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SHARP LOCAL ISOPERIMETRIC INEQUALITIES INVOLVING THE SCALAR CURVATURE

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Abstract. We provide sharp local isoperimetric inequalities on Riemannian manifolds involving the scalar curvature, and thus answer a question asked by Johnson and Morgan.

1. Introduction and statement of the results

Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$ with sectional curvature $K_g \leq K_0$. A long-standing conjecture, a formulation of which can be found in [1], asserts that for any $x \in M$, there exists $r_x > 0$ such that for any Ω contained in the geodesic ball of center x and radius r_x,

$$|\partial\Omega|_g \geq |\partial B|_{g_0}$$

where $|\,.\,|_g$ (resp. $|\,.\,|_{g_0}$) denotes the volume with respect to g (resp. g_0) and B is a ball of volume $|\Omega|_g$ in the model space (M_0, g_0) of constant sectional curvature K_0. A compact version of this conjecture was proved, with an additional assumption on the Gauss-Bonnet-Chern integrand in even dimensions, in the very nice Johnson and Morgan [10]. A natural question that Johnson and Morgan [10] asked is the following: is the result still true if we assume that the scalar curvature of (M, g) is such that $S_g < n(n-1)K_0$ instead of assuming that $K_g \leq K_0$? We answer this question in the affirmative and prove the following:

Theorem 1. *Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$ and let $x \in M$. Assume that $S_g(x) < n(n-1)K_0$ for some $K_0 \in \mathbf{R}$. Then there exists $r_x > 0$ such that for any Ω contained in the geodesic ball of center x and radius r_x,*

$$|\partial\Omega|_g > |\partial B|_{g_0}$$

where B is a ball of volume $|\Omega|_g$ in the model space (M_0, g_0) of constant sectional curvature K_0.

In the compact setting, the situation that was actually considered by Johnson and Morgan [10], we have the following:

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Theorem 2. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 2$ with scalar curvature $S_g < n(n-1)K_0$. There exists $V > 0$ such that for any subset Ω of M of volume less than or equal to V,*

$$|\partial\Omega|_g > |\partial B|_{g_0}$$

where B is a ball of volume $|\Omega|_g$ in the model space (M_0, g_0) of constant sectional curvature K_0.

These results are optimal in the following sense: if we only assume that the Ricci curvature of M verifies $Ric_g \leq (n-1)K_0$, the above isoperimetric comparison fails. Indeed, for any n-manifold M which is Ricci-flat but not flat (see [3] for examples of such manifolds), one may find a ball B_r in M of radius r as small as we want which verifies

$$|\partial B_r|_g < |\partial B|_\xi$$

where B is a ball of volume $|B_r|_g$ in the Euclidean space $(\mathbf{R}^n, \xi)$. The above comparison result is also false on $S^2 \times S^2$, as noticed in [10]. The proof of Theorem 1 is based on the study of local optimal Sobolev inequalities. The proof relies on PDE techniques and a fine asymptotic analysis of solutions of quasi-elliptic equations involving the p-Laplacian. Theorem 2 is a consequence of Theorem 1 thanks to geometric measure theory. The relevance of the scalar curvature when studying the validity of sharp Sobolev inequalities was noticed first by the author in [4] and underlined by Hebey in [9].

2. Sobolev inequalities and proof of Theorem 1

Let B be a ball in the model space (M_0, g_0) of constant sectional curvature K_0. It is not difficult to check that, for balls of small volume,

$$|\partial B|_{g_0}^2 = K(n,1)^{-2}|B|_{g_0}^{2\frac{n-1}{n}} - \frac{n}{n+2}\left(n(n-1)K_0\right)|B|_{g_0}^2 + o\left(|B|_{g_0}^2\right).$$

Here, $n = dim\ M_0$ and

$$K(n,1)^{-1} = n\left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n}}.$$

Now, let (M,g) be a complete Riemannian manifold of dimension $n \geq 2$ and let $x_0 \in M$. In order to prove Theorem 1, it is clearly sufficient to prove that for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for any $\Omega \subset B_g(x_0, r_\varepsilon)$,

$$|\partial\Omega|_g^2 \geq K(n,1)^{-2}|\Omega|_g^{2\frac{n-1}{n}} - \left(\frac{n}{n+2}S_g(x_0) + \varepsilon\right)|\Omega|_g^2. \tag{2.1}$$

