subject Classification.* Primary 22A05, 22B05, 54D25, 54H11; Secondary 54A35, 54B30, 54D30, 54H13.

Key words and phrases. Countably compact, precompact, sequentially complete, variety of topological groups, Martin's Axiom.

The first author was partially supported by Research Grant of the Italian MURST in the framework of the project "Nuove prospettive nella teoria degli anelli, dei moduli e dei gruppi abeliani" 2000.

The second author was partially supported by the Mexican National Council of Sciences and Technology (CONACyT), grant no. 400200-5-3012PE. He also thanks the hosts for the hospitality and generous support during his visit to the Università di Udine, Italy in December, 1999.

The opposite possibility is that *every* precompact topological group can be embedded as a closed subgroup into a product of countably compact groups. However, this is not the case. Recall that a topological group G is called *sequentially complete* [DT2, DT3] if G is sequentially closed in its completion $\widetilde{G}$, that is, no sequence in G converges to a point of $\widetilde{G}\setminus G$. It is easy to see that the class of sequentially complete groups is closed under taking arbitrary direct products and closed subgroups [DT2], and hence is an $\overline{S}C$*-variety* (see the definition below). Since countably compact groups are sequentially complete, we infer that every closed subgroup of a direct product of countably compact groups must be sequentially complete. On the other hand, sequential completeness coincides with compactness for metrizable groups (the completion of a metrizable topological group is metrizable), so no proper dense subgroup of a compact metrizable group can be embedded as a closed subgroup in a product of countably compact groups. In particular, the torsion subgroup of the circle group $\mathbb{T}$ is not topologically isomorphic to a closed subgroup of a product of countably compact groups.

Let us take one more step and consider continuous isomorphic images of closed subgroups lying in direct products of countably compact (Abelian) groups. Will these coincide with all precompact (Abelian) groups? This is the main problem we study in the article. In fact, we restrict ourselves to considering the Abelian case of the problem. Surprisingly, MA enables us to represent in this way every precompact Abelian group G of size $\le 2^{\aleph_0}$ (Theorem A).

The problem under consideration suggests using the language of varieties. In the next subsection we give the necessary definitions and briefly discuss the relations between different types of varieties.

1.1. **Varieties of countably compact groups and sequential completeness.** In this paper we consider several types of varieties of Abelian *topological* groups. Here are the relevant definitions, introduced (except for the last one) by Morris [Mo1, Mo2] and Taylor [Tay].

Definition 1.1. A class $\mathcal{V}$ of topological Abelian groups is called a:

(1) Q*-variety* if $\mathcal{V}$ is stable under taking quotient groups;
(2) $\overline{S}$*-variety* if $\mathcal{V}$ is stable under taking closed subgroups;
(3) C*-variety* if $\mathcal{V}$ is stable under taking arbitrary Cartesian (direct) products;
(4) H*-variety* if $\mathcal{V}$ is stable under taking continuous homomorphic images;
(5) I*-variety* if $\mathcal{V}$ is stable under taking continuous isomorphic images.

Clearly $\mathcal{V}$ is an H-variety iff it is a Q- and I-variety. Taking the intersection of a Q-variety with a C-variety, we obtain a QC-variety. Similarly, one defines $I\overline{S}C$-, $Q\overline{S}C$- and $H\overline{S}C$-varieties. Typical examples are the $H\overline{S}C$-variety $\mathcal{TA}$ of all topological Abelian groups, and its $H\overline{S}C$-subvarieties $\mathcal{CA}$ and $\mathcal{PA}$ of compact and precompact groups, respectively. Note that the complete groups form an $\overline{S}C$-variety which is neither an I-variety nor a Q-variety, and the same holds for the sequentially complete groups (cf. [DT2]). We shall deal mainly with $\overline{S}C$- and $I\overline{S}C$-varieties in this paper.

It is clear that the intersection of Q-, $\overline{S}$-, C-, H- and I-varieties is again a Q-, $\overline{S}$-, C-, H- or I-variety, respectively. Therefore, for every $X \in \{Q, \overline{S}, C, H, I\}$, a class $\mathcal{U}$ of topological groups generates a smallest X-variety $X(\mathcal{U})$ containing the class $\mathcal{U}$, which we call the X*-variety generated by* $\mathcal{U}$. Similarly, one defines $HC(\mathcal{U})$, $\overline{S}C(\mathcal{U})$, $I\overline{S}C(\mathcal{U})$, etc.

Denote by $\mathcal{CCA}$ the class of all countably compact Abelian groups and by $\mathcal{SCPA}$ the class of all sequentially complete precompact Abelian groups. Clearly, $\mathcal{SCPA} = \overline{S}C(\mathcal{SCPA})$ is a $\overline{S}C$-variety and

$$\overline{S}C(\mathcal{CCA}) \subseteq \mathcal{SCPA} \subseteq I(\mathcal{SCPA}).$$

It would be interesting to find out whether the first two of these $\overline{S}C$-varieties coincide (see Problems 3.1 (d) and 3.2). We show in Theorem B that the third one coincides with $\mathcal{PA}$.

The main objective of this paper is the $I\overline{S}C$-variety $\mathfrak{C} = I\overline{S}C(\mathcal{CCA})$.

