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# ON THE CELLULAR DECOMPOSITION OF THE EXCEPTIONAL LIE GROUP $G_2$

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ABSTRACT. The present note is to give a cellular decomposition of the compact connected exceptional Lie group $G_2$.

## 1. INTRODUCTION

Let us denote by $G_2$ the compact connected exceptional Lie group of rank 2. By definition $G_2$ is the automorphism group $\mathrm{Aut}(\mathfrak{C})$, where $\mathfrak{C}$ is the Cayley algebra. It has been long known that $G_2$ has the homotopy type of

$$S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$

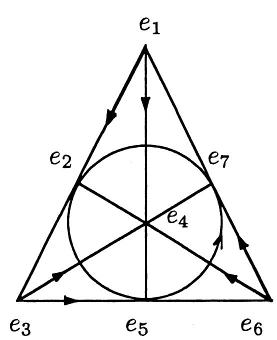
The purpose of the present note is to give a cellular decomposition of $G_2$:

$$G_2 = S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$

## 2. PRELIMINARY

Let us recall known results on $G_2$ which will be needed later.

The Cayley algebra $\mathfrak{C}$ is isomorphic to $\mathbb{R}^8$ as an $\mathbb{R}$-module and we denote its basis by $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Notice that the multiplication is not associative. The element $e_0$ is the unit of the algebra which we denote by 1. The multiplication of the remaining basis is given in the following diagram:

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Then the exceptional Lie group $G_2$ is defined to be

$$G_2 = \{g \in SO(8) \mid g(x)g(y) = g(xy), x, y \in \mathfrak{C}\} = \mathrm{Aut}(\mathfrak{C}).$$

Since $g(1) = 1$ for any $g \in G_2$, we may regard $G_2$ as a subgroup of $SO(7)$. So from now on, we express an element of $G_2$ as that of $SO(7)$. The subgroup of elements in $G_2$ fixing $e_1$ is known to be isomorphic to $SU(3)$. The subgroup of elements in $G_2$ fixing $e_1$ and $e_2$ is known to be isomorphic to $SU(2)$. Thus we regard $SU(3)$ and $SU(2)$ as subgroups of $G_2$. Let $S^6$ be the unit sphere of $\mathbb{R}^7$ whose basis is $\{e_i \mid 1 \leq i \leq 7\}$ and $S^5$ be the unit sphere of $\mathbb{R}^6$ whose basis is $\{e_i \mid 2 \leq i \leq 7\}$. Then there are two principal fibre bundles over them:

$$\begin{aligned} SU(3) &\longrightarrow G_2 \xrightarrow{p_1} S^6, \\ SU(2) &\longrightarrow SU(3) \xrightarrow{p_2} S^5, \end{aligned}$$

where $p_i(g) = g(e_i)$ for $i = 1, 2$. Let $H$ be the subgroup of $G_2$ defined by

$$H = G_2 \cap (SO(3) \oplus SO(4)).$$

**Lemma 2.1.** *$hgh^{-1} \in SU(2)$ for any $h \in H$ and $g \in SU(2)$.*

*Proof.* It is obvious, since $SU(2) = G_2 \cap (\{1\} \oplus SO(4))$, where $\{1\}$ denotes the subgroup of $SO(3)$ consisting of the identity element. $\square$

In the remainder of the section, we will construct cells of $G_2$. Let $D^i$ $(1 \leq i \leq 3)$ be the $i$-dimensional discs defined respectively by

$$\begin{aligned} D^3 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}, \\ D^2 &= \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\}, \\ D^1 &= \{z_1 \in \mathbb{R} \mid z_1^2 \leq 1\}. \end{aligned}$$

We define $V^3, V^5$ and $V^6$ as follows:

$$V^3 = D^3, \quad V^5 = D^3 \times D^2, \quad V^6 = D^3 \times D^2 \times D^1,$$

and put $X, Y$ and $Z$ as

$$X = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}, \quad Y = \sqrt{1 - y_1^2 - y_2^2}, \quad Z = \sqrt{1 - z_1^2}.$$