It is now well known that (2.1) is a consequence of the following Sobolev inequality: for any $u \in C_c^\infty(B_g(x_0, r_\varepsilon))$,

$$\|u\|_{\frac{n}{n-1}}^2 \leq K(n,1)^2\left(\|\nabla u\|_1^2\right) + \left(\frac{n}{n+2}S_g(x_0) + \varepsilon\right)\|u\|_1^2$$

where $\|\,.\,\|_p$ denotes the L^p-norm with respect to the Riemannian volume element dv_g. Indeed, $\Omega \subset B_g(x_0, r_\varepsilon)$ being given, one may find a sequence (u_i) of smooth functions with compact support in $B_g(x_0, r_\varepsilon)$ such that for any $q \geq 1$,

$$\lim_{i\to+\infty}\int_{B_g(x_0,r_\varepsilon)} |u_i|^q\, dv_g = |\Omega|_g$$

and

$$\lim_{i\to+\infty}\int_{B_g(x_0,r_\varepsilon)}|\nabla u_i|_g\,dv_g=|\partial\Omega|_g.$$

Before starting the proof of the above Sobolev inequality, we must set up some notations. For any $1\le p<n$, we let

$$K(n,p)^{-p}=\inf_{u\in C_c^\infty(\mathbf{R}^n),u\not\equiv 0}\frac{\int_{\mathbf{R}^n}|\nabla u|_\xi^p\,dv_\xi}{\left(\int_{\mathbf{R}^n}|u|^{p^\star}\,dv_\xi\right)^{\frac{p}{p^\star}}}$$

where $p^\star=\frac{np}{n-p}$ is the critical exponent for the Sobolev embeddings and ξ is the Euclidean metric. The value of $K(n,p)$ is explicitly known (see [1] or [15]) but the only point of interest to us is that

$$\lim_{p\to 1}K(n,p)=K(n,1)=\frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}}.$$

We also let, for $1\le p<n$, $H_1^p(\mathbf{R}^n)$ be the standard Sobolev space of order p, that is the completion of $C_c^\infty(\mathbf{R}^n)$ for the norm

$$\|u\|_{H_1^p}=\left(\int_{\mathbf{R}^n}|\nabla u|_\xi^p\,dv_\xi\right)^{\frac{1}{p}}.$$

At last, we let $BV(\mathbf{R}^n)$ be the space of functions with bounded variations in $\mathbf{R}^n$, defined as the completion of $C_c^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|u\|_{BV}=\sup\left\{-\int_{\mathbf{R}^n}u\,div\,(X)\;dv_\xi,\;\|X\|_{L^\infty(\mathbf{R}^n)}\le 1,\;div\,(X)\in L^n(\mathbf{R}^n)\right\}$$

where $div\,(X)=\partial^i X_i$. Basic facts about $BV(\mathbf{R}^n)$ can be found in [7] or [16].

As already mentioned, Theorem 1 reduces to the following proposition:

Proposition. *Let (M,g) be a complete Riemannian manifold of dimension $n\ge 2$ and let $x_0\in M$. For any $\varepsilon>0$, there exists $r_\varepsilon>0$ such that for any u in $C_c^\infty(B_g(x_0,r_\varepsilon))$,*

$$\|u\|^2_{\frac{n}{n-1}}\le K(n,1)^2\left(\|\nabla u\|_1^2+\alpha_\varepsilon\|u\|_1^2\right)$$

where $\alpha_\varepsilon=\frac{n}{n+2}S_g(x_0)+\varepsilon$.

We prove the Proposition in what follows.

