PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 8, Pages 2343–2349
S 0002-9939(02)06426-2
Article electronically published on March 8, 2002

WIGNER'S THEOREM IN HILBERT C^*-MODULES OVER C^*-ALGEBRAS OF COMPACT OPERATORS

DAMIR BAKIĆ AND BORIS GULJAŠ

(Communicated by David R. Larson)

ABSTRACT. Let W be a Hilbert C^*-module over the C^*-algebra $\mathcal{A} \neq \boldsymbol{C}$ of all compact operators on a Hilbert space. It is proved that any function $T : W \to W$ which preserves the absolute value of the $\mathcal{A}$-valued inner product is of the form $Tv = \varphi(v)Uv$, $v \in W$, where φ is a phase function and U is an $\mathcal{A}$-linear isometry. The result generalizes Molnár's extension of Wigner's classical unitary-antiunitary theorem.

1. INTRODUCTION

Wigner's unitary-antiunitary theorem states that each bijective function $T : H \to H$ acting on a complex Hilbert space $(H, (\cdot, \cdot))$ which satisfies $|(Tx, Ty)| = |(x, y)|$, $x, y \in H$, must be of the form $Tx = \varphi(x)Ux$, $x \in H$, where $U : H \to H$ is either a unitary or antiunitary operator and $\varphi : H \to \boldsymbol{C}$ is a phase function (i.e. its values are of modulus 1).

Wigner's theorem was first published in 1931 ([14]), while in the 1960's several authors began working on the proof in order to make the argument rigorous (see [11] and references therein).

In [8] and [9] a new, algebraic approach to this theorem is used in proving a natural generalization of Wigner's theorem for Hilbert C^*-modules over matrix algebras M_d. It should also be observed that in the statement of the Theorem in [9] even the surjectivity of the transformation in question is not assumed.

In the present note we give a further generalization of Wigner's theorem to Hilbert C^*-modules over C^*-algebras of compact operators.

A (left) Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module W equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ which is linear over $\mathcal{A}$ in the first variable and conjugate linear in the second, such that W is a Banach space with the norm $\|w\| = \|\langle w, w \rangle\|^{1/2}$. Hilbert C^*-modules are introduced and first investigated in [6], [10] and [12]. A Hilbert C^*-module is said to be full if the two sided ideal generated by all products $\langle v, w \rangle$, $v, w \in W$, is dense in $\mathcal{A}$. The basic theory of Hilbert C^*-modules can be found in [7] and [13].

We denote by $\boldsymbol{B}_a(W)$ the C^*-algebra of all adjointable operators on W (i.e. of all maps $A : W \to W$ such that there exists $A^* : W \to W$ with the property $\langle Av, w \rangle = \langle v, A^*w \rangle$, $\forall v, w \in W$). It is well known that each adjointable operator is

Received by the editors October 2, 2000 and, in revised form, March 12, 2001.
1991 *Mathematics Subject Classification.* Primary 46C05, 46C50; Secondary 39B42, 47J05.
Key words and phrases. C^*-algebra, Hilbert C^*-module, compact operator, Wigner's theorem.

necessarily bounded and $\mathcal{A}$-linear in the sense $A(av) = aAv$, $\forall a \in \mathcal{A}, \forall v \in W$. In general, bounded $\mathcal{A}$-linear operators may fail to possess an adjoint. However, if W is a Hilbert C^*-module over the C^*-algebra $\mathcal{A}$ of all compact operators on a Hilbert space, then it is known that each bounded $\mathcal{A}$-linear operator on W is necessarily adjointable (see for example [3], Remark 5).

Let us also note that an operator $U \in \boldsymbol{B}_a(W)$ on an arbitrary Hilbert C^*-module W is an isometry if and only if $U^*U = I$ (with I denoting the identity operator on W). Indeed, $U^*U = I$ obviously implies that U is an isometry, while the converse can be proved repeating the nice argument from Theorem 3.5 in [7].