Conjecture 1.2. *$\mathfrak{C}$ coincides with the class $\mathcal{PA}$ of all precompact Abelian groups.*

The main result of the article confirms Conjecture 1.2 for "small" groups:

Theorem A ([MA]). *Every precompact Abelian group of cardinality $\leq 2^{\aleph_0}$ belongs to $\mathfrak{C}$.*

We see now that the above conjecture becomes valid when $\mathfrak{C}$ is replaced by the formally larger $I\overline{S}C$-variety $I(\mathcal{SCPA})$. Therefore, if the equality $\overline{S}C(\mathcal{CCA}) = \mathcal{SCPA}$ holds, then $\mathfrak{C} = \mathcal{PA}$. For an abstract Abelian group G, we denote by $G^{\#}$ the underlying group G endowed with the maximal precompact group topology [Do2, DTT].

Theorem B. *$I(\mathcal{SCPA}) = Q(\mathcal{SCPA}) = \mathcal{PA}$.*

Proof. To prove that $I(\mathcal{SCPA}) = \mathcal{PA}$, let G be an infinite precompact Abelian group, and let τ_G be the topology of G. Clearly, the topology τ of $G^{\#}$ is finer than τ_G, so the identity map $id\colon G^{\#} \to G$ is a continuous isomorphism. By Flor's theorem in [Flo], the group $G^{\#}$ contains no convergent sequences other than trivial ones. Therefore, the group $G^{\#}$ is sequentially complete by Proposition 2.2 of [DT1]. This implies that $G \in I(\mathcal{SCPA})$.

It follows from [DT2, Theorem 5.4] that every precompact Abelian group G is a quotient of a precompact sequentially complete Abelian group G^* (to be precise, this assertion requires a brief analysis of the proof given in [DT2]: the group G^* turns out to be a subgroup of $G \times \mathbb{T}^{\lambda}$ for some infinite cardinal λ). This proves that $Q(\mathcal{SCPA}) = \mathcal{PA}$. □

It is also known that every Abelian topological group is isomorphic to a quotient of a complete Abelian group, so $Q\overline{S}C$(complete Abelian groups) $= \mathcal{TA}$. Moreover, every (precompact) topological group is isomorphic to a quotient of a zero-dimensional (precompact) group [Ar2, Sha, D].

Notation and terminology. The symbols $\mathbb{N}$ and $\mathbb{Z}$ are used for the set of positive integers and the group of integers, respectively. We put $\mathbb{N}^* = \mathbb{N} \setminus \{1\}$. The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual compact topology. The cyclic group of order n is denoted by $\mathbb{Z}(n)$. For $d, n \in \mathbb{N}$, the fact that d divides n abbreviates to $d|n$.

We consider only Abelian groups, so the additive notation is used here. The symbol 0_G (or simply 0) stands for the neutral element of an Abelian group G. We write $H \leq G$ if H is a subgroup of G. If α is an ordinal, we use G^{α} and $G^{(\alpha)}$ to denote the direct product and direct sum of α copies of the group G, respectively.

Let G be a group. For a non-zero element $b \in G$, we denote by $ord(b)$ the order of b (so $ord(b) \in \mathbb{N}^*$ or $ord(b) = \infty$). The cyclic subgroup of G generated by b is denoted by $\langle b\rangle$. Clearly, $ord(b) = |\langle b\rangle|$. For every $n \in \mathbb{N}$, we put $G[n] = \{g \in G : ng = 0\}$. For convenience we extend this notation also to the symbol ∞ by putting $G[\infty] = G$.

All topological groups are assumed to be Hausdorff. For a topological group G, we denote by $w(G)$ the weight of G.

Let I be a non-empty set, and suppose that for every $i \in I$, $f_i\colon G \to G_i$ is a homomorphism. Then we denote by $f = \triangle_{i\in I} f_i$ the diagonal product of the family $\{f_i : i \in I\}$. Clearly, $f\colon G \to \prod_{i\in I} G_i$ is the unique homomorphism satisfying $p_i \circ f = f_i$ for each $i \in I$, where $p_i\colon \prod_{j\in I} G_j \to G_i$ is the natural projection.

The symbol $\mathfrak{c}$ stands for the cardinality of the continuum, so $\mathfrak{c} = 2^{\aleph_0}$. We abbreviate Martin's Axiom to MA. It is known (see [Juh]) that MA is equivalent to the following topological statement: *In a countably cellular compact space X, the intersection of less than $\mathfrak{c}$ open dense sets is dense in X.*

2. Proof of Theorem A

We start with three auxiliary results that appeared in [DT4]. The proofs of the first and the second one are given in [DT4], so we only prove Lemma 2.6 (the only one that requires MA).

The notion of HFD subsets of infinite products was introduced by Hajnal and Juhász in [HJ]. It plays an important role in the proof of Theorem 2.7.

Definition 2.1. Let λ be an infinite cardinal. An infinite subset Y of a topological product $X = \prod_{\alpha<\lambda} X_\alpha$ is called *finally dense* in X if there exists $\beta < \lambda$ such that $\pi_{\lambda\setminus\beta}(Y)$ is dense in $\prod_{\beta\le\alpha<\lambda} X_\alpha$, where $\pi_{\lambda\setminus\beta}$ is the projection of X onto $\prod_{\beta\le\alpha<\lambda} X_\alpha$. If every infinite subset of Y is finally dense in X, then Y is called *hereditarily finally dense* (HFD) in X.

The first lemma goes back to [HJ] and, in the present form, it appears in [DT4, Prop. 3.2].