Proof of the Proposition. Clearly, we may assume, without loss of generality, that $M=\mathbf{R}^n$ and that $x_0=0$. We let, for any $r>0$, any $p>1$ and any $\varepsilon>0$,

$$\lambda_{p,r}=\inf_{u\in C_c^\infty(B_g(0,r)),\;u\not\equiv 0}\frac{\left(\int_{B_g(0,r)}|\nabla u|_g^p\,dv_g\right)^{\frac{2}{p}}+\alpha_\varepsilon\left(\int_{B_g(0,r)}|u|^p\,dv_g\right)^{\frac{2}{p}}}{\left(\int_{B_g(0,r)}|u|^{p^\star}\,dv_g\right)^{\frac{2}{p^\star}}}.$$

We proceed by contradiction. We assume that there exists $\varepsilon_0>0$ such that for any $r>0$,

$$\lambda_{1,r}<K(n,1)^{-2}.$$

Then, since $\limsup_{p\to 1} \lambda_{p,r} \le \lambda_{1,r}$, one easily gets that for any $r > 0$, there exists $p_r > 1$ such that

$$\lambda_{p_r,r} < K(n,1)^{-2} \left(\frac{n-p_r}{p_r(n-1)} \right)^2, \lambda_{p_r,r} < K(n,p_r)^{-2}. \tag{2.2}$$

We may assume that $r \to 0$ and we may choose p_r decreasing when r is decreasing. Then we get a sequence $p > 1$ going to 1 and a sequence $r_p > 0$ going to 0 as p goes to 1 which verify (2.2). It is by now classical that the second inequality in (2.2) ensures the existence of a minimizer u_p which satisfies the following:

$$\begin{aligned} &C_p \Delta_p u_p + \alpha \|u_p\|_p^{2-p} u_p^{p-1} = \lambda_p u_p^{p^\star - 1} \quad \text{in } B_g(0, r_p), \qquad (2.3)\\ &u_p \in C^{1,\eta}(B_g(0,r_p)) \quad \text{for some } \eta > 0,\\ &u_p > 0 \quad \text{in } B_g(0,r_p), \ u_p = 0 \quad \textit{on } \partial B_g(0,r_p), \end{aligned}$$

$$\int_{B_g(0,r_p)} u_p^{p^\star}\, dv_g = 1, \tag{2.4}$$

$$\lambda_p < K(n,p)^{-2}, \ \lambda_p < K(n,1)^{-2} \left(\frac{n-p}{p(n-1)} \right)^2, \tag{2.5}$$

$$C_p = \left(\int_{B_g(0,r_p)} |\nabla u_p|_g^p \, dv_g \right)^{\frac{2-p}{p}}. \tag{2.6}$$

In the above equations, Δ_p is the p-laplacian with respect to g, that is $\Delta_p u = -div_g \left(|\nabla u|_g^{p-2} \nabla u \right)$, and we have set

$$\alpha = \frac{n}{n+2} S_g(0) + \varepsilon_0.$$

Now the aim is to study this sequence (u_p) as $p \to 1$. We let x_p be a point in $B_g(0,r_p)$ where u_p achieves its maximum and we also let

$$u_p(x_p) = \mu_p^{1-\frac{n}{p}}.$$

We have

$$1 = \int_{B_g(0,r_p)} u_p^{p^\star}\, dv_g \le Vol_g(B_g(0,r_p))\, \mu_p^{-n}$$

and since r_p goes to 0, μ_p goes to 0 as p goes to 1. In the same way, thanks to Hölder's inequalities, we get

$$\lim_{p\to 1} \int_{B_g(0,r_p)} u_p^p \, dv_g = 0. \tag{2.7}$$

Step 1. We first claim that

$$\lim_{p\to 1} \lambda_p = K(n,1)^{-2} \tag{2.8}$$

and that

$$\lim_{p\to 1} \int_{B_g(0,r_p)} |\nabla u_p|_g^p \, dv_g = K(n,1)^{-1}. \tag{2.9}$$

Indeed (see for instance [8] for an exposition in book form) for any $\varepsilon > 0$ there exists $B_\varepsilon > 0$ such that for any $p > 1$,

$$\begin{aligned}\left(\int_{B_g(0,r_p)} u_p^{p^\star} dv_g\right)^{2\frac{n-1}{n}} &\leq (K(n,1)+\varepsilon)^2 \left(\int_{B_g(0,r_p)} |\nabla \left(u_p^{\frac{p(n-1)}{n-p}}\right)|_g \, dv_g\right)^2 \\ &\quad + B_\varepsilon \left(\int_{B_g(0,r_p)} u_p^{\frac{p(n-1)}{n-p}} \, dv_g\right)^2\end{aligned}$$

which gives with (2.3), (2.4) and Hölder's inequalities

$$1 \leq (K(n,1)+\varepsilon)^2 \left(\frac{p(n-1)}{n-p}\right)^2 \left(\lambda_p - \alpha \|u_p\|_p^2\right) + B_\varepsilon \|u_p\|_p^2.$$