We define three maps $A, B, C$ respectively of $D^i$ to $G_2$ for $1 \le i \le 3$ as follows:

$$A(x_1,x_2,x_3) = \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1-2X^2 & -2x_1X & -2x_2X & -2x_3X \\ &&& 2x_1X & 1-2X^2 & 2x_3X & -2x_2X \\ &&& 2x_2X & -2x_3X & 1-2X^2 & 2x_1X \\ &&& 2x_3X & 2x_2X & -2x_1X & 1-2X^2 \end{pmatrix},$$

$$B(y_1,y_2) = \begin{pmatrix} 1 &&&&&& \\ & y_1 & -y_2 & -Y & 0 && \\ & y_2 & y_1 & 0 & -Y && \\ & Y & 0 & y_1 & y_2 && \\ & 0 & Y & -y_2 & y_1 && \\ &&&&& 1 & \\ &&&&&& 1 \end{pmatrix},$$

$$C(z_1) = \begin{pmatrix} z_1 & 0 & -Z &&&& \\ 0 & 1 & 0 &&&& \\ Z & 0 & z_1 &&&& \\ &&& 1 &&& \\ &&&& z_1 & 0 & -Z \\ &&&& 0 & 1 & 0 \\ &&&& Z & 0 & z_1 \end{pmatrix},$$

where blanks consist of the zero element. After having prepared these definitions, we will construct some cells of $G_2$. Let $\varphi_i$ be a map of $V^i$ to $G_2$ for $i = 3, 5, 6$ defined respectively by

$$\begin{aligned} &\varphi_3(x_1,x_2,x_3) = A(x_1,x_2,x_3), \\ &\varphi_5(x_1,x_2,x_3,y_1,y_2) = B(y_1,y_2)A(x_1,x_2,x_3)B(y_1,y_2)^{-1}, \\ &\varphi_6(x_1,x_2,x_3,y_1,y_2,z_1) = C(z_1)B(y_1,y_2)A(x_1,x_2,x_3)B(y_1,y_2)^{-1}C(z_1)^{-1}. \end{aligned}$$

We define eight cells $e^j$ for $j = 0, 3, 5, 6, 8, 9, 11, 14$ as follows:

$$\begin{array}{llll} e^0 = \{1\}, & e^3 = \mathrm{Im}\,\varphi_3, & e^5 = \mathrm{Im}\,\varphi_5, & e^6 = \mathrm{Im}\,\varphi_6, \\ e^8 = e^5e^3, & e^9 = e^6e^3, & e^{11} = e^6e^5, & e^{14} = e^6e^5e^3. \end{array}$$

We denote the boundary and the interior of a cell $e^i$ simply by $\dot{e}^i$ and by $\mathring{e}^i$ respectively.

## 3. A cellular decomposition of $SU(3)$

Yokota [Y1], [Y3] constructed a cellular decomposition of $SU(n)$. In this section, we reconstruct a cellular decomposition of $SU(3)$ for our purpose, which is essentially the same as Yokota's decomposition.

As is known, $SU(2)$ is homeomorphic to $S^3$, and hence $e^0 \cup e^3$ is a cellular decomposition of $SU(2)$.

**Lemma 3.1.** *The composite map $p_2\varphi_5 : (V^5, \partial V^5) \to (S^5, \{e_2\})$ is a relative homeomorphism.*

*Proof.* We express the map $(p_2\varphi_5)|_{V^5\setminus\partial V^5}$ as follows:

$$\begin{pmatrix} 0 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = p_2\varphi_5(x_1,x_2,x_3,y_1,y_2) = \begin{pmatrix} 0 \\ 1-2X^2Y^2 \\ 2x_1XY^2 \\ 2Y(y_1X^2 - x_1y_2X) \\ -2Y(x_1y_1X + y_2X^2) \\ -2x_2XY \\ -2x_3XY \end{pmatrix},$$

and hence

$$\begin{pmatrix} 0 \\ 1-a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = 2XY \begin{pmatrix} 0 \\ XY \\ x_1Y \\ y_1X - x_1y_2 \\ -x_1y_1 - y_2X \\ -x_2 \\ -x_3 \end{pmatrix}.$$