Lemma 2.2. *Let λ be an uncountable regular cardinal, and let $X = \prod_{\alpha<\lambda} X_\alpha$ be a product of compact metrizable spaces X_α. Suppose that Y is a subset of X such that $\pi_\beta(Y) = \prod_{\alpha<\beta} X_\alpha$ for each $\beta < \lambda$, and $S \subseteq Y$ is an HFD set in X. Then S has a cluster point in Y. In particular, if Y is HFD in X, then it is a countably compact dense subspace of X.*

Items (b) and (c) of the following notion (the latter under a different name) were introduced in [DT4, Definition 3.3].

Definition 2.3. Let G be an Abelian group, and $n \in \mathbb{N}^*$. A countably infinite subset S of G is called:

(a) *d-flat*, if it meets some coset of the subgroup $G[d]$ in an infinite subset;
(b) *n-round*, if $S \subseteq G[n]$ and S is not d-flat for any proper divisor d of n;
(c) *∞-round*, if S is not d-flat for each $d \in \mathbb{N}^*$.

One important property of n-round sets in Abelian groups is immediate.

Lemma 2.4. *Every infinite set in an Abelian group G contains either an ∞-round subset or a subset of the form $S + z$, where $z \in G$ and S is an n-round set for some integer $n \ge 2$.*

For a discrete Abelian group G, let $\widehat{G}$ be the group of all homomorphisms $f\colon G \to \mathbb{T}$ endowed with the topology of pointwise convergence. It is well known that the topological group $\widehat{G}$ is compact [HR, DPS]. The group of homomorphisms of G to $\mathbb{T}^\omega$ is naturally identified with $\widehat{G}^\omega$. We shall also identify $\mathbb{Z}(n)$ with the subgroup of $\mathbb{T}$ generated by the coset $1/n + \mathbb{Z}$.

The following result is a simple combination of Lemma 3.7 and Lemma 4.2 of [DT4]. To formulate it properly, we put $T_\infty = \mathbb{T}^\omega$ and $T_n = \mathbb{Z}(n)^\omega$ for every $n \in \mathbb{N}^*$. The groups $\mathbb{T}^\omega$ and $\mathbb{Z}(n)^\omega \le \mathbb{T}^\omega$ carry their usual compact group topologies. This notation will be also used in Lemma 2.6 and in the proof of Theorem 2.7.

Lemma 2.5. *Let S be an n-round subset of an Abelian group G, $n \in \mathbb{N}^* \cup \{\infty\}$. Then the set*

$$H_S = \{h \in \widehat{G}^\omega : h(S) \text{ is dense in } T_n\}$$

is an intersection of countably many open dense subsets of $\widehat{G}^\omega$, hence dense in $\widehat{G}^\omega$.

The next lemma is our main technical tool in the proof of Theorem 2.7.

Lemma 2.6 ([MA])**.** *Let G be an abstract Abelian group, $\alpha < \mathfrak{c}$ an ordinal, and x^* a non-zero element of G. For every $\gamma < \alpha$, let $f_\gamma\colon G \to G_\gamma$ be a homomorphism of G to a non-discrete topological group G_γ. Suppose that for every $\gamma < \alpha$,*

(a) *a subset S_γ of G is n_γ-round, for $n_\gamma \in \mathbb{N}^* \cup \{\infty\}$; and*
(b) *$f_\gamma(S_\gamma)$ is dense in G_γ and $w(G_\gamma) < \mathfrak{c}$.*

Then there exists a homomorphism $f\colon G \to T_\infty$ such that $f(x^*) \neq 0$ and for every $\gamma < \alpha$, the image $(f_\gamma \triangle f)(S_\gamma)$ is dense in $G_\gamma \times T_{n_\gamma}$.

Proof. According to (b), there exists a base $\mathcal{B}_\gamma$ of non-empty open sets for G_γ such that $|\mathcal{B}_\gamma| < \mathfrak{c}$. Let also $\mathcal{B}$ be a countable base for $T_\infty = \mathbb{T}^\omega$. For $\gamma < \alpha$ and $U \in \mathcal{B}_\gamma$, put $S_\gamma(U) = S_\gamma \cap f_\gamma^{-1}(U \cap f_\gamma(S_\gamma))$. Since $f_\gamma(S_\gamma)$ is dense in the non-discrete group G_γ, the family

$$\mathcal{S}_\gamma = \{S_\gamma(U) : U \in \mathcal{B}_\gamma\}$$

consists of infinite subsets of G, and $|\mathcal{S}_\gamma| \le |\mathcal{B}_\gamma| < \mathfrak{c}$. Note that $S_\gamma(U)$ is n_γ-round for every $U \in \mathcal{B}_\gamma$ as an infinite subset of the n_γ-round set S_γ. Denote by H_γ the set of all homomorphisms $h\colon G \to T_\infty$ such that $h(S)$ is dense in T_{n_γ} for each $S \in \mathcal{S}_\gamma$. By Lemma 2.5, H_γ is the intersection of at most $|\mathcal{S}_\gamma| \cdot \omega < \mathfrak{c}$ open dense subsets of the compact group $\widehat{G}^\omega$. Since $\alpha < \mathfrak{c}$, the set $H = \bigcap_{\gamma<\alpha} H_\gamma$ is the intersection of less than $\mathfrak{c}$ open dense subsets of $\widehat{G}^\omega$ (here we use the fact that $\mathfrak{c}$ is a regular cardinal under MA). The dual group $\widehat{G}$ of the discrete Abelian group G is compact, hence dyadic [Kuz], so the cellularity of $\widehat{G}^\omega$ is countable.