Before stating the main result let us fix the rest of our notation.

Throughout, $|a|$ denotes the unique positive square root of a^*a. We denote by $\boldsymbol{K}(H)$ the C^*-algebra of all compact operators on a Hilbert space H. The field of complex numbers is denoted by $\boldsymbol{C}$.

Theorem 1. *Let W be a Hilbert C^*-module over the C^*-algebra $\mathcal{A}$ of all compact operators on a Hilbert space H with $\dim H > 1$. Let $T : W \to W$ be a function satisfying*

$$|\langle Tv, Tw\rangle| = |\langle v, w\rangle|,\ \forall v, w \in W. \tag{1}$$

Then there exist an $\mathcal{A}$-linear isometry $U \in \boldsymbol{B}_a(W)$ and a phase function $\varphi : W \to \boldsymbol{C}$ such that

$$Tv = \varphi(v)Uv,\ \forall v \in W.$$

Let us observe that the corresponding result for the case $\dim H = 1$ is in fact the classical Wigner's theorem. In this situation one cannot exclude the antilinear alternative from the assertion of the theorem. On the other hand, if $\dim H > 1$, then, as noted in [9], the nonappearance of antilinear isometries is the consequence of the noncommutativity of the underlying C^*-algebra.

We also note that our theorem includes the Theorem from [9] since the C^*-algebra of all compact operators $\boldsymbol{K}(H)$ with $\dim H = d$ is in fact the algebra of all $d \times d$ complex matrices M_d. Thus our theorem generalizes the Theorem in [9] from a finite to an arbitrary dimension of the underlying Hilbert space.

The proof of Theorem 1 basically depends on the presence of an orthonormal basis in each Hilbert C^*-module W over the C^*-algebra of compact operators $\boldsymbol{K}(H)$.

Recall from [3] that a vector $v \in W$ is said to be a basic vector if $e = \langle v, v\rangle$ is a minimal (i.e. one-dimensional) projection in $\boldsymbol{K}(H)$. A system (v_λ), $\lambda \in \Lambda$, is an orthonormal system if each v_λ is a basic vector and $\langle v_\lambda, v_\mu\rangle = 0$ for all $\lambda \neq \mu$. An orthonormal system (v_λ) in W is said to be an orthonormal basis for W if it generates a dense submodule of W.

Originally, the concept is introduced in [4] for Hilbert H^*-modules. It is proved in [3], Theorem 2, that each Hilbert $\boldsymbol{K}(H)$-module possesses an orthonormal basis. We also note that by Theorem 1 from [3] the following properties of an orthonormal system (v_λ), $\lambda \in \Lambda$, are mutually equivalent:

(a) (v_λ) is an orthonormal basis for W.
(b) $w = \sum_{\lambda\in\Lambda}\langle w, v_\lambda\rangle v_\lambda,\ \ \forall w \in W$.
(c) $\langle w, w\rangle = \sum_{\lambda\in\Lambda}\langle w, v_\lambda\rangle\langle v_\lambda, w\rangle,\ \ \forall w \in W$.
(d) $\langle w_1, w_2\rangle = \sum_{\lambda\in\Lambda}\langle w_1, v_\lambda\rangle\langle v_\lambda, w_2\rangle,\ \ \forall w_1, w_2 \in W$.

Notice that each orthonormal basis is in fact a standard normalized tight frame in the sense of [5] (cf. Definition 2.1 and Proposition 2.2 in [5]). However, we shall use the name orthonormal basis to emphasize the fact that its members v_λ are supported by minimal projections $\langle v_\lambda, v_\lambda\rangle$.

2. Proof

In order to prove Theorem 1 we first state an independent lemma. Although our lemma is in fact proved in [8], p. 363, we include the sketch of a more direct proof.