Since $X > 0$, $Y > 0$ and $1 - a_2 > 0$, an easy calculation from the second component in the above equation gives the following equation:

$$XY = \frac{\sqrt{1-a_2}}{\sqrt{2}}, \tag{3.1}$$

from which we easily obtain

$$x_2 = \frac{-a_6}{\sqrt{2(1-a_2)}}, \tag{3.2}$$

$$x_3 = \frac{-a_7}{\sqrt{2(1-a_2)}}. \tag{3.3}$$

Further we obtain two more equalities from the above equation:

$$\begin{aligned} &(1-a_2)^2 + a_3^2 = 4X^2Y^4(x_1^2 + X^2), \\ &a_4^2 + a_5^2 = 4X^2Y^2(y_1^2 + y_2^2)(x_1^2 + X^2) = 4X^2Y^2(1-Y^2)(x_1^2 + X^2). \end{aligned}$$

Using these two equalities, we obtain

$$Y^2 = \frac{(1-a_2)^2 + a_3^2}{(1-a_2)^2 + a_3^2 + a_4^2 + a_5^2}. \tag{3.4}$$

It follows from (3.1) and (3.4) that

$$X^2 = \frac{(1-a_2)((1-a_2)^2 + a_3^2 + a_4^2 + a_5^2)}{2((1-a_2)^2 + a_3^2)}. \tag{3.5}$$

It follows from (3.2), (3.3) and (3.5) that

$$x_1^2 = \frac{a_3^2((1-a_2)^2 + a_3^2 + a_4^2 + a_5^2)}{2(1-a_2)((1-a_2)^2 + a_3^2)}. \tag{3.6}$$

Since $Y > 0$, (3.4) implies that

$$Y = \frac{\sqrt{(1-a_2)^2 + a_3^2}}{\sqrt{(1-a_2)^2 + a_3^2 + a_4^2 + a_5^2}}. \tag{3.7}$$

Since $X > 0$, (3.5) implies that

$$X = \frac{\sqrt{(1-a_2)((1-a_2)^2 + a_3^2 + a_4^2 + a_5^2)}}{\sqrt{2((1-a_2)^2 + a_3^2)}}. \tag{3.8}$$

Since the signs of $x_1$ and $a_3$ are the same, (3.6) implies that

$$x_1 = \frac{a_3\sqrt{(1-a_2)^2 + a_3^2 + a_4^2 + a_5^2}}{\sqrt{2(1-a_2)((1-a_2)^2 + a_3^2)}}. \tag{3.9}$$

Now we determine $y_2$; we have

$$a_4 x_1 + a_5 X = -2XY(x_1^2 + X^2)y_2.$$

Substituting the equations (3.1), (3.8) and (3.9) in the above equation, we obtain

$$y_2 = \frac{-a_3 a_4 - (1-a_2)a_5}{\sqrt{((1-a_2)^2 + a_3^2)((1-a_2)^2 + a_3^2 + a_4^2 + a_5^2)}}. \tag{3.10}$$

Finally we determine $y_1$; we have

$$a_4 X - a_5 x_1 = 2XY(x_1^2 + X^2)y_1.$$

Substituting the equations (3.1), (3.8) and (3.9) in the above equation, we obtain

$$y_1 = \frac{(1-a_2)a_4 - a_3 a_5}{\sqrt{((1-a_2)^2 + a_3^2)((1-a_2)^2 + a_3^2 + a_4^2 + a_5^2)}}. \tag{3.11}$$

Thus we have expressed $x_1, x_2, x_3, y_1, y_2$ in terms of $a_2, \cdots, a_7$, that is, the inverse map has been constructed, which completes the proof. $\square$

**Proposition 3.2.** *$e^0 \cup e^3 \cup e^5 \cup e^8$ is a cellular decomposition of $SU(3)$.*

*Proof.* First we will show that $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$. We consider the following three cases:

(1) For the case where $i = 0$ and $j = 3$, it is obvious that $\mathring{e}^0 \cap \mathring{e}^3 = \emptyset$ since $e^0 \cup e^3$ is a cellular decomposition of $SU(2)$.
(2) For the case where $i \in \{0, 3\}$ and $j \in \{5, 8\}$, we have $p_2(\mathring{e}^i) = e_2$ and $p_2(\mathring{e}^j) = S^5 \backslash \{e_2\}$. Then we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$.
(3) For the case where $i = 5$ and $j = 8$, suppose that $A \in \mathring{e}^5 \cap \mathring{e}^8$. Since $\mathring{e}^8 = \mathring{e}^5\mathring{e}^3$, we can put $A = A_1 A_2$ where $A_1 \in \mathring{e}^5$ and $A_2 \in \mathring{e}^3$. We have $A = A_1$ since $p_2(A) = p_2(A_1 A_2) = p_2(A_1)$ and $p_2|_{\mathring{e}^5}$ is monic. Then we have $A_2 = 1 \in \mathring{e}^3$, which is a contradiction. Thus $\mathring{e}^5 \cap \mathring{e}^8 = \emptyset$.