Consider the non-empty open subset

$$W = \{h \in \widehat{G}^\omega : h(x^*) \neq 0\}$$

of $\widehat{G}^\omega$. Then $W \cap H \neq \emptyset$, so there exists an element $f \in W \cap H$. Clearly, $f(x^*) \neq 0$. For every $\gamma < \alpha$, denote by h_γ the diagonal product of f_γ and f. We claim that $h_\gamma(S_\gamma)$ is dense in $G_\gamma \times T_{n_\gamma}$ for each $\gamma < \alpha$. Indeed, let $U \times V$ be a non-empty open subset of $G_\gamma \times T_\infty$, where $U \in \mathcal{B}_\gamma$, $V \in \mathcal{B}$ and $V \cap T_{n_\gamma} \neq \emptyset$. Since $S = S_\gamma(U) \in \mathcal{S}_\gamma$ and $f \in H$, the set $f(S)$ is dense in T_{n_γ}. Choose a point $y \in S$ such that $f(y) \in V \cap T_{n_\gamma}$. Then $h_\gamma(y) = (f_\gamma(y), f(y)) \in U \times V$, whence $h_\gamma(S) \cap (U \times V) \neq \emptyset$. Since $S \subseteq S_\gamma$, we conclude that $h_\gamma(S_\gamma)$ is dense in $G_\gamma \times T_{n_\gamma}$. □

The next theorem is the main step in the proof of Theorem A.

Theorem 2.7. *Under MA, every precompact Abelian group H with $w(H) \leq \aleph_0$ belongs to $\mathfrak{C}$.*

Proof. Our argument follows the line of the proof of [DT4, Theorem 4.4], but several important modifications are necessary. Without loss of generality we can assume that the group H is infinite. Then $w(H) = \aleph_0$, and hence $|H| \leq 2^{\aleph_0} = \mathfrak{c}$. Put $M = \mathbb{Z}^{(\mathfrak{c})} \oplus \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)^{(\mathfrak{c})}$ and $G = H \oplus M$. For every $\beta < \mathfrak{c}$, put $M_\beta = \mathbb{Z}^{(\mathfrak{c}\setminus\beta)} \oplus \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)^{(\mathfrak{c}\setminus\beta)}$, and let $G_\beta = H \oplus M_\beta$. Note that $H \cong H \times \{0_M\} = \bigcap_{\beta<\mathfrak{c}} G_\beta$. Our main idea is to define a group topology τ on G which will additionally make the subgroup G_β of G countably compact for each $\beta < \mathfrak{c}$, and such that the topology τ_H^G on H induced from G will be finer than the original topology τ_H of H. Once this is done, the group (H, τ_H) can be represented as a continuous isomorphic image of the closed diagonal subgroup $\triangle_H$ of the direct product $\prod_{\beta<\mathfrak{c}} G_\beta$. Let us show that such a construction is possible under MA.

Put $T_n = \mathbb{Z}(n)^\omega$ if $n \in \mathbb{N}^*$, and $T_\infty = \mathbb{T}^\omega$. Note that $T_n \leq T_\infty$ for each $n \in \mathbb{N}^*$. We also put $T = T_\infty$ to avoid a frequent use of the subscript ∞. For every $\alpha < \mathfrak{c}$, denote by π_α the projection of $T^\mathfrak{c}$ onto T^α. We shall construct an injective homomorphism $h\colon G \to T^\mathfrak{c}$ satisfying the following conditions for all $\alpha, \beta < \mathfrak{c}$ and $n \in \mathbb{N}^* \cup \{\infty\}$ (recall that by definition, $G_\beta[\infty] = G_\beta$):

(A) if S is an n-round subset of G, then $h(S)$ is finally dense in $T_n^\mathfrak{c}$;
(B) $\pi_\alpha(h(G_\beta[n])) = T_n^\alpha$.

The claim below partially explains the role of the conditions (A) and (B).

Claim 1. *If the homomorphism $h\colon G \to T^\mathfrak{c}$ satisfies* (A) *and* (B), *then for every $\beta < \mathfrak{c}$, the subgroup $h(G_\beta)$ of $T^\mathfrak{c}$ is countably compact and has no non-trivial convergent sequences.*

Indeed, consider a countably infinite subset C of G. By Lemma 2.4, C contains either a set of the form $S+z$, where S is a d-round subset of G for an integer $d > 1$, or an ∞-round subset S'. In the first case, the image $h(S)$ is finally dense in $T_d^\mathfrak{c}$ by (A), and in the second case (A) implies that $h(S')$ is finally dense in $T^\mathfrak{c}$. In either case, $h(C)$ cannot converge since compact countable subsets of $T^\mathfrak{c}$ or $T_d^\mathfrak{c}$ are not finally dense.