This leads with (2.7) to

$$1 \leq \left(1 + \varepsilon K(n,1)^{-1}\right)^2 \liminf_{p\to 1} \left(\lambda_p K(n,1)^2\right).$$

Since it is valid for any $\varepsilon > 0$, we obtain $\liminf_{p\to 1} \lambda_p \geq K(n,1)^{-2}$. By (2.5), we get that (2.8) is proved. Then (2.9) is an obvious consequence of (2.3), (2.4), (2.7) and (2.8).

Step 2. We let $\Omega_p = \mu_p^{-1} \exp_{x_p}^{-1}\left(B_g(0,r_p)\right) \subset \mathbf{R}^n$ and we set

$$g_p(x) = \exp_{x_p}^\star g\left(\mu_p x\right) \quad \text{for } x \in \Omega_p$$

and

$$v_p(x) = \mu_p^{\frac{n}{p}-1} u_p \left(\exp_{x_p}\left(\mu_p x\right)\right) \quad \text{for } x \in \Omega_p,\ v_p(x) = 0 \quad \text{for } x \in \mathbf{R}^n \backslash \Omega_p.$$

Clearly we have

$$C_p \Delta_{p,g_p} v_p + \alpha \mu_p^2 \|v_p\|_p^{2-p} v_p^{p-1} = \lambda_p v_p^{p^\star - 1} \quad \text{in } \Omega_p \tag{2.10}$$

with $v_p = 0$ on $\partial\Omega_p$ and

$$\int_{\Omega_p} v_p^{p^\star} \, dv_{g_p} = 1. \tag{2.11}$$

We also let

$$\tilde{v}_p(x) = v_p(x)^{\frac{p(n-1)}{n-p}}. \tag{2.12}$$

By the Cartan expansion of a metric in the exponential chart, there exists $C > 1$ such that

$$\begin{aligned} dv_{g_p} &\geq \left(1 - \frac{1}{C}\mu_p^2\right) dv_\xi, \\ |\nabla \tilde{v}_p|_{g_p} dv_{g_p} &\leq \left(1 + C\mu_p^2\right) |\nabla \tilde{v}_p|_\xi dv_\xi \end{aligned}$$

where ξ is the Euclidean metric. This easily leads with (2.9), (2.11) and Hölder's inequalities to

$$\lim_{p\to 1} \frac{\int_{\mathbf{R}^n} |\nabla \tilde{v}_p|_\xi^p \, dv_\xi}{\left(\int_{\mathbf{R}^n} v_p^{p^\star} \, dv_\xi\right)^{\frac{n-1}{n}}} = K(n,1)^{-1}. \tag{2.13}$$

Remember here that $r_p \to 0$ as $p \to 1$. Since $(\tilde{v}_p)$ is bounded in $H_1^1(\mathbf{R}^n)$, there exists $v_0 \in BV(\mathbf{R}^n)$ such that

$$\lim_{p\to 1} \tilde{v}_p = v_0 \quad \text{weakly in } BV(\mathbf{R}^n).$$

If we apply the concentration-compactness principle of P.L. Lions ([11], [12], see also [14] for an exposition in book form) to $|v_p|^{p^\star} dv_\xi$, four situations may occur: compactness, concentration, dichotomy or vanishing. Dichotomy is classically forbidden by (2.13). Concentration cannot happen since $\sup_{\Omega_p} v_p = v_p(0) = 1$. As for vanishing, since v_p is bounded in L^∞, by applying Moser's iterative scheme to (2.10), one gets the existence of some $C > 0$ such that for any $p > 1$,

$$1 = \sup_{\Omega_p \cap B_{g_p}(0,1/2)} v_p \le C \left(\int_{\Omega_p \cap B_{g_p}(0,1)} v_p^{p^\star} \, dv_\xi \right)^{\frac{1}{p^\star}}.$$