Lemma 1. *Let W be a Hilbert C^*-module over the C^*-algebra $\mathcal{A} = \boldsymbol{K}(H)$ of all compact operators on a Hilbert space H. Suppose that $x, y \in W$ satisfy $\langle x, v\rangle\langle v, x\rangle = \langle y, v\rangle\langle v, y\rangle$, $\forall v \in W$. Then there exists a complex number λ such that $|\lambda| = 1$ and $y = \lambda x$.*

Proof. First observe that the corresponding statement is obviously true for Hilbert spaces.

Let (u_j) be an orthonormal basis for W. By Theorem 1 from [3],

$$\langle x, x\rangle = \sum_j \langle x, u_j\rangle\langle u_j, x\rangle = \sum_j \langle y, u_j\rangle\langle u_j, y\rangle = \langle y, y\rangle.$$

Now $\langle x, x\rangle$, being a positive compact operator on H, can be written in the form

$$\langle x, x\rangle = \langle y, y\rangle = \sum_i \gamma_i e_i \tag{2}$$

where $\gamma_i > 0$ and (e_i) is a family of pairwise orthogonal minimal projections in $\boldsymbol{K}(H)$. This in turn implies

$$x = \sum_i e_i x,\ \ y = \sum_i e_i y. \tag{3}$$

Let e be any minimal orthogonal projection in $\boldsymbol{K}(H)$. By assumption we have

$$\langle ex, ev\rangle\langle ev, ex\rangle = \langle ey, ev\rangle\langle ev, ey\rangle,\ \forall v \in W. \tag{4}$$

Recall from [3] (see also [2]) that $W_e := eW$ is a Hilbert space contained in W with a scalar product $(\cdot,\cdot)$ such that $\langle\cdot,\cdot\rangle = (\cdot,\cdot)e$. The above equality gives

$$(ex, ev)(ev, ex) = (ey, ev)(ev, ey),\ \forall v \in W.$$

By the correspondent assertion of Lemma 1 for Hilbert spaces one gets $ey = \lambda_e ex$ for some $\lambda_e \in \boldsymbol{C}$ such that $|\lambda_e| = 1$ which, together with (3), implies

$$x = \sum_i e_i x,\ \ y = \sum_i \lambda_i e_i x,\ \ |\lambda_i| = 1,\ \forall i. \tag{5}$$

It remains to show $\lambda_i = \lambda_j$, $\forall i, j$. To do this, it suffices to substitute $v = ax$ in the hypothesis of the lemma, where $a = e_i + e_j + h + h^*$ and h is the partial isometry such that $h^*h = e_i, hh^* = e_j$. We omit the details. □

Notice that W, being nontrivial (as we tacitly assume), must be a full $\boldsymbol{K}(H)$-module since $\boldsymbol{K}(H)$ has no nontrivial closed two sided ideals. This implies that the subspace W_e used in the above proof cannot be trivial. Indeed, $ev = 0$, $\forall v \in W$ would imply $e\langle v, v\rangle = 0$, $\forall v \in W$ and finally $e = 0$ since W is full.

Proof of Theorem 1. We assume $\dim H = \infty$ since the case $\dim H < \infty$ is already proved in [9]. Let $e \in \mathcal{A} = \boldsymbol{K}(H)$ be an arbitrary minimal projection. Notice that

$e\boldsymbol{K}(H)e = \boldsymbol{C}e$. Consider again $W_e = eW$ which is a Hilbert space with the scalar product $(x, y) = \mathrm{tr}(\langle x, y\rangle)$, $x, y \in W_e$. Now the equality ([2], Lemma 2)

$$W_e = \{x \in W : \langle x, x\rangle \in \boldsymbol{C}e\} \tag{6}$$

and the assumed property of T imply that W_e is invariant for T. Indeed, take any $x \in W_e$ and observe $\langle x, x\rangle = \alpha e$ for some $\alpha \geq 0$. Since T preserves inner squares (this follows immediately from (1)), we have $\langle Tx, Tx\rangle = \langle x, x\rangle = \alpha e$ and, by (6), $Tx \in W_e$.