Let us show that $h(G_\beta)$ is countably compact for each $\beta < \mathfrak{c}$. First, we claim that $h(G_\beta[n])$ is countably compact for each $n \in \mathbb{N}$. This is trivial for $n = 1$. The next step is to see that $h(G_\beta[p])$ is countably compact for a prime $n = p$. Indeed, by (A), $h(G_\beta[p])$ is an HFD subgroup of $T_p^\mathfrak{c}$ (every infinite subset of $G_\beta[p]$ is p-round), and by (B), the projections of $h(G_\beta[p])$ fill all faces T_p^α, $\alpha < \mathfrak{c}$. Therefore, Lemma 2.2 implies that $h(G_\beta[p])$ is countably compact. Suppose that $n \in \mathbb{N}$ is compound and we have proved countable compactness of $h(G_\beta[d])$ for every proper divisor d of n. Consider a countably infinite subset S of $G_\beta[n]$. If S is n-round, then every infinite subset of S is also n-round, so (A) implies that $h(S)$ is an HFD subset of $T_n^\mathfrak{c}$. Since $h(G_\beta[n])$ fills all "initial" faces of $T_n^\mathfrak{c}$, again Lemma 2.2 implies that $h(S)$ has a cluster point in $h(G_\beta[n])$. If S is not n-round, we apply Lemma 2.4 to find a proper divisor d of n, an element $z \in G_\beta[n]$ and a d-round subset S_0 of S such that $S_0 + z \subseteq S$. Since $h(G_\beta[d])$ is countably compact by the inductive hypothesis, $h(S_0)$ has a cluster point $h(y)$ for some $y \in G_\beta[d]$. Therefore, $h(z+y)$ is an accumulation point of $h(S)$. This proves that $h(G_\beta[n])$ is countably compact.

Suppose now that C is a countably infinite subset of G_β. If C contains a subset of the form $S+z$, where $z \in G_\beta$ and S is d-round in G_β for some $d \in \mathbb{N}^*$, then $S \subseteq G_\beta[d]$, and hence $h(S)$ has a cluster point $h(x)$ for some $x \in G_\beta[d]$. Therefore, $h(x+z)$ is a cluster point of $h(C)$. Otherwise C contains an ∞-round subset S' by Lemma 2.4, so (A) implies that $h(S')$ is finally dense in $T^{\mathfrak{c}}$. In fact, every infinite subset of S' is ∞-round, so $h(S')$ is an HFD subset of $T^{\mathfrak{c}}$. From (B) and Lemma 2.2 it follows that $h(S') \subseteq h(S)$ has a cluster point in $h(G_\beta)$, so $h(G_\beta)$ is countably compact. This proves Claim 1.

By Claim 1, we obtain the required countably compact group topology on G identifying the group G with its image $h(G) \subseteq T^{\mathfrak{c}}$.

We shall construct the monomorphism $h\colon G \to T^{\mathfrak{c}}$ by recursion of length $\mathfrak{c}$. For every $\xi < \mathfrak{c}$, put $N_\xi = H \oplus \mathbb{Z}^{(\xi)} \oplus \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)^{(\xi)}$. Then $G = \bigcup_{\xi<\mathfrak{c}} N_\xi$, and the subgroups N_ξ of G satisfy the following conditions for all $\xi < \mathfrak{c}$ (we put $\mathbb{Z}(\infty) = \mathbb{Z}$ in (c) below):

(a) $N_0 = H$, and $N_\zeta \subseteq N_\xi$ if $\zeta < \xi$;
(b) $N_\xi = \bigcup_{\zeta<\xi} N_\zeta$ if ξ is an infinite limit ordinal;
(c) for every $n \in \mathbb{N}^* \cup \{\infty\}$, $N_{\xi+1} \cap G_\xi$ contains a copy of the group $\mathbb{Z}(n)$ which trivially intersects N_ξ.

We shall use the decomposition $G = \bigcup_{\xi<\mathfrak{c}} N_\xi$ in order to consequently define homomorphisms $h_{\alpha,\xi}$ of N_ξ to T in such a way that $h_{\alpha,\xi}$ extends $h_{\alpha,\zeta}$ if $\zeta < \xi < \mathfrak{c}$ and $\alpha < \mathfrak{c}$. Therefore, for every $\alpha < \mathfrak{c}$, we shall finally get the homomorphism $h_\alpha = \bigcup_{\xi<\mathfrak{c}} h_{\alpha,\xi}$ of G to T. The homomorphism $h\colon G \to T^{\mathfrak{c}}$ will be the diagonal product of the family $\{h_\alpha : \alpha < \mathfrak{c}\}$.

Denote by $\mathcal{S}$ the family of all countably infinite subsets of $G \setminus \{0\}$ such that every $S \in \mathcal{S}$ is n-round for some $n \in \mathbb{N}^* \cup \{\infty\}$. It is clear that $|\mathcal{S}| = \mathfrak{c}$, so there exists an enumeration $\mathcal{S} = \{S_\mu : 0 < \mu < \mathfrak{c}\}$ such that $S_\mu \subseteq N_\mu$ whenever $0 < \mu < \mathfrak{c}$. Then S_μ is d_μ-round for some $d_\mu \in \mathbb{N}^* \cup \{\infty\}$.

Let Σ be the subgroup of $T^{\mathfrak{c}}$ consisting of all $x \in T^{\mathfrak{c}}$ that satisfy

$$|\{\alpha < \mathfrak{c} : x(\alpha) \neq 0\}| < \mathfrak{c}.$$

MA implies that the cardinality of Σ is equal to $\sum_{\lambda<\mathfrak{c}} 2^\lambda = \mathfrak{c}$, so there exists an enumeration $\Sigma \setminus \{0\} = \{b_\nu : \nu < \mathfrak{c}\}$. Finally, for every $x \in G$, denote by $\xi(x)$ the minimal ordinal $\xi < \mathfrak{c}$ such that $x \in N_\xi$. Then $\xi(x)$ is either zero or non-limit.

