SHARP LOCAL ISOPERIMETRIC INEQUALITIES INVOLVING THE SCALAR CURVATURE

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Abstract. We provide sharp local isoperimetric inequalities on Riemannian manifolds involving the scalar curvature, and thus answer a question asked by Johnson and Morgan.

1. Introduction and statement of the results

Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\) with sectional curvature \(K_g \leq K_0\). A long-standing conjecture, a formulation of which can be found in [1], asserts that for any \(x \in M\), there exists \(r_x > 0\) such that for any \(\Omega\) contained in the geodesic ball of center \(x\) and radius \(r_x\),

\[|\partial \Omega|_g \geq |\partial B|_{g_0}\]

where \(\cdot|_g\) (resp. \(\cdot|_{g_0}\)) denotes the volume with respect to \(g\) (resp. \(g_0\)) and \(B\) is a ball of volume \(|\Omega|_g\) in the model space \((M_0, g_0)\) of constant sectional curvature \(K_0\).

A compact version of this conjecture was proved, with an additional assumption on the Gauss-Bonnet-Chern integrand in even dimensions, in the very nice Johnson and Morgan [10]. A natural question that Johnson and Morgan [10] asked is the following: is the result still true if we assume that the scalar curvature of \((M, g)\) is such that \(S_g < n(n-1)K_0\) instead of assuming that \(K_g \leq K_0\)? We answer this question in the affirmative and prove the following:

**Theorem 1.** Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\) and let \(x \in M\). Assume that \(S_g(x) < n(n-1)K_0\) for some \(K_0 \in \mathbb{R}\). Then there exists \(r_x > 0\) such that for any \(\Omega\) contained in the geodesic ball of center \(x\) and radius \(r_x\),

\[|\partial \Omega|_g > |\partial B|_{g_0}\]

where \(B\) is a ball of volume \(|\Omega|_g\) in the model space \((M_0, g_0)\) of constant sectional curvature \(K_0\).

In the compact setting, the situation that was actually considered by Johnson and Morgan [10], we have the following:
Theorem 2. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\) with scalar curvature \(S_g < n(n-1)K_0\). There exists \(V > 0\) such that for any subset \(\Omega\) of \(M\) of volume less than or equal to \(V\),

\[
|\partial \Omega|_g > |\partial B|_{g_0}
\]

where \(B\) is a ball of volume \(|\Omega|_g\) in the model space \((M_0, g_0)\) of constant sectional curvature \(K_0\).

These results are optimal in the following sense: if we only assume that the Ricci curvature of \(M\) verifies \(\text{Ric}_g \leq (n-1)K_0\), the above isoperimetric comparison fails. Indeed, for any \(n\)-manifold \(M\) which is Ricci-flat but not flat (see \([3]\) for examples of such manifolds), one may find a ball \(B_r\) in \(M\) of radius \(r\) as small as we want which verifies

\[
|\partial B_r|_g < |\partial B|_\xi
\]

where \(B\) is a ball of volume \(|B_r|_g\) in the Euclidean space \((\mathbb{R}^n, \xi)\). The above comparison result is also false on \(S^2 \times S^2\), as noticed in \([10]\). The proof of Theorem 1 is based on the study of local optimal Sobolev inequalities. The proof relies on PDE techniques and a fine asymptotic analysis of solutions of quasi-elliptic equations involving the \(p\)-Laplacian. Theorem 2 is a consequence of Theorem 1 thanks to geometric measure theory. The relevance of the scalar curvature when studying the validity of sharp Sobolev inequalities was noticed first by the author in \([3]\] and underlined by Hebey in \([9]\).