Thus vanishing cannot happen. Compactness together with (2.13) just gives

$$\lim_{p\to 1} \tilde{v}_p = v_0 \quad \text{strongly in } BV(\mathbf{R}^n). \tag{2.14}$$

Then v_0 is a minimizer for the H_1^1 Euclidean Sobolev inequality which verifies $\int_{\mathbf{R}^n} v_0^{\frac{n}{n-1}} \, dv_\xi = 1$. Thus there exists $y_0 \in \mathbf{R}^n$, $\lambda_0 > 0$ and $R_0 > 0$ such that

$$v_0 = \lambda_0 \mathbf{1}_{B(y_0,R_0)} \tag{2.15}$$

where $\mathbf{1}_{B(y_0,R_0)}$ denotes the characteristic function of the Euclidean ball $B(y_0, R_0)$. Moreover, since $v_p \le 1$ in Ω_p, we obtain with (2.14) that $v_p \to v_0$ in any $L^q(\mathbf{R}^n)$, $q \ge \frac{n}{n-1}$. One can deduce from this that $\lambda_0 = 1$. At last, we have:

$$Vol_\xi (B(y_0, R_0)) = \frac{\omega_{n-1}}{n} R_0^n = 1. \tag{2.16}$$

Up to changing x_p into $exp_{x_p}(\mu_p y_0)$ in the definition of v_p, Ω_p and g_p, we may assume that $y_0 = 0$. We have thus obtained that

$$\lim_{p\to 1} \tilde{v}_p = \mathbf{1}_{B(0,R_0)} \quad \text{strongly in } BV(\mathbf{R}^n).$$

This means in particular that

$$\lim_{p\to 1} \tilde{v}_p = \mathbf{1}_{B(0,R_0)} \quad \text{strongly in } L^{\frac{n}{n-1}}(\mathbf{R}^n) \tag{2.17}$$

and that for any $\varphi \in C_c^\infty(\mathbf{R}^n)$,

$$\lim_{p\to 1} \int_{\mathbf{R}^n} |\nabla \tilde{v}_p|_\xi \varphi \, dv_\xi = \int_{\partial B(0,R_0)} \varphi \, d\sigma_\xi. \tag{2.18}$$

If we set

$$V_p(x) = \left(1 + \left(\frac{|x|}{R_0} \right)^{\frac{p}{p-1}} \right)^{1-n}, \quad x \in \mathbf{R}^n, \tag{2.19}$$

a simple application of the concentration-compactness principle, using what we just proved, gives

$$\lim_{p\to 1} \int_{\mathbf{R}^n} |\nabla (\tilde{v}_p - V_p)|_\xi \, dv_\xi = 0. \tag{2.20}$$

Applying Moser's iterative scheme to (2.10) with the help of (2.17), we also get that for any $R > R_0$,

$$\lim_{p\to 1} \sup_{\Omega_p\setminus B(0,R)} v_p = 0. \tag{2.21}$$

Step 3. The aim is to transform the $L^{\frac{n}{n-1}}$-estimate (2.17) into a pointwise estimate. We follow here [6] (see also [5]). We let

$$w_p(z) = |z|^{\frac{n}{p}-1} v_p(z)$$

and we let $z_p \in \Omega_p$ be a point where w_p achieves its maximum. Let us assume by contradiction that

$$\lim_{p\to 1} w_p(z_p) = +\infty.$$

We set

$$\nu_p^{1-\frac{n}{p}} = v_p\left(z_p\right)$$

so that

$$\lim_{p\to 1} \frac{|z_p|}{\nu_p} = +\infty. \tag{2.22}$$

Independently, since $v_p \le 1$ in Ω_p,

$$\lim_{p\to 1} |z_p| = +\infty. \tag{2.23}$$

Thanks to (2.22) and (2.23), one proves then that $(\nu_p^{\frac{n}{p}-1} v_p(exp_{z_p}(\nu_p x)))$ is bounded in $L^\infty(B(0,1))$. This allows us to apply Moser's iterative scheme to the equation verified by $\left(\nu_p^{\frac{n}{p}-1} v_p\left(exp_{z_p}\left(\nu_p x\right)\right)\right)$ and to get the existence of some $C > 0$ such that

$$\liminf_{p\to 1} \int_{B_{g_p}(z_p,\nu_p)\cap\Omega_p} v_p^{p^\star}\, dv_g > 0.$$