Our aim now is to define a family $\mathcal{H} = \{h_{\alpha,\nu} : \alpha, \nu < \mathfrak{c}\}$ satisfying the following conditions for all $\alpha, \nu < \mathfrak{c}$:

(1) $h_{\alpha,\nu}\colon N_\nu \to T$ is a homomorphism and $h_{\alpha,\nu}$ extends $h_{\alpha,\mu}$ if $\mu < \nu$;
(2) for every μ with $0 < \mu < \alpha$, the image $\triangle_{\mu\le\gamma<\alpha} h_{\gamma,\mu}(S_\mu)$ is dense in $T_{d_\mu}^{\alpha\setminus\mu}$;
(3) there exists a point $x \in N_{\nu+1} \cap G_\nu[ord(b_\nu)]$ such that $h_{\gamma,\nu+1}(x) = b_\nu(\gamma)$ for each $\gamma \le \nu$;
(4) if $\alpha > 0$ and $\xi = \min\{\xi(x) : x \in S_\alpha\}$, then there exists a point $z \in S_\alpha \cap N_\xi$ such that $h_{\alpha,\alpha}(z) \neq 0$.

Let us show that the homomorphism $h = \triangle_{\alpha<\mathfrak{c}} h_\alpha\colon G \to T^{\mathfrak{c}}$ is injective. Suppose that $x \in G$ is a non-zero element and put $\xi = \xi(x)$. Then choose an ∞-round subset S of G such that $x \in S$ and $\xi(y) > \xi$ for each $y \in S$ distinct from x. Clearly, $S = S_\alpha$ for some $\alpha > 0$. By (4), there exists $z \in S_\alpha \cap N_\xi$ such that $h_{\alpha,\alpha}(z) \neq 0$. However, $S_\alpha \cap N_\xi = \{x\}$, so $z = x$. This proves that $h_{\alpha,\alpha}(x) \neq 0$.

Conditions (1)–(3) imply the validity of (A) and (B). Let us verify that (A) holds. If $d \in \mathbb{N}^* \cup \{\infty\}$ and S is a d-round subset of G, then $S = S_\mu$ for some $\mu < \mathfrak{c}$. For every α satisfying $\mu < \alpha < \mathfrak{c}$, let $f_{\mu,\alpha} = \triangle_{\mu \le \gamma < \alpha} h_{\gamma,\alpha}$ be the diagonal product of the family $\{h_{\gamma,\alpha} : \mu \le \gamma < \alpha\}$. From the equality $\pi_{\alpha \setminus \mu} \circ h = f_{\mu,\alpha}$ and (1), (2) it follows that $\pi_{\alpha\setminus\mu}(h(S_\mu))$ is dense in $T_d^{\alpha\setminus\mu}$ for each $\alpha > \mu$, and hence $h(S_\mu)$ is dense in $T_d^{\mathfrak{c}\setminus\mu}$.

Let us show that (B) holds. Suppose that $\alpha, \beta < \mathfrak{c}$, $n \in \mathbb{N}^* \cup \{\infty\}$ and $z \in T_n^\alpha$, $z \neq 0$. Denote by d the order of z. Then either $d = n = \infty$ or $d|n$. Since $|\{\nu < \mathfrak{c} : ord(b_\nu) = d \ \&\ \pi_\alpha(b_\nu) = z\}| = \mathfrak{c}$, there exists $\nu < \mathfrak{c}$ such that $\alpha \le \nu$, $\beta \le \nu$, $ord(b_\nu) = d$ and $\pi_\alpha(b_\nu) = z$. By (3), there exists a point $x \in N_{\nu+1} \cap G_\nu[d]$ such that $h_{\gamma,\nu+1}(x) = b_\nu(\gamma)$ for each $\gamma < \nu$. This and the definition of h together imply that $\pi_\alpha(h(x)) = z$. Since z is an arbitrary point of T_n^α and $G_\nu \subseteq G_\beta$, we have proved that $\pi_\alpha(h(G_\beta[n])) = T_n^\alpha$.

Therefore, it remains to prove the following:

Claim 2. *The family $\mathcal{H}$ satisfying* (1)–(4) *exists under MA.*

Indeed, since $w(H) = \aleph_0$, there exists a topological monomorphism $h_{0,0} \colon H \to T = \mathbb{T}^\omega$, and we put $\mathcal{H}_0 = \{h_{0,0}\}$. Let α be an ordinal with $0 < \alpha < \mathfrak{c}$, and suppose that for every $\delta < \alpha$, we have defined a family $\mathcal{H}_\delta = \{h_{\gamma,\nu} : \gamma, \nu \le \delta\}$ satisfying (1)–(4).

If $\alpha = \beta+1$, the family $\mathcal{H}_\beta$ has been defined to satisfy (1)–(4). We have to extend the homomorphisms $h_{\gamma,\beta}$ (with $\gamma \le \beta$) over N_α, thus obtaining the homomorphisms $h_{\gamma,\alpha}$, and define a homomorphism $h_{\alpha,\alpha} \colon N_\alpha \to T$.

Let us start with extensions of homomorphisms $h_{\gamma,\beta}$. Suppose that the element $b_\beta \in \Sigma \setminus \{0\}$ has order $n \in \mathbb{N}^* \cup \{\infty\}$. Then by (c), there exists an element $x \in N_\alpha \cap G_\beta$ of order n such that $\langle x \rangle \cap N_\beta = \{0\}$. For every $\gamma \le \beta$, define a homomorphism $h_{\gamma,\alpha} \colon N_\alpha \to T$ extending $h_{\gamma,\beta}$ and satisfying $h_{\gamma,\alpha}(x) = b_\beta(\gamma)$.