2. Sobolev inequalities and proof of Theorem 1

Let \(B\) be a ball in the model space \((M_0, g_0)\) of constant sectional curvature \(K_0\). It is not difficult to check that, for balls of small volume,

\[
|\partial B|_{g_0}^2 = K(n, 1)^{-2}|B|_{g_0}^2 \frac{n}{n+2} - \frac{n}{n+2} (n(n-1)K_0)|B|_{g_0}^2 + o(|B|_{g_0}^2).
\]

Here, \(n = \dim M_0\) and

\[K(n, 1)^{-1} = n \left(\frac{\omega_{n-1}}{n}\right)^\frac{n}{n-1}.\]

Now, let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\) and let \(x_0 \in M\). In order to prove Theorem 1, it is clearly sufficient to prove that for any \(\varepsilon > 0\), there exists \(r_\varepsilon > 0\) such that for any \(\Omega \subset B_g(x_0, r_\varepsilon)\),

\[
|\partial \Omega|_g^2 \geq K(n, 1)^{-2}|\Omega|_g^2 \frac{n}{n+2} - \left(\frac{n}{n+2} S_g(x_0) + \varepsilon\right) |\Omega|_g^2.
\]

It is now well known that (2.1) is a consequence of the following Sobolev inequality: for any \(u \in C_c^\infty(B_g(x_0, r_\varepsilon))\),

\[
\|u\|_{\frac{n+2}{n-1}}^2 \leq K(n, 1)^2 \left(\|\nabla u\|_1^2 + \left(\frac{n}{n+2} S_g(x_0) + \varepsilon\right) \|u\|_1^2\right)
\]

where \(\|\cdot\|_p\) denotes the \(L^p\)-norm with respect to the Riemannian volume element \(dv_g\). Indeed, \(\Omega \subset B_g(x_0, r_\varepsilon)\) being given, one may find a sequence \((u_i)\) of smooth functions with compact support in \(B_g(x_0, r_\varepsilon)\) such that for any \(q \geq 1\),

\[
\lim_{i \to +\infty} \int_{B_g(x_0, r_\varepsilon)} |u_i|^q dv_g = |\Omega|_g
\]
and
\[
\lim_{i \to +\infty} \int_{B_g(x_0, r_i)} |\nabla u_i|_g \, dv_g = |\partial \Omega|_g.
\]

Before starting the proof of the above Sobolev inequality, we must set up some notations. For any \(1 < p < n\), let
\[
K(n, p)^{-p} = \inf_{u \in C^\infty_c(\mathbb{R}^n), u \neq 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^p \, dv_g}{\left(\int_{\mathbb{R}^n} |u|^{p^*} \, dv_g\right)^{\frac{p}{p^*}}},
\]
where \(p^* = \frac{np}{n-p}\) is the critical exponent for the Sobolev embeddings and \(\xi\) is the Euclidean metric. The value of \(K(n, p)\) is explicitly known (see [11] or [15]) but the only point of interest to us is that
\[
\lim_{p \to 1} K(n, p) = K(n, 1) = \frac{1}{n} \left(\frac{n}{\omega_{n-1}}\right)^{\frac{n}{n-1}}.
\]

We also let, for \(1 < p < n\), \(H^p(\mathbb{R}^n)\) be the standard Sobolev space of order \(p\), that is the completion of \(C^1_c(\mathbb{R}^n)\) for the norm
\[
\|u\|_{H^p} = \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dv_g\right)^{\frac{1}{p}}.
\]

At last, we let \(BV(\mathbb{R}^n)\) be the space of functions with bounded variations in \(\mathbb{R}^n\), defined as the completion of \(C^\infty_c(\mathbb{R}^n)\) with respect to the norm
\[
\|u\|_{BV} = \sup \left\{-\int_{\mathbb{R}^n} u \, div (X) \, dv_g, \|X\|_{L^\infty(\mathbb{R}^n)} \leq 1, \, div (X) \in L^n(\mathbb{R}^n)\right\},
\]
where \(div (X) = \partial X_1\). Basic facts about \(BV(\mathbb{R}^n)\) can be found in [7] or [16].

As already mentioned, Theorem 1 reduces to the following proposition:

Proposition. Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\) and let \(x_0 \in M\). For any \(\varepsilon > 0\), there exists \(r_\varepsilon > 0\) such that for any \(u\) in \(C^\infty_c(B_g(x_0, r_\varepsilon))\),
\[
\|u\|^2_{L^2} \leq K(n, 1)^2 \left(\|\nabla u\|^2_1 + \alpha \|u\|^2_1\right)
\]
where \(\alpha = \frac{n}{n+2} S_g(x_0) + \varepsilon\).

We prove the Proposition in what follows.