The contradiction then easily follows from (2.17), (2.22) and (2.23). Thus we have the existence of some $C > 0$ such that for any $p > 1$, any $z \in \Omega_p$,

$$|z|^{\frac{n}{p}-1} v_p(z) \le C. \tag{2.24}$$

In the same way, using (2.24), one proves thanks to (2.21) that for any $R > R_0$,

$$\lim_{p\to 1} \sup_{\Omega_p\setminus B_{g_p}(0,R)} |z|^{\frac{n}{p}-1} v_p(z) = 0. \tag{2.25}$$

We refer the reader to [6] for details on such claims.
Step 4. We let L_p be the following operator:

$$L_p u = C_p \Delta_{p,g_p} u + \alpha\mu_p^2 \|v_p\|_p^{2-p} u^{p-1} - \lambda_p v_p^{p^\star - p} u^{p-1}.$$

We fix $0 < \nu < n-1$ and we set

$$G_p(x) = \theta_p |x|^{-\frac{n-p-\nu}{p-1}}$$

where θ_p is some positive constant to be fixed later. Easy computations lead to

$$\begin{aligned}|x|^{n-\nu}\frac{L_pG_p(x)}{G_p(x)^{p-1}} &\geq C_p\nu\left(\frac{n-p-\nu}{p-1}\right)^{p-1}-C\mu_p^2|x|^2\\ &\quad+\alpha\mu_p^2\|v_p\|_p^{2-p}|x|^p-\lambda_p|x|^pv_p^{p^\star-p}\end{aligned}$$

in $\Omega_p\backslash\{0\}$. Here C denotes some constant independent of p. Thanks to (2.7), (2.8), (2.9), (2.25) and the fact that $r_p\to 0$ as $p\to 1$, one gets that for any $R>R_0$,

$$L_pG_p(x)\geq 0 \quad \text{in } \Omega_p\backslash B_{g_p}(0,R)$$

for p small enough. On the other hand,

$$L_pv_p=0 \quad \text{in } \Omega_p.$$

At last, it is not difficult to check with (2.21) that

$$v_p\leq\theta_pG_p \quad \text{on } \partial B_{g_p}(0,R)$$

if we take $\theta_p=R^{\frac{n-p-\nu}{p-1}}$. Now we may apply the maximum principle as stated for instance in [2] (lemma 3.4) to get, for p small enough,

$$v_p(y)\leq\left(\frac{R}{|y|}\right)^{\frac{n-p-\nu}{p-1}} \quad \text{in } \Omega_p\backslash B_{g_p}(0,R)\,.$$

Since this inequality obviously holds on $B_{g_p}(0,R)$, we have finally obtained the following: for any $\nu>0$ and any $R>R_0$, there exists $C(R,\nu)>0$ such that for any $p>1$ and any $y\in\Omega_p$,

$$\left(\frac{|y|}{R}\right)^{\frac{n-p-\nu}{p-1}}v_p(y)\leq C(R,\nu)\,. \tag{2.26}$$

Step 5. We conclude the proof of the Proposition. We apply the H_1^1 Euclidean Sobolev inequality to $\tilde v_p$:

$$\left(\int_{\Omega_p}\tilde v_p^{\frac{n}{n-1}}\,dv_\xi\right)^{\frac{n-1}{n}}\leq K(n,1)\int_{\Omega_p}|\nabla\tilde v_p|_\xi\,dv_\xi. \tag{2.27}$$