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. It is obvious that the boundary $\dot{e}^3$ is included in $e^0$. Observe that the boundary $\dot{e}^5$ is a union of the following two sets:

$$\{BAB^{-1} \mid A \in A(\dot{D}^3), B \in B(D^2)\},$$
$$\{BAB^{-1} \mid A \in A(D^3), B \in B(\dot{D}^2)\}.$$

The first set contains only the identity element since $A$ is the identity element. Lemma 2.1 implies that the second set is contained in $SU(2)$ since $B$ is contained in $H$. Thus we have $\dot{e}^5 \subset e^3$. Further we have $\dot{e}^8 = \dot{e}^5 e^3 \cup e^5 \dot{e}^3 \subset e^3 e^3 \cup e^5 e^0 = e^3 \cup e^5$.

Finally, we will show that the inclusion map $e^0 \cup e^3 \cup e^5 \cup e^8 \to SU(3)$ is epic. Let $g \in SU(3)$. If $p_2(g) = e_2$, then $g$ is contained in $SU(2) = e^0 \cup e^3$. Suppose that $p_2(g) \neq e_2$. There is an element $h \in e^5$ such that $p_2(h) = p_2(g)$. Then we have

$h^{-1}g \in SU(2) = e^0 \cup e^3$, since $p_2(h^{-1}g) = e_2$. Therefore we have $g \in h(e^0 \cup e^3) \subset e^0 \cup e^3 \cup e^5 \cup e^8$. □

## 4. A cellular decomposition of $G_2$

First we need to show

**Lemma 4.1.** *The composite map* $p_1\varphi_6 : (V^6, \partial V^6) \to (S^6, \{e_1\})$ *is a relative homeomorphism.*

*Proof.* We express the map $(p_1\varphi_6)|_{V^6\setminus\partial V^6}$ as follows:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = p_1\varphi_6(x_1, x_2, x_3, y_1, y_2, z_1) = \begin{pmatrix} 1 - 2X^2Y^2Z^2 \\ 2x_1XY^2Z \\ 2z_1X^2Y^2Z \\ -2XYZ(x_1y_1 + y_2X) \\ -2XYZ(y_1z_1X - x_1y_2z_1 + x_2Z) \\ -2XYZx_3 \\ -2XYZ(y_1XZ - x_1y_2Z - x_2z_1) \end{pmatrix},$$

and hence

$$\begin{pmatrix} 1 - a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = 2XYZ \begin{pmatrix} XYZ \\ x_1Y \\ z_1XY \\ -x_1y_1 - y_2X \\ -y_1z_1X + x_1y_2z_1 - x_2Z \\ -x_3 \\ -y_1XZ + x_1y_2Z + x_2z_1 \end{pmatrix}.$$

We set for simplicity

$$\alpha_1 = (1-a_1)^2 + a_3^2, \quad \alpha_2 = (1-a_1)^2 + a_2^2 + a_3^2, \quad \alpha_3 = (1-a_1)^2 + a_2^2 + a_3^2 + a_4^2,$$
$$\beta_1 = a_3a_5 + (1-a_1)a_7, \qquad \beta_2 = a_2a_4 + a_3a_5 + (1-a_1)a_7.$$

Since $1 - a_1 > 0$, we have $\alpha_i > 0$ for $i = 1, 2, 3$. By an easy calculation one can obtain the following three equations:

$$Z^2 = \frac{(1-a_1)^2}{\alpha_1}, \tag{4.1}$$

$$X^2Y^2 = \frac{1-a_1}{2Z^2} = \frac{\alpha_1}{2(1-a_1)}, \tag{4.2}$$

$$z_1^2 = 1 - Z^2 = \frac{a_3^2}{\alpha_1}. \tag{4.3}$$

Since $Z \geq 0$ and $1 - a_1 \geq 0$, (4.1) implies that

$$Z = \frac{1-a_1}{\sqrt{\alpha_1}}. \tag{4.4}$$

Since the signs of $z_1$ and $a_3$ are the same, (4.3) implies that

$$z_1 = \frac{a_3}{\sqrt{\alpha_1}}. \tag{4.5}$$

We easily have

$$x_3 = \frac{-a_6}{2XYZ} = \frac{-a_6}{\sqrt{2(1-a_1)}}. \tag{4.6}$$

Next we determine $X$ and $Y$; we have

$$\begin{aligned}(a_4^2 + (a_5 z_1 + a_7 Z)^2)Y^2 &= 4X^2Y^4Z^2(y_1^2 + y_2^2)(x_1^2 + X^2)\\ &= 4X^2Y^4Z^2(1-Y^2)(x_1^2+X^2),\\ a_2^2(1-Y^2) = 4X^2Y^4Z^2(1-Y^2)x_1^2.&\end{aligned}$$

It follows from these two equalities that

$$(a_4^2 + (a_5 z_1 + a_7 Z)^2)Y^2 - a_2^2(1-Y^2) = 4X^4Y^4Z^2(1-Y^2).$$

Substituting the equations (4.2), (4.4) and (4.5) in the above equation, we obtain

$$a_4^2Y^2 + \frac{(a_3a_5 + (1-a_1)a_7)^2}{\alpha_1}Y^2 + a_2^2Y^2 - a_2^2 = \alpha_1(1-Y^2),$$

whence we have

$$Y^2 = \frac{\alpha_1\alpha_2}{\alpha_1\alpha_3 + \beta_1^2}.$$

Since $Y \geq 0$, we have

$$Y = \frac{\sqrt{\alpha_1\alpha_2}}{\sqrt{\alpha_1\alpha_3 + \beta_1^2}}. \tag{4.7}$$

Since $X \geq 0$, (4.2) and (4.7) imply that

$$X = \frac{\sqrt{\alpha_1\alpha_3 + \beta_1^2}}{\sqrt{2(1-a_1)\alpha_2}}. \tag{4.8}$$

We easily have

$$x_1 = \frac{a_2}{2XY^2Z} = \frac{a_2\sqrt{\alpha_1\alpha_3 + \beta_1^2}}{\sqrt{2(1-a_1)\alpha_1\alpha_2}}. \tag{4.9}$$

Now we determine $y_1$ and $y_2$; we have

$$a_4x_1 + a_5z_1X + a_7XZ = -2XYZ(x_1^2 + X^2)y_1,$$

into which substituting (4.4), (4.5), (4.7), (4.8) and (4.9) we obtain

$$y_1 = \frac{-\beta_2\sqrt{\alpha_1}}{\sqrt{\alpha_2(\alpha_1\alpha_3 + \beta_1^2)}}. \tag{4.10}$$

Quite similarly the equation

$$a_4X - a_5x_1z_1 - a_7x_1Z = -2XYZ(x_1^2 + X^2)y_2$$

gives rise to

$$y_2 = \frac{a_2\beta_1 - a_4\alpha_1}{\sqrt{\alpha_2(\alpha_1\alpha_3 + \beta_1^2)}}. \tag{4.11}$$

Finally we determine $x_2$; we have

$$a_5Z - a_7z_1 = -2XYZx_2,$$

which gives

$$x_2 = \frac{a_3 a_7 - (1-a_1)a_5}{\sqrt{2(1-a_1)\alpha_1}}. \tag{4.12}$$

Thus we have expressed $x_1, x_2, x_3, y_1, y_2, z_1$ in terms of $a_1, \cdots, a_7$, that is, the inverse map has been constructed and this completes the proof. □

The following is our main result.