Proof of the Proposition. Clearly, we may assume, without loss of generality, that \(M = \mathbb{R}^n\) and that \(x_0 = 0\). We let, for any \(r > 0\), any \(p > 1\) and any \(\varepsilon > 0\),
\[
\lambda_{p, r} = \inf_{u \in C^\infty_c(B_g(0, r)), u \neq 0} \frac{\left(\int_{B_g(0, r)} |\nabla u|^p \, dv_g\right)^{\frac{1}{p}} + \alpha \left(\int_{B_g(0, r)} |u|^{p^*} \, dv_g\right)^{\frac{1}{p^*}}}{\left(\int_{B_g(0, r)} |u|^p \, dv_g\right)^{\frac{1}{p}}}.
\]

We proceed by contradiction. We assume that there exists \(\varepsilon_0 > 0\) such that for any \(r > 0\),
\[
\lambda_1 < K(n, 1)^{-2}.
\]
Then, since \( \limsup_{p \to 1} \lambda_{p,r} \leq \lambda_{1,r} \), one easily gets that for any \( r > 0 \), there exists \( p_r > 1 \) such that

\[
\lambda_{p_r,r} < K(n,1)^{-2} \left( \frac{n - p_r}{p_r (n - 1)} \right)^2, \quad \lambda_{p_r,r} < K(n,p_r)^{-2}.
\]

We may assume that \( r \to 0 \) and we may choose \( p_r \) decreasing when \( r \) is decreasing. Then we get a sequence \( p > 1 \) going to 1 and a sequence \( r_p > 0 \) going to 0 as \( p \) goes to 1 which verify (2.2). It is by now classical that the second inequality in (2.2) ensures the existence of a minimizer \( u_p \) which satisfies the following:

(2.3) \[
C_p \Delta_p u_p + \alpha u_p \chi_n^{2-p} u_p^{p-1} = \lambda_p u_p^{p-1} \quad \text{in } B_g(0,r_p),
\]

\( u_p \in C^{1,\eta}(B_g(0,r_p)) \) for some \( \eta > 0 \),

\( u_p > 0 \) in \( B_g(0,r_p) \), \( u_p = 0 \) on \( \partial B_g(0,r_p) \),

(2.4) \[
\int_{B_g(0,r_p)} u_p^p \, dv_g = 1,
\]

(2.5) \[
\lambda_p < K(n,p)^{-2}, \quad \lambda_p < K(n,1)^{-2} \left( \frac{n - p}{p (n - 1)} \right)^2.
\]

(2.6) \[
C_p = \left( \int_{B_g(0,r_p)} |\nabla u_p|^p \, dv_g \right)^{\frac{2}{np}}.
\]

In the above equations, \( \Delta_p \) is the \( p \)-laplacian with respect to \( g \), that is \( \Delta_p u = -\text{div}_g (|\nabla u|^p - 2 \nabla u) \), and we have set

\[
\alpha = \frac{n}{n+2} S_g(0) + \varepsilon_0.
\]

Now the aim is to study this sequence \( (u_p) \) as \( p \to 1 \). We let \( x_p \) be a point in \( B_g(0,r_p) \) where \( u_p \) achieves its maximum and we also set

\[
u_p(x_p) = u_p(x_p)^{1 - \frac{p}{np}}.
\]

We have

\[
1 = \int_{B_g(0,r_p)} u_p^p \, dv_g \leq Vol_g(B_g(0,r_p)) \mu_p^{-n}
\]

and since \( r_p \) goes to 0, \( \mu_p \) goes to 0 as \( p \) goes to 1. In the same way, thanks to Hölder’s inequalities, we get

(2.7) \[
\lim_{p \to 1} \int_{B_g(0,r_p)} u_p^p \, dv_g = 0.
\]