By the Cartan expansion of g_p around 0, we have

$$dv_\xi=\left(1+\frac{\mu_p^2}{6}Ric_g(y_p)_{ij}x^ix^j+o\left(\mu_p^2|x|^2\right)\right)dv_{g_p} \tag{2.28}$$

where Ric_g denotes the Ricci curvature of g in the exp_{y_p}-map. Thus, by (2.11),

$$\int_{\Omega_p}\tilde v_p^{\frac{n}{n-1}}\,dv_\xi=1+\frac{\mu_p^2}{6}Ric_g(y_p)_{ij}\int_{\Omega_p}x^ix^jv_p^{p^\star}\,dv_{g_p}+o\left(\mu_p^2\int_{\Omega_p}|x|^2v_p^{p^\star}\,dv_{g_p}\right).$$

Using (2.17) and (2.26), one gets

$$\int_{\Omega_p}\tilde v_p^{\frac{n}{n-1}}\,dv_\xi=1+\frac{S_g(0)}{6n(n+2)}\omega_{n-1}R_0^{n+2}\mu_p^2+o\left(\mu_p^2\right). \tag{2.29}$$

By the Cartan expansion of g_p around 0, since $r_p\to 0$ as $p\to 1$, we also have

$$|\nabla\tilde v_p|_\xi^p=|\nabla\tilde v_p|_{g_p}^p\left[1-\frac{\mu_p^2}{6}|\nabla\tilde v_p|_{g_p}^{-2}Rm_g(y_p)(\nabla\tilde v_p,x,x,\nabla\tilde v_p)+o\left(\mu_p^2|x|^2\right)\right]$$

where Rm_g denotes the Riemann curvature of g in the exp_{y_p}-map. Then, using (2.28), we get

$$\begin{aligned}\int_{\Omega_p} |\nabla \tilde{v}_p|_\xi \, dv_\xi &= \int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p} + \frac{\mu_p^2}{6} Ric_g\left(y_p\right)_{ij} \int_{\Omega_p} x^i x^j |\nabla \tilde{v}_p|_\xi \, dv_\xi \\ &\quad -\frac{\mu_p^2}{6} \int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p}^{-1} Rm_g\left(y_p\right)\left(\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p\right) \, dv_{g_p} \\ &\quad + o\left(\mu_p^2 \int_{\Omega_p} |x|^2 |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p}\right).\end{aligned} \tag{2.30}$$

Let us now look at the different terms of (2.30). First, by equation (2.10) and relation (2.5), we have

$$\begin{aligned}\int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p} &= \frac{p(n-1)}{n-p} \int_{\Omega_p} v_p^{\frac{n(p-1)}{n-p}} |\nabla v_p|_{g_p} \, dv_{g_p} \\ &\leq \frac{p(n-1)}{n-p} \left(\int_{\Omega_p} |\nabla v_p|_{g_p}^p \, dv_{g_p}\right)^{\frac{1}{p}} \\ &\leq K(n,1)^{-1}\left(1 - \alpha \mu_p^2 \lambda_p^{-1} \|v_p\|_p^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since, by (2.17) and (2.26), $\|v_p\|_p = 1 + o\left(1\right)$, we get

$$\int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p} \leq K(n,1)^{-1} - \frac{\alpha}{2} K(n,1) \mu_p^2 + o\left(\mu_p^2\right). \tag{2.31}$$

Independently, by Hölder's inequalities, we have

$$\int_{\Omega_p} |x|^2 |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p} \leq \frac{p(n-1)}{n-p} \left(\int_{\Omega_p} |x|^{2p} |\nabla v_p|_{g_p}^p \, dv_{g_p}\right)^{\frac{1}{p}}.$$

By equation (2.10), one gets

$$\begin{aligned}\int_{\Omega_p} |x|^{2p} |\nabla v_p|_{g_p}^p \, dv_{g_p} &\leq \int_{\Omega_p} |\nabla v_p|_{g_p}^{p-2} \left(\nabla\left(|x|^{2p} v_p\right), \nabla v_p\right)_{g_p} \, dv_{g_p} \\ &\quad + C \int_{\Omega_p} |x|^{2p-1} |\nabla v_p|_{g_p}^{p-1} v_p \, dv_{g_p} \\ &\leq C + C \left(\int_{\Omega_p} |x|^{2p} |\nabla v_p|_{g_p}^p \, dv_{g_p}\right)^{\frac{p-1}{p}} \left(\int_{\Omega_p} |x|^p v_p^p \, dv_{g_p}\right)^{\frac{1}{p}}\end{aligned}$$

where C denotes some constant independent of p. Using (2.26) and Young's inequalities, one deduces that

$$\int_{\Omega_p} |x|^{2p} |\nabla v_p|_{g_p}^p \, dv_{g_p} = O\left(1\right). \tag{2.32}$$