Thus we may consider the induced function $T_e = T|W_e : W_e \to W_e$. Moreover, since $\langle x, y\rangle = (x, y)e$, $\forall x, y \in W_e$, we conclude $|(Tx, Ty)| = |(x, y)|$, $\forall x, y \in W_e$. Now we can apply the classical Wigner's theorem on the Hilbert space W_e: there exist an isometry (either linear or antilinear) U_e on W_e and a phase function $\varphi_e : W_e \to \boldsymbol{C}$ such that

$$Tx = \varphi_e(x)U_e x,\ \forall x \in W_e. \tag{7}$$

We claim that the isometry U_e must be linear. To see this, we shall "add one more dimension" and apply the same trick. Let f be a minimal projection in $\boldsymbol{K}(H)$ orthogonal to e and let $b \in \boldsymbol{K}(H)$ be the partial isometry such that $b^*b = e, bb^* = f$. Let us denote by $M_2 \subseteq \boldsymbol{K}(H)$ the algebra spanned by matrix units e, f, b, b^*. Now define

$$X = \{x \in W : \langle x, x\rangle \in M_2\}. \tag{8}$$

Obviously, X is closed and, by (6), contains W_e. Denoting $p = e + f$ one finds $px = x$, $\forall x \in X$ and $X = pW$. Now it is easy to conclude that X is a Hilbert module over the matrix algebra M_2 with the M_2-valued inner product inherited from W. We omit the details.

Since T by assumption preserves the inner squares, (8) implies that X is invariant for T. Now we apply the Theorem from [9] to the induced function $T_2 = T|X : X \to X$: there exist a phase function $\varphi_2 : X \to \boldsymbol{C}$ and a M_2-linear (!) isometry $U_2 : X \to X$ such that $T_2x = \varphi_2(x)U_2x$, $\forall x \in X$. By M_2-linearity W_e remains invariant under the action of U_2 since $W_e = eW$. This shows that the pair (φ_e, U_e) from the first part of the proof can actually be chosen by restricting the action of φ_2 and U_2 to W_e.

After all, we can apply Theorem 5 from [3] (see also Theorem 1 in [2]): there exists a unique isometry $U \in \boldsymbol{B}_a(W)$ such that $U|W_e = U_e$. Define a new function T' on W putting $T' = U^*T$.

We claim

$$\langle T'v, w\rangle e\langle w, T'v\rangle = \langle v, w\rangle e\langle w, v\rangle,\ \forall v, w \in W. \tag{9}$$

Indeed,

$$\langle T'v, w\rangle e\langle w, T'v\rangle = \langle T'v, w\rangle\overline{\varphi_e(ew)}e\varphi_e(ew)e\langle w, T'v\rangle$$

$$= \langle U^*Tv, \varphi_e(ew)ew\rangle\langle\varphi_e(ew)ew, U^*Tv\rangle = \langle Tv, T(ew)\rangle\langle T(ew), Tv\rangle$$

$$= \langle v, ew\rangle\langle ew, v\rangle = \langle v, w\rangle e\langle w, v\rangle.$$

Now observe that each minimal projection $f \in \boldsymbol{K}(H)$ can be written in the form $f = a^*ea$ for a suitably chosen partial isometry $a \in \boldsymbol{K}(H)$. Substituting aw for w in (9) we get

$$\langle T'v, w\rangle f\langle w, T'v\rangle = \langle v, w\rangle f\langle w, v\rangle,\ \forall v, w \in W. \tag{10}$$

Since there exists an approximate unit (u_λ) for $\boldsymbol{K}(H)$ whose elements are finite linear combinations of minimal projections, this gives

$$\langle T'v, w\rangle u_\lambda \langle w, T'v\rangle = \langle v, w\rangle u_\lambda \langle w, v\rangle,\ \forall\lambda,\ \forall v, w \in W. \tag{11}$$