**Step 1.** We first claim that

(2.8) \[
\lim_{p \to 1} \lambda_p = K(n,1)^{-2}
\]

and that

(2.9) \[
\lim_{p \to 1} \int_{B_g(0,r_p)} |\nabla u_p|^p \, dv_g = K(n,1)^{-1}.
\]
Indeed (see for instance [8] for an exposition in book form) for any $\varepsilon > 0$ there exists $B_\varepsilon > 0$ such that for any $p > 1$,

$$
\left( \int_{B_p(0, r_p)} u_p^p \, dv_g \right)^{2\frac{n-1}{n}} \leq (K(n, 1) + \varepsilon)^2 \left( \int_{B_p(0, r_p)} \left| \nabla \left( \frac{u_p^{(n-1)}}{u_p} \right) \right|^2 \, dv_g \right) + B_\varepsilon \left( \int_{B_p(0, r_p)} \frac{u_p^{(n-1)}}{u_p^p} \, dv_g \right)^2
$$

which gives with (2.3), (2.4) and Hölder’s inequalities

$$
1 \leq (K(n, 1) + \varepsilon)^2 \left( \frac{p(n-1)}{n-p} \right)^2 (\lambda_p - \alpha \| u_p \|^2_2) + B_\varepsilon \| u_p \|^2_2.
$$

This leads with (2.7) to

$$
1 \leq (1 + \varepsilon K(n, 1)^{-1})^2 \liminf_{p \to 1} (\lambda_p K(n, 1)^2).
$$

Since it is valid for any $\varepsilon > 0$, we obtain $\liminf_{p \to 1} \lambda_p \geq K(n, 1)^{-2}$. By (2.5), we get that (2.8) is proved. Then (2.9) is an obvious consequence of (2.3), (2.4), (2.7) and (2.8).

**Step 2.** We let $\Omega_p = \mu_p^{-1} \exp_{x_p}^{-1} (B_p(0, r_p)) \subset \mathbb{R}^n$ and we set

$$
g_p(x) = \exp_{x_p}^* g(\mu_p x) \quad \text{for } x \in \Omega_p
$$

and

$$
v_p(x) = \mu_p^{\frac{2}{p-1}} u_p \left( \exp_{x_p} (\mu_p x) \right) \quad \text{for } x \in \Omega_p, \quad v_p(x) = 0 \quad \text{for } x \in \mathbb{R}^n \setminus \Omega_p.
$$

Clearly we have

$$
C_p \Delta_{\Omega_p, g_p} v_p + \alpha \mu_p^2 \| v_p \|^2_{p-1} v_p^{p-2} = \lambda_p v_p^{p-1} \quad \text{in } \Omega_p
$$

with $v_p = 0$ on $\partial \Omega_p$ and

$$
\int_{\Omega_p} v_p^{p-1} \, dv_{g_p} = 1.
$$

We also let

$$
\tilde{v}_p(x) = v_p(x)^{\frac{n-1}{n-p}}.
$$

By the Cartan expansion of a metric in the exponential chart, there exists $C > 1$ such that

$$
\begin{align*}
\left| \nabla \tilde{v}_p \right|_{g_p} \, dv_{g_p} & \leq (1 + C \mu_p^2) \left| \nabla \tilde{v}_p \right|_{g}\, dv_{g}, \\
\left| \nabla \tilde{v}_p \right|_{g} \, dv_{g_P} & \leq \left( 1 + C \mu_p^2 \right) \left| \nabla \tilde{v}_p \right|_{g} \, dv_{g}
\end{align*}
$$

where $\xi$ is the Euclidean metric. This easily leads with (2.9), (2.11) and Hölder’s inequalities to

$$
\lim_{p \to 1} \frac{\int_{\mathbb{R}^n} |\nabla \tilde{v}_p|_{g}^{\frac{n}{n-1}} \, dv_{g}}{\left( \int_{\mathbb{R}^n} \tilde{v}_p^{\frac{p}{p-1}} \, dv_{g} \right)^{\frac{n-1}{n}}} = K(n, 1)^{-1}.
$$
Remember here that \( r_p \to 0 \) as \( p \to 1 \). Since \((\tilde{v}_p)\) is bounded in \( H^1_1(\mathbb{R}^n) \), there exists \( v_0 \in BV(\mathbb{R}^n) \) such that
\[
\lim_{p \to 1} \tilde{v}_p = v_0 \quad \text{weakly in } BV(\mathbb{R}^n).
\]