Finally, we define a homomorphism $h_{\alpha,\alpha} \colon N_\alpha \to T$. First, choose an element $z \in S_\alpha$ with the minimal possible $\xi(z)$. For every $\mu \le \beta$, denote by $f_{\mu,\alpha}$ the diagonal product of the homomorphisms $h_{\gamma,\beta}$ with $\mu \le \gamma \le \beta$. Then $f_{\mu,\alpha} \colon N_\alpha \to T^{\alpha\setminus\mu}$. Apply Lemma 2.6 to find a homomorphism $f \colon N_\alpha \to T$ such that $f(z) \neq 0$ and for every $\mu \le \beta$, the image $(f_{\mu,\alpha} \triangle f)(S_\mu)$ is dense in $T_{d_\mu}^{\alpha\setminus\mu} \times T_{d_\mu}$. It remains to put $h_{\alpha,\alpha} = f$ and $h_{\alpha,\nu} = f|_{N_\nu}$ for each $\nu < \alpha$.

If α is limit, then $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$, so by (1), for every $\gamma < \alpha$ there exists a homomorphism $h_{\gamma,\alpha} \colon N_\alpha \to T$ such that $h_{\gamma,\alpha}|_{N_\nu} = h_{\gamma,\nu}$ for each $\nu < \alpha$. The existence of a homomorphism $h_{\alpha,\alpha} \colon N_\alpha \to T$ satisfying (4) can be established similarly to the case of a non-limit α above. Put $h_{\alpha,\nu} = h_{\alpha,\alpha}|_{N_\nu}$ for each $\nu < \alpha$ to complete the construction of $\mathcal{H}_\alpha$. This also finishes our recursive construction of the family $\mathcal{H} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{H}_\alpha$.

One easily verifies that the family $\mathcal{H}$ satisfies (1)–(4). This proves Claim 2 and the theorem. □

Our next step settles the case of Abelian groups $G^\#$ with $|G| \le \mathfrak{c}$.

Proposition 2.8 ([MA])**.** *$G^\# \in \overline{S}C(\mathcal{CCA})$ for every Abelian group G of cardinality $\le \mathfrak{c}$.*

Proof. Let G be an abstract Abelian group G with $|G| \le \mathfrak{c}$ and let $\widehat{G} = \{\chi_i\}_{i\in I}$ be a one-to-one enumeration of the dual group $\widehat{G}$. Clearly, the topology τ_B of $G^\#$ is induced by the monomorphism $g = \triangle_{i\in I}\chi_i : G \to \mathbb{T}^I$. First we prove that $G^\# \in \mathfrak{C}$.

There exists an isomorphic embedding $j : G \to \mathbb{T}^\omega$ [Fuc] that induces a second countable precompact group topology τ on G which is obviously coarser than τ_B. For every $i \in I$, denote by τ_i the topology induced on G by $j\triangle\chi_i : G \to \mathbb{T}^\omega \times \mathbb{T}$. Then τ_i is finer than τ and $w(G, \tau_i) = \aleph_0$, so $(G, \tau_i) \in \mathfrak{C}$ by Theorem 2.7. Hence also $H = \prod_{i\in I}(G, \tau_i) \in \mathfrak{C}$. The diagonal subgroup Δ of H consisting of all points $(x, x, \dots)$, with $x \in G$, is closed, whence $\Delta \in \mathfrak{C}$. Since τ_i is precompact, the identity map $\nu_i : G^\# \to (G, \tau_i)$ is continuous for every $i \in I$. Hence $\nu = \triangle_{i\in I}\nu_i : G \to H$ is continuous and $\nu(G) = \Delta$. Since the restriction p of the natural projection $(\mathbb{T}^\omega \times \mathbb{T})^I \to \mathbb{T}^I$ to Δ is continuous and $g = p \circ \nu \colon G^\# \to g(G)$ is a topological isomorphism, we conclude that ν is a topological isomorphism too. Hence $G^\# \cong \Delta$ and so $G^\# \in \mathfrak{C}$.

Finally, let $f \colon K \to G^\#$ be a continuous isomorphism, where $K \in \overline{S}C(\mathcal{CCA})$. The group K is necessarily precompact, and since $G^\#$ carries the maximal precompact group topology, f has to be a topological isomorphism. Therefore, $G^\# \in \overline{S}C(\mathcal{CCA})$. □

Proof of Theorem A. Suppose that a precompact Abelian group G satisfies $|G| \le \mathfrak{c}$. Since the original topology of G is coarser than the Bohr topology of the abstract group G, the identity map $G^\# \to G$ is continuous. By Proposition 2.8, $G^\# \in \overline{S}C(\mathcal{CCA})$, so $G \in I\overline{S}C(\mathcal{CCA}) = \mathfrak{C}$. □

3. Open problems

Here we formulate two problems concerning the relations between the classes $\mathfrak{C} = I\overline{S}C(\mathcal{CCA})$, $\overline{S}C(\mathcal{CCA})$, $Q\overline{S}C(\mathcal{CCA})$, $H\overline{S}C(\mathcal{CCA})$ and the class $\mathcal{PA}$ of all precompact Abelian groups.

Problem 3.1. Which of the following equalities holds:

(a) $\mathfrak{C} = \mathcal{PA}$;
(b) $Q\overline{S}C(\mathcal{CCA}) = \mathcal{PA}$;
(c) $H\overline{S}C(\mathcal{CCA}) = \mathcal{PA}$;
(d) $\overline{S}C(\mathcal{CCA}) = \mathcal{SCPA}$?