Finally, (11) obviously implies

$$\langle T'v, w\rangle \langle w, T'v\rangle = \langle v, w\rangle \langle w, v\rangle,\ \forall v, w \in W. \tag{12}$$

Now we apply Lemma 1. There exists a complex number (depending on v) $\lambda_v =: \varphi(v)$ of modulus 1 such that

$$T'v = \varphi(v)v \quad \text{or} \quad U^*Tv = \varphi(v)v,\ \forall v \in W.$$

This gives

$$UU^*Tv = \varphi(v)Uv,\ \forall v \in W$$

and, because UU^* is the projection to $\operatorname{Im} U$, the proof will be finished by showing $T(W) \subseteq \operatorname{Im} U$.

To do this, let us take an orthonormal basis (u_j) for W such that $\langle u_j, u_j\rangle = e, \forall j$. (There exists an orthonormal basis for W with this property by [2], Proposition 2 and [3], Theorem 2.) Observe that the system (Tu_j) is also an orthonormal system in W. Indeed, $\langle Tu_i, Tu_j\rangle = \langle u_i, u_j\rangle = \delta_{ij}e$, $\forall i, j$ (with δ denoting the Kronecker symbol). Moreover, for each $v \in W$ we have (see [3], Theorem 1)

$$\langle Tv, Tv\rangle = \langle v, v\rangle = \sum_j \langle v, u_j\rangle\langle u_j, v\rangle = \sum_j |\langle u_j, v\rangle|^2 = \sum_j |\langle Tu_j, Tv\rangle|^2$$
$$= \sum_j \langle Tv, Tu_j\rangle\langle Tu_j, Tv\rangle.$$

This is enough to conclude

$$Tv = \sum_j \langle Tv, Tu_j\rangle Tu_j. \tag{13}$$

Notice that $\langle u_j, u_j\rangle = e$ implies $u_j = eu_j \in eW = W_e$, hence we may use (7) to rewrite (13):

$$Tv = \sum_j \langle Tv, \varphi_e(u_j)U_e(u_j)\rangle \varphi_e(u_j)U_e(u_j) = \sum_j \langle Tv, Uu_j\rangle Uu_j$$
$$= \sum_j U(\langle Tv, Uu_j\rangle u_j).$$

Since U is an isometry, its image is closed. This completes the proof. □

Remark 1. Let W be a Hilbert C^*-module over an arbitrary C^*-algebra $\mathcal{A}$ of compact operators and let $T : W \to W$ be a function satisfying condition (1) from Theorem 1.

It is well known ([1], Theorem 1.4.5) that $\mathcal{A}$ must be of the form

$$\mathcal{A} = \bigoplus_{j\in J} \boldsymbol{K}(H_j), \tag{14}$$

i.e. $\mathcal{A}$ is a direct sum of C^*-algebras $\boldsymbol{K}(H_j)$ of all compact operators acting on Hilbert spaces H_j, $j \in J$. We may assume that W is a full Hilbert $\mathcal{A}$-module by dropping unnecessary summands (i.e. those $\boldsymbol{K}(H_i)$ which act on W as the zero operator) from the above decomposition of $\mathcal{A}$. For each $j \in J$ consider the associated ideal submodule $W_j = \overline{[\boldsymbol{K}(H_j)W]}$. Notice that W_j, regarded as a Hilbert $\boldsymbol{K}(H_j)$-module, is full. Obviously, (W_j) is a family of pairwise orthogonal closed submodules of W and it is well known (cf. [3]) that W admits a decomposition into the (outer) direct sum

$$W = \bigoplus_{j \in J} W_j, \quad W_j = \overline{[\boldsymbol{K}(H_j)W]}. \tag{15}$$

Now observe that each W_j satisfies

$$W_j = \{w \in W : \langle w, w \rangle \in \boldsymbol{K}(H_j)\}. \tag{16}$$

Since each function satisfying condition (1) preserves $\mathcal{A}$-valued inner squares, (16) shows that each W_j is invariant for T. Applying Theorem 1 to all induced functions $T_j = T|W_j : W_j \to W_j$ we obtain the factorization $T_j(v) = \varphi_j(v)U_j v, \forall v \in W_j$ on each component W_j.