If we apply the concentration-compactness principle of P.L. Lions ([11], [12], see also [13] for an exposition in book form) to \( |v_p|^p \, dv_{\xi} \), four situations may occur: compactness, concentration, dichotomy or vanishing. Dichotomy is classically forbidden by (2.13). Concentration cannot happen since \( \sup \Delta \tilde{v}_p = v_p(0) = 1 \). As for vanishing, since \( v_p \) is bounded in \( L^1 \), by applying Moser’s iterative scheme to (2.10), one gets the existence of some \( C > 0 \) such that for any \( p > 1 \),
\[
1 = \sup_{\Omega_p \cap B_{\Delta_p}(0,1/2)} v_p \leq C \left( \int_{\Omega_p \cap B_{\Delta_p}(0,1)} v_p^p \, dv_{\xi} \right)^{1/p}.
\]
Thus vanishing cannot happen. Compactness together with (2.13) just gives
\[
\lim_{p \to 1} \tilde{v}_p = v_0 \quad \text{strongly in } BV(\mathbb{R}^n).
\]

Then \( v_0 \) is a minimizer for the \( H^1_1 \) Euclidean Sobolev inequality which verifies
\[
J_{\Omega_p} v_0 = 1.\quad \text{Thus there exists } y_0 \in \mathbb{R}^n, \; \lambda_0 > 0 \; \text{and } R_0 > 0 \; \text{such that}
\]
\[
(2.15) \quad v_0 = \lambda_0 1_{B(y_0,R_0)}
\]
where \( 1_{B(y_0,R_0)} \) denotes the characteristic function of the Euclidean ball \( B(y_0,R_0) \). Moreover, since \( v_p \leq 1 \) in \( \Omega_p \), we obtain with (2.14) that \( v_p \to v_0 \) in any \( L^q(\mathbb{R}^n) \), \( q \geq \frac{n}{n-1} \). One can deduce from this that \( \lambda_0 = 1 \). At last, we have:
\[
(2.16) \quad \text{Vol}_{\xi} (B(y_0,R_0)) = \frac{\omega_{n-1}}{n} R_0^n = 1.
\]

Up to changing \( x_p \) into \( exp_{x_p}(\mu_p y_0) \) in the definition of \( v_p \), \( \Omega_p \) and \( g_p \), we may assume that \( y_0 = 0 \). We have thus obtained that
\[
\lim_{p \to 1} \tilde{v}_p = 1_{B(0,R_0)} \quad \text{strongly in } BV(\mathbb{R}^n).
\]

This means in particular that
\[
\lim_{p \to 1} v_p = 1_{B(0,R_0)} \quad \text{strongly in } L^{\frac{n}{n-1}}(\mathbb{R}^n)
\]
and that for any \( \varphi \in C_{c}^{\infty}(\mathbb{R}^n) \),
\[
(2.18) \quad \lim_{p \to 1} \int_{\mathbb{R}^n} |\nabla v_p| |\varphi| \, dv_{\xi} = \int_{\partial B(0,R_0)} \varphi \, d\sigma_{\xi}.
\]

If we set
\[
(2.19) \quad V_p(x) = 1 + \left( \frac{|x|}{R_0} \right)^{\frac{p}{p-1}} \left( \frac{p-1}{n-p} \right)^{1-n} \quad x \in \mathbb{R}^n,
\]
a simple application of the concentration-compactness principle, using what we just proved, gives
\[
(2.20) \quad \lim_{p \to 1} \int_{\mathbb{R}^n} |\nabla (\tilde{v}_p - V_p)| \, dv_{\xi} = 0.
\]
Applying Moser’s iterative scheme to (2.10) with the help of (2.17), we also get that for any $R > R_0$,
\begin{equation}
\lim_{p \to 1} \sup_{\Omega_p \setminus B(0, R)} v_p = 0.
\end{equation}