**Theorem 4.2.** *The cell complex $e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}$ thus constructed gives a cellular decomposition of $G_2$.*

*Proof.* First we show that $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$. We consider the following three cases:

(1) For the case where $i, j \in \{0, 3, 5, 8\}$, both cells $e^i$ and $e^j$ are in $SU(3)$ and $e^0 \cup e^3 \cup e^5 \cup e^8$ is a cellular decomposition of $SU(3)$. Then we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$.
(2) For the case where $i \in \{0, 3, 5, 8\}$ and $j \in \{6, 9, 11, 14\}$, we have $p_1(\mathring{e}^i) = \{e_1\}$ and $p_1(\mathring{e}^j) = S^6 \backslash \{e_1\}$. Then we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$.
(3) For the case where $i, j \in \{6, 9, 11, 14\}$, suppose that $A \in \mathring{e}^i \cap \mathring{e}^j$. Since $\mathring{e}^i = \mathring{e}^6 \mathring{e}^{i-6}$ and $\mathring{e}^j = \mathring{e}^6 \mathring{e}^{j-6}$, we can put $A = A_1 A_2 = A_1' A_2'$ where $A_1, A_1' \in \mathring{e}^6$, $A_2 \in \mathring{e}^{i-6}$ and $A_2' \in \mathring{e}^{j-6}$. We have $A_1 = A_1'$, since $p_1(A_1) = p_1(A_1 A_2) = p_1(A_1' A_2') = p_1(A_1')$ and $p_1|_{\mathring{e}^6}$ is monic. Then we have $A_2 = A_2'$ and the first case shows that $i - 6 = j - 6$, that is, $i = j$. Thus $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$.

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. In the proof of Proposition 3.2, it is proved that the boundaries $\dot{e}^3$, $\dot{e}^5$ and $\dot{e}^8$ are included in the lower dimensional cells. Observe that the boundary $\dot{e}^6$ is a union of the following three sets:

$$\{CBAB^{-1}C^{-1} \mid A \in A(\dot{D}^3), B \in B(D^2), C \in C(D^1)\},$$
$$\{CBAB^{-1}C^{-1} \mid A \in A(D^3), B \in B(\dot{D}^2), C \in C(D^1)\},$$
$$\{CBAB^{-1}C^{-1} \mid A \in A(D^3), B \in B(D^2), C \in C(\dot{D}^1)\}.$$

The first set contains only the identity element, since $A$ is the identity element. Lemma 2.1 implies that the second set is contained in $SU(2)$, since $B$ and $C$ are contained in the subgroup $H$. We consider the third set. If $C = C(1) = 1$, it is obvious that $CBAB^{-1}C^{-1} = BAB^{-1} \in e^5$. Suppose that $C = C(-1)$. It is easy to check that

$$CB(y_1, y_2)C^{-1} = B(y_1, -y_2),$$
$$CA(x_1, x_2, x_3)C^{-1} = A(-x_1, x_2, -x_3).$$

Thus the third set is contained in $e^5$, since we have

$$\begin{aligned} &CB(y_1, y_2)A(x_1, x_2, x_3)B(y_1, y_2)^{-1}C^{-1} \\ =&(CB(y_1, y_2)C^{-1})(CA(x_1, x_2, x_3)C^{-1})(CB(y_1, y_2)^{-1}C^{-1}) \\ =&B(y_1, -y_2)A(-x_1, x_2, -x_3)B(y_1, -y_2)^{-1}. \end{aligned}$$

We have $\dot{e}^9 = e^6 \dot{e}^3 \cup \dot{e}^6 e^3 \subset e^6 e^0 \cup e^5 e^3 = e^6 \cup e^8$. We also have $\dot{e}^{11} = \dot{e}^6 e^5 \cup e^6 \dot{e}^5 \subset e^5 e^5 \cup e^6 e^3 = e^5 \cup e^9$, and $\dot{e}^{14} = \dot{e}^6 e^5 e^3 \cup e^6 \dot{e}^5 e^3 \cup e^6 e^5 \dot{e}^3 \subset e^5 e^5 e^3 \cup e^6 e^3 e^3 \cup e^6 e^5 = e^8 \cup e^9 \cup e^{11}$.

Finally, we will show that the inclusion map $e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14} \to G_2$ is epic. Let $g \in G_2$. If $p_1(g) = e_1$, then $g$ is contained in $SU(3) = e^0 \cup e^3 \cup e^5 \cup e^8$. Suppose that $p_1(g) \neq e_1$. There is an element $h \in e^6$ such that $p_1(h) = p_1(g)$. Thus we have $h^{-1}g \in SU(3) = e^0 \cup e^3 \cup e^5 \cup e^8$ since $p_1(h^{-1}g) = e_1$. Therefore we have $g \in h(e^0 \cup e^3 \cup e^5 \cup e^8) \subset e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}$. $\square$

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