Now, for some $R > R_0$, we get by (2.18) that

$$\int_{\Omega_p} |\nabla \tilde{v}_p|_\xi x^i x^j \, dv_\xi = O\left(\int_{\Omega_p \setminus B(0,R)} |x|^2 |\nabla \tilde{v}_p|_\xi \, dv_\xi\right) + \int_{\partial B(0,R_0)} x^i x^j d\sigma_\xi + o\left(1\right).$$

Using equation (2.10) and relation (2.26), it is easy to check that

$$\lim_{p\to 1}\int_{\Omega_p\setminus B(0,R)} |x|^2|\nabla\tilde{v}_p|_\xi \, dv_\xi = 0$$

so that

$$\lim_{p\to 1} Ric_g\left(y_p\right)_{ij}\int_{\Omega_p} |\nabla\tilde{v}_p|_\xi x^i x^j \, dv_\xi = \frac{\omega_{n-1}}{n} R_0^{n+1} S_g(0). \tag{2.33}$$

At last, since ∇V_p, V_p as in (2.19), and x are pointwise colinear vector fields, we have

$$Rm_g\left(y_p\right)\left(\nabla\tilde{v}_p, x, x, \nabla\tilde{v}_p\right) \le C|x|^2|\nabla\tilde{v}_p|_\xi |\nabla\left(\tilde{v}_p - V_p\right)|_\xi$$

so that, by (2.10), (2.20) and (2.26),

$$\lim_{p\to 1}\int_{\Omega_p} |\nabla\tilde{v}_p|_{g_p}^{-1} Rm_g\left(y_p\right)\left(\nabla\tilde{v}_p, x, x, \nabla\tilde{v}_p\right) \, dv_{g_p} = 0. \tag{2.34}$$

Coming back to (2.27) with (2.29)-(2.34), we obtain, after easy computations using in particular (2.16),

$$\left(\alpha - \frac{n}{n+2} S_g(0)\right)\mu_p^2 + o\left(\mu_p^2\right) \le 0.$$

This gives the desired contradiction by letting p go to 0. Remember here that $\alpha - \frac{n}{n+2}S_g(0) = \varepsilon_0 > 0$. This ends the proof of the Proposition, hence the proof of Theorem 1. ☐

3. The compact case - Proof of Theorem 2

In order to prove Theorem 2, we let (M, g) be a compact Riemannian manifold of dimension $n \ge 2$. We assume that $S_g < n(n-1)K_0$. If we apply Theorem 1 with some x in M and K_0, we get some $r_x > 0$ such that the isoperimetric comparison (with the model space form of curvature K_0) holds for sets contained in the geodesic ball of center x and radius r_x. It is clear that r_x is continuous with respect to x. Thus, there exists $d > 0$ such that for any subset Ω of M of diameter less than or equal to d,

$$|\partial\Omega|_g > |\partial B|_{g_0} \tag{3.1}$$

where B is a ball of volume $|\Omega|_g$ in the model space of constant curvature K_0. For $0 < V < |M|_g$, we let

$$h(V) = \inf\left\{|\partial\Omega|_g,\ \Omega \subset M,\ |\Omega|_g = V\right\}.$$

There exists some $\Omega_V \subset M$ such that

$$|\partial\Omega_V|_g = h(V).$$

The boundary $\partial\Omega_V$ of Ω_V is a smooth hypersurface of constant mean curvature up to a compact set of Hausdorff dimension at most $n-8$ (see for instance [13]). Now, as a consequence of the work of Johnson and Morgan [10], we know that

$$diam\left(\Omega_V\right) \to 0$$

as $V \to 0$. In fact, Johnson and Morgan proved that Ω_V is asymptotically, as $V \to 0$, a ball. In particular, for some V_0 small enough, any Ω_V for $V \le V_0$ has a diameter less than or equal to d. We may then apply (3.1) to end the proof of Theorem 2. ☐

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