**Step 3.** The aim is to transform the $L^{\infty}$-estimate (2.17) into a pointwise estimate. We follow here [6] (see also [5]). We let
\begin{equation}
w_p(z) = |z|^{\frac{n}{p} - 1} v_p(z)
\end{equation}
and we let $z_p \in \Omega_p$ be a point where $w_p$ achieves its maximum. Let us assume by contradiction that
\begin{equation}
\lim_{p \to 1} w_p(z_p) = +\infty.
\end{equation}
We set
\begin{equation}
v_p^{1 - \frac{n}{p}} = v_p(z_p)
\end{equation}
so that
\begin{equation}
\lim_{p \to 1} \frac{|z_p|}{v_p} = +\infty.
\end{equation}
Independently, since $v_p \leq 1$ in $\Omega_p$,
\begin{equation}
\lim_{p \to 1} |z_p| = +\infty.
\end{equation}
Thanks to (2.22) and (2.23), one proves then that $(v_p^{\frac{n}{p} - 1} v_p(exp_{z_p}(\nu_p x)))$ is bounded in $L^{\infty}(B(0, 1))$. This allows us to apply Moser’s iterative scheme to the equation verified by $(v_p^{\frac{n}{p} - 1} v_p(exp_{z_p}(\nu_p x)))$ and to get the existence of some $C > 0$ such that
\begin{equation}
\liminf_{p \to 1} \int_{B_p(z_p, \nu_p) \cap \Omega_p} v_p^{p^*} \, dv_g > 0.
\end{equation}
The contradiction then easily follows from (2.17), (2.22) and (2.23). Thus we have the existence of some $C > 0$ such that for any $p > 1$, any $z \in \Omega_p$,
\begin{equation}
|z|^{\frac{n}{p} - 1} v_p(z) \leq C.
\end{equation}
In the same way, using (2.24), one proves thanks to (2.21) that for any $R > R_0$,
\begin{equation}
\lim_{p \to 1} \sup_{\Omega_p \setminus B_p(0, R)} \frac{|z|^{\frac{n}{p} - 1} v_p(z)}{v_p} = 0.
\end{equation}
We refer the reader to [6] for details on such claims.

**Step 4.** We let $L_p$ be the following operator:
\begin{equation}
L_p u = C_p \Delta_{p, \nu_p} u + \alpha \delta_p^2 \|u\|_{p, B}^{2-p} u^{p-1} - \lambda_p p^{p^*} - p u^{p-1}.
\end{equation}
We fix $0 < \nu < n - 1$ and we set
\begin{equation}
G_p(x) = \theta_p |x|^{\frac{n}{p} - 1 - \nu}.
\end{equation}
where $\theta_p$ is some positive constant to be fixed later. Easy computations lead to
\[ |x|^{n-\nu} \frac{L_p G_p(x)}{G_p(x)^{p-1}} \geq C_p \nu \left( \frac{n-p-\nu}{p-1} \right)^{p-1} - C \mu_p^2 |x|^2 + \alpha \mu_p^2 \|v_p\|^{2-p} |x|^p - \lambda_p |x|^p v_p^{p-p} \]
in $\Omega_p \setminus \{0\}$. Here $C$ denotes some constant independent of $p$. Thanks to (2.7), (2.8), (2.9), (2.25) and the fact that for $p \to 0$ as $p \to 1$, one gets that for any $R > R_0$,
\[ L_p G_p(x) \geq 0 \quad \text{in } \Omega_p \setminus B_{\theta_p}(0, R) \]
for $p$ small enough. On the other hand,
\[ L_p v_p = 0 \quad \text{in } \Omega_p. \]
At last, it is not difficult to check with (2.21) that
\[ v_p \leq \theta_p G_p \quad \text{on } \partial B_{\theta p}(0, R) \]
if we take $\theta_p = R^{\frac{n-p}{p-1}}$. Now we may apply the maximum principle as stated for instance in [2] (lemma 3.4) to get, for $p$ small enough,
\[ v_p(y) \leq \left( \frac{R}{|y|} \right)^{\frac{n-p-\nu}{p-1}} \quad \text{in } \Omega_p \setminus B_{\theta p}(0, R). \]
Since this inequality obviously holds on $B_{\theta p}(0, R)$, we have finally obtained the following: for any $\nu > 0$ and any $R > R_0$, there exist $C(R, \nu) > 0$ such that for any $p > 1$ and any $y \in \Omega_p$,
\[ (2.26) \quad \left( \frac{|y|}{R} \right)^{\frac{n-p-\nu}{p-1}} v_p(y) \leq C(R, \nu). \]