Our second problem is a special case of Problem 3.1 (d).

Problem 3.2. Is every precompact sequentially complete Abelian group of cardinality $\le \mathfrak{c}$ in $\overline{S}C(\mathcal{CCA})$?

References

[Ar1] A. V. Arhangel'skiĭ, Cardinal invariants of topological groups. Embeddings and condensations, *Soviet Math. Dokl.* **20** (1979), 783–787. Russian original in: *Dokl. AN SSSR* **247** (1979), 779–782. MR **80m:**54005

[Ar2] A. V. Arhangel'skiĭ, Every topological group is a quotient group of a zero-dimensional topological group, *Dokl. AN SSSR* **258** (1981), 1037–1040 (*in Russian*). MR **83e:**22002

[CM] W. W. Comfort and J. van Mill, On the existence of free topological groups, *Topology Appl.* **29** (1988), 245–265. MR **90e:**22001

[CR] W. W. Comfort and K. A. Ross, Pseudocompactness and uniform continuity in topological groups, *Pacific J. Math.* **16** (1966), 483–496. MR **34:**7699

[D] D. Dikranjan, Quotients of zero-dimensional precompact abelian groups, *Topology Appl.* **86** (1998), no. 1, 47–62. MR **99e:**54028

[DPS] D. Dikranjan, I. Prodanov and L. Stoyanov, *Topological groups (Characters, Dualities and Minimal group topologies).* Marcel Dekker, Inc., New York–Basel, 1990. MR **91e:**22001

[DT1] D. Dikranjan and M. G Tkachenko, Sequentially complete groups: dimension and minimality, *J. Pure Appl. Algebra* **157** (2001), 215–239. MR **2001m:**22002

[DT2] D. Dikranjan and M. G. Tkachenko, Sequential completeness of quotient groups, *Bull. Austral. Math. Soc.* **61** (2000), 129–151. CMP 2001:10

[DT3] D. Dikranjan and M. G. Tkachenko, Weakly complete free topological groups, *Topology Appl.* **112** (2001), no. 3, 259–287. MR **2001:**11

[DT4] D. Dikranjan and M. G. Tkachenko, Algebraic structure of small countably compact Abelian groups, Forum Math, to appear.

[DTT] D. Dikranjan, M. G. Tkachenko and V. V. Tkachuk, Topological groups with thin generating sets, *J. Pure Appl. Algebra,* **145** (2000) 123–148. MR **2000m:**22003

[Do1] E. van Douwen, The product of two countably compact topological groups, *Trans. Amer. Math. Soc.* **262** (1980), 417–427. MR **82b:**22002

[Do2] E. van Douwen, The maximal totally bounded group topology on G and the biggest minimal G-space for Abelian groups G, *Topology Appl.* **34** (1990), 69–91. MR **91d:**54044

[Flo] P. Flor, Zur Bohr-Konvergenz von Folgen, *Math. Scand.* **23** (1968), 169–170. MR **40:**4685

[Fuc] L. Fuchs, *Infinite Abelian groups*, Vol. I, Academic Press, New York, 1970. MR **41:**333

[HJ] A. Hajnal and I. Juhász, A normal separable group need not be Lindelöf, *Gen. Topol. Appl.* **6** (1976), 199–205. MR **55:**4088

[HR] E. Hewitt and K. Ross, Abstract Harmonic Analysis I (Springer-Verlag, Berlin-Göttingen-Heidelberg 1979). MR **81k:**43001

[HM] K. Hart and J. van Mill, A countably compact group H such that $H \times H$ is not countably compact, *Trans. Amer. Math. Soc.* **323** (1991), 811–821. MR **91e:**54025

[Juh] I. Juhász, *Cardinal functions in Topology,* Math. Centre Tracts 34, Amsterdam 1971. MR **49:**4778

[Kuz] V. Kuz'minov, On a hypothesis of P. S. Alexandrov in the theory of topological groups, *Doklady Akad. Nauk SSSR* **125** (1959), 727–729 (*in Russian*). MR **21:**3506

[Mo1] S. A. Morris, Varieties of topological groups, *Bull. Austral. Math. Soc.* **1** (1969), 145–160. MR **41:**3655

[Mo2] S. A. Morris, Varieties of topological groups. A survey. *Colloq. Math.* **46** (1982), 147–165. MR **84e:**22005

[Sha] D. Shakhmatov, Imbeddings into topological groups preserving dimensions, *Topology Appl.* **36** (1990), 181–204. MR **91i:**54028

[Tay] W. Taylor, Varieties of topological algebras, *J. Austral. Math. Soc.* **23** (1977), 207–241. MR **56:**5392

[Tk] M. G. Tkachenko, Introduction to topological groups, *Topology Appl.* **86** (1998), 179–231. MR **99b:**54064

[TY] M. G. Tkachenko and Iv. Yaschenko, Independent group topologies on Abelian groups, *Topology Appl.*, to appear.

[To1] A. H. Tomita, On finite powers of countably compact groups, *Comment. Math. Univ. Carolin.* **37** (1996), no. 3, 617–626. MR **98a:**54033

[To2] A. H. Tomita, A group under $MA_{countable}$ whose square is countably compact but whose cube is not, *Topology Appl.* **91** (1999), 91–104. MR **2000d:**54039

Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze 206, 33100 Udine, Italy

E-mail address: dikranja@dimi.uniud.it

Departamento de Matemáticas, Universidad Autónoma Metropolitana, México

E-mail address: mich@xanum.uam.mx