Observe that the family (U_j) defines an isometry $U \in \boldsymbol{B}_a(W)$ ([3], Theorem 8). However, the complex numbers $\varphi_j(w)$ might be different for fixed w and varying $j \in J$, so a unique global choice for φ as a scalar-valued function might be impossible. Therefore we cannot obtain a global factorization for T as in Theorem 1.

Remark 2. Let us note that our Theorem 1 (as well as the above Remark 1) also holds true for functions satisfying condition (1) which are defined on H^*-modules. Namely, if W is an arbitrary H^*-module over the H^*-algebra of all Hilbert Schmidt operators acting on some Hilbert space, then the same proof applies using the corresponding results concerned with operators on H^*-modules (cf. [2]).

To conclude, we shall mention a possible extension of Wigner's theorem to more general Hilbert C^*-modules. A good candidate is the class of Hilbert C^*-modules over concrete C^*-algebras which contain the ideal of all compact operators. There is some evidence along this line.

Acknowledgment

The authors wish to thank the referee for several valuable suggestions for improving the original manuscript.

References

1. W.B. Arveson, *An invitation to C^*-algebras*, GTM 39, Springer Verlag, Berlin, 1976. MR **58:**23621
2. D. Bakić, B. Guljaš, *Operators on Hilbert H^*-modules*, accepted for publication in the Journal of Operator Theory.
3. D. Bakić, B. Guljaš, *Hilbert C^*-modules over C^*-algebras of compact operators*, accepted for publication in Acta Sci. Math. (Szeged).
4. M. Cabrera, J. Martínez, A. Rodríguez, *Hilbert modules revisited: Orthonormal bases and Hilbert-Schmidt operators*, Glasgow Math. J. 37 (1995), 45-54. MR **96c:**46051
5. M. Frank, D. R. Larson, *Frames in Hilbert C^*-modules and C^*-algebras*, preprint, University of Houston, Houston, and Texas A&M University, College Station, Texas, USA, 1998.
6. I. Kaplansky, *Modules over operator algebras*, Amer. J. Math. 75(1953), 839-853. MR **15:**327f
7. C. Lance, *Hilbert C^*-modules*, London Mat. Soc. Lecture Notes Series, 210, Cambridge University Press, Cambridge, 1995.
8. L. Molnár, *An algebraic approach to Wigner's unitary-antiunitary theorem*, J. Austral. Math. Soc. 65(1998), 354-369. MR **99k:**46031
9. L. Molnár, *A generalization of Wigner's unitary-antiunitary theorem to Hilbert modules*, J. Math. Phys. 40(1999), 5544-5554. MR **2000j:**46112

10. W. Paschke, *Inner product modules over B^*-algebras,* Trans. Amer. Math. Soc. 182(1973), 443-468. MR **50:**8087
11. J. Rätz, *On Wigner's theorem: remarks, complements, comments and corollaries*, Aequationes Math. 52(1996), 1-9. MR **98b:**39022
12. M.A. Rieffel, *Induced representations of C^*-algebras,* Advances in Math. 13(1974), 176-257. MR **50:**5489
13. N.E. Wegge-Olsen, *K-theory and C^*-algebras - a friendly approach*, Oxford University Press, Oxford, 1993. MR **95c:**46116
14. E. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren*, Vieweg, Braunschweig, 1931. MR **6:**39g (reprint)

Department of Mathematics, University of Zagreb, Bijenička c. 30, 10000 Zagreb, Croatia
E-mail address: bakic@math.hr

Department of Mathematics, University of Zagreb, Bijenička c. 30, 10000 Zagreb, Croatia
E-mail address: guljas@math.hr