Step 5. We conclude the proof of the Proposition. We apply the $H^1_1$ Euclidean Sobolev inequality to $\tilde{v}_p$:
\[ (2.27) \quad \left( \int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} \, dv_\xi \right)^{\frac{n}{n-1}} \leq K(n, 1) \int_{\Omega_p} |\nabla \tilde{v}|_{\xi} \, dv_\xi. \]
By the Cartan expansion of $g_p$ around 0, we have
\[ (2.28) \quad dv_\xi = \left( 1 + \frac{\mu_p^2}{6} R \det g_i)_{ij} x^i x^j + o \left( \mu_p^2 |x|^2 \right) \right) \, dv g_p \]
where $Ric_g$ denotes the Ricci curvature of $g$ in the $exp_{g_p}$-map. Thus, by (2.11),
\[ \int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} \, dv_\xi = 1 + \frac{\mu_p^2}{6} R \det g_i j_{ij} x^i x^j v_p^{n-2} + o \left( \mu_p^2 |x|^2 \right) \, dv g_p. \]
Using (2.17) and (2.26), one gets
\[ (2.29) \quad \int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} \, dv_\xi = 1 + \frac{S_p(0)}{6n(n+2)} \omega_{n-1} R_0^{n+2} \mu_p^2 + o \left( \mu_p^2 \right). \]
By the Cartan expansion of $g_p$ around 0, since $r_p \to 0$ as $p \to 1$, we also have
\[ |\nabla \tilde{v}_p|_{g_p}^{\frac{n}{n-1}} \left[ 1 - \frac{\mu_p^2}{6} |\nabla \tilde{v}_p|_{g_p}^{-2} R \det g_i (\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) + o \left( \mu_p^2 |x|^2 \right) \right] \]
where $Rm_g$ denotes the Riemann curvature of $g$ in the $exp_{y_p}$-map. Then, using (2.28), we get

$$
\int_{\Omega_p} |\nabla \tilde{v}_p| dz\xi = \int_{\Omega_p} |\nabla \tilde{v}_p| dz_{g_p} + \frac{\mu_p^2}{6} Ric_g (y_p)_{ij} \int_{\Omega_p} x^i x^j |\nabla \tilde{v}_p| dz\xi
$$

$$
- \frac{\mu_p^2}{6} \int_{\Omega_p} |\nabla \tilde{v}_p|^{-1} Rm_g (y_p) (\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) dz_{g_p}
$$

$$
+ o \left( \mu_p^2 \int_{\Omega_p} |x|^2 |\nabla \tilde{v}_p| dz_{g_p} \right).
$$

Let us now look at the different terms of (2.30). First, by equation (2.10) and relation (2.5), we have

$$
\int_{\Omega_p} |\nabla \tilde{v}_p| dz_{g_p} = \frac{p(n-1)}{n-p} \int_{\Omega_p} v_p^{\frac{n-1}{n-p}} |\nabla v_p| dz_{g_p}
$$

$$
\leq \frac{p(n-1)}{n-p} \left( \int_{\Omega_p} |\nabla v_p|^p dz_{g_p} \right)^{\frac{1}{p}}
$$

$$
\leq K(n,1)^{-1} (1 - \alpha \mu_p^2 \lambda_p^{-1} \|v_p\|_{L_p}^2)^{\frac{1}{p}}.
$$

Since, by (2.17) and (2.26), $\|v_p\|_p = 1 + o(1)$, we get

$$
\int_{\Omega_p} |\nabla \tilde{v}_p| dz_{g_p} \leq K(n,1)^{-1} - \frac{\alpha}{2} K(n,1) \mu_p^2 + o (\mu_p^2).
$$

Independently, by Hölder’s inequalities, we have

$$
\int_{\Omega_p} |x|^2 |\nabla v_p| dz_{g_p} \leq \frac{p(n-1)}{n-p} \left( \int_{\Omega_p} |x|^{2p} |\nabla v_p|^p dz_{g_p} \right)^{\frac{1}{p}}.
$$

By equation (2.10), one gets

$$
\int_{\Omega_p} |x|^{2p} |\nabla v_p|^p dz_{g_p} \leq \int_{\Omega_p} |\nabla v_p|^{p-2} (\nabla (|x|^{2p} v_p), \nabla v_p)_{g_p} dz_{g_p}
$$

$$
+ C \int_{\Omega_p} |x|^{2p-1} |\nabla v_p|^{p-1} v_p dz_{g_p}
$$

$$
\leq C + C \left( \int_{\Omega_p} |x|^{2p} |\nabla v_p|^p dz_{g_p} \right)^{\frac{p-1}{p}} \left( \int_{\Omega_p} |x|^p v_p^p dz_{g_p} \right)^{\frac{1}{p}}
$$

where $C$ denotes some constant independent of $p$. Using (2.26) and Young’s inequalities, one deduces that

$$
\int_{\Omega_p} |x|^{2p} |\nabla v_p|^p dz_{g_p} = O(1).
$$

Now, for some $R > R_0$, we get by (2.18) that

$$
\int_{\Omega_p} |\nabla \tilde{v}_p| dz \xi x^1 x^2 d\xi = O \left( \int_{\Omega_p \setminus B(0,R)} |x|^2 |\nabla \tilde{v}_p| dz\xi \right) + \int_{\partial B(0,R_0)} x^1 x^2 d\sigma \xi + o(1).
$$
Using equation (2.10) and relation (2.26), it is easy to check that

\[
\lim_{p \to 1} \int_{\Omega_p \setminus B(0,R)} |x|^2 |\nabla \tilde{v}_p| \xi \, dx = 0
\]

so that

\[
(2.33) \quad \lim_{p \to 1} \int_{\Omega_p} |\nabla \tilde{v}_p| \xi x^i x^j \, dx = \frac{\omega_{n-1}}{n} R_0^{n+1} S_g(0).
\]

At last, since \(\nabla V_p, V_p\) as in (2.19), and \(x, V_p\) are pointwise colinear vector fields, we have

\[
Rm_g (y_p) (\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) \leq C |x|^2 |\nabla \tilde{v}_p| |\nabla (\tilde{v}_p - V_p)| \xi
\]

so that, by (2.10), (2.20) and (2.26),

\[
(2.34) \quad \lim_{p \to 1} \int_{\Omega_p} |\nabla \tilde{v}_p|^{-1} Rm_g (y_p) (\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) \, dv_{g_p} = 0.
\]

Coming back to (2.27) with (2.29)-(2.34), we obtain, after easy computations using in particular (2.16),

\[
\left( \alpha - \frac{n}{n+2} S_g(0) \right) \mu_p^2 + o(\mu_p^2) \leq 0.
\]

This gives the desired contradiction by letting \(p \to 0\). Remember here that

\[
\alpha - \frac{n}{n+2} S_g(0) = \varepsilon_0 > 0.
\]

This ends the proof of the Proposition, hence the proof of Theorem 1.

\[\square\]

3. THE COMPACT CASE - PROOF OF THEOREM 2

In order to prove Theorem 2, we let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\). We assume that \(S_g < n(n-1)K_0\). If we apply Theorem 1 with some \(x \in M\) and \(K_0\), we get some \(r_x > 0\) such that the isoperimetric comparison (with the model space form of curvature \(K_0\)) holds for sets contained in the geodesic ball of center \(x\) and radius \(r_x\). It is clear that \(r_x\) is continuous with respect to \(x\). Thus, there exists \(d > 0\) such that for any subset \(\Omega\) of \(M\) of diameter less than or equal to \(d\),

\[
(3.1) \quad |\partial \Omega|_g > |\partial B|_{g_0}
\]

where \(B\) is a ball of volume \(|\Omega|_g\) in the model space of constant curvature \(K_0\).

For \(0 < V < |M|_g\), we let

\[
h(V) = \inf \left\{ |\partial \Omega|_g, \Omega \subset M, |\Omega|_g = V \right\}.
\]

There exists some \(\Omega_V \subset M\) such that

\[
|\partial \Omega_V|_g = h(V).
\]

The boundary \(\partial \Omega_V\) of \(\Omega_V\) is a smooth hypersurface of constant mean curvature up to a compact set of Hausdorff dimension at most \(n - 8\) (see for instance [13]). Now, as a consequence of the work of Johnson and Morgan [10], we know that

\[
diam (\Omega_V) \to 0
\]

as \(V \to 0\). In fact, Johnson and Morgan proved that \(\Omega_V\) is asymptotically, as \(V \to 0\), a ball. In particular, for some \(V_0\) small enough, any \(\Omega_V\) for \(V \leq V_0\) has a diameter less than or equal to \(d\). We may then apply (3.1) to end the proof of Theorem 2.

\[\square\]
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REFERENCES


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