ON THE CELLULAR DECOMPOSITION OF THE EXCEPTIONAL LIE GROUP $G_2$

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Abstract. The present note is to give a cellular decomposition of the compact connected exceptional Lie group $G_2$.

1. Introduction

Let us denote by $G_2$ the compact connected exceptional Lie group of rank 2. By definition $G_2$ is the automorphism group $\text{Aut}(\mathfrak{C})$, where $\mathfrak{C}$ is the Cayley algebra. It has been long known that $G_2$ has the homotopy type of $S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}$.

The purpose of the present note is to give a cellular decomposition of $G_2$:

\[ G_2 = S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}. \]

2. Preliminary

Let us recall known results on $G_2$ which will be needed later.

The Cayley algebra $\mathfrak{C}$ is isomorphic to $\mathbb{R}^8$ as an $\mathbb{R}$-module and we denote its basis by $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Notice that the multiplication is not associative. The element $e_0$ is the unit of the algebra which we denote by 1. The multiplication of the remaining basis is given in the following diagram:

![Diagram of Cayley algebra elements]

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Then the exceptional Lie group $G_2$ is defined to be

$$G_2 = \{ g \in SO(8) \mid g(x)g(y) = g(xy), x, y \in \mathbb{C} \} = \text{Aut}(\mathbb{C}).$$

Since $g(1) = 1$ for any $g \in G_2$, we may regard $G_2$ as a subgroup of $SO(7)$. So from now on, we express an element of $G_2$ as that of $SO(7)$. The subgroup of elements in $G_2$ fixing $e_1$ is known to be isomorphic to $SU(3)$. The subgroup of elements in $G_2$ fixing $e_1$ and $e_2$ is known to be isomorphic to $SU(2)$. Thus we regard $SU(3)$ and $SU(2)$ as subgroups of $G_2$. Let $S^6$ be the unit sphere of $\mathbb{R}^7$ whose basis is $\{e_i \mid 1 \leq i \leq 7\}$ and $S^5$ be the unit sphere of $\mathbb{R}^6$ whose basis is $\{e_i \mid 2 \leq i \leq 7\}$. Then there are two principal fibre bundles over them:

$$SU(3) \longrightarrow G_2 \overset{p_1}{\longrightarrow} S^6,$$

$$SU(2) \longrightarrow SU(3) \overset{p_2}{\longrightarrow} S^5,$$

where $p_i(g) = g(e_i)$ for $i = 1, 2$. Let $H$ be the subgroup of $G_2$ defined by

$$H = G_2 \cap (SO(3) \oplus SO(4)).$$

**Lemma 2.1.** $hgh^{-1} \in SU(2)$ for any $h \in H$ and $g \in SU(2)$.

**Proof.** It is obvious, since $SU(2) = G_2 \cap (\{1\} \oplus SO(4))$, where $\{1\}$ denotes the subgroup of $SO(3)$ consisting of the identity element. \hfill \square

In the remainder of the section, we will construct cells of $G_2$. Let $D^i$ $(1 \leq i \leq 3)$ be the $i$-dimensional discs defined respectively by

$$D^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\},$$

$$D^2 = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\},$$

$$D^1 = \{z_1 \in \mathbb{R} \mid z_1^2 \leq 1\}.$$

We define $V^3$, $V^5$ and $V^6$ as follows:

$$V^3 = D^3, \quad V^5 = D^3 \times D^2, \quad V^6 = D^3 \times D^2 \times D^3,$$

and put $X$, $Y$ and $Z$ as

$$X = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}, \quad Y = \sqrt{1 - y_1^2 - y_2^2}, \quad Z = \sqrt{1 - z_1^2}.$$
We define three maps $A, B, C$ respectively of $D_i$ to $G_2$ for $1 \leq i \leq 3$ as follows:

\[
A(x_1, x_2, x_3) = \begin{pmatrix}
1 & 1 & 1 \\
1 - 2X^2 & -2x_1X & -2x_2X & -2x_3X \\
2x_1X & 1 - 2X^2 & 2x_3X & -2x_2X \\
2x_2X & -2x_3X & 1 - 2X^2 & 2x_1X \\
2x_3X & 2x_2X & -2x_1X & 1 - 2X^2 \\
\end{pmatrix},
\]

\[
B(y_1, y_2) = \begin{pmatrix}
1 & y_1 - y_2 & -Y & 0 \\
y_2 & y_1 & 0 & -Y \\
Y & 0 & y_1 & y_2 \\
0 & Y & -y_2 & y_1 \\
1 & 1 & 1 & 1 \\
\end{pmatrix},
\]

\[
C(z_1) = \begin{pmatrix}
z_1 & 0 & -Z \\
0 & 1 & 0 \\
Z & 0 & z_1 \\
1 & z_1 & 0 & -Z \\
0 & 1 & 0 \\
Z & 0 & z_1 \\
\end{pmatrix},
\]

where blanks consist of the zero element. After having prepared these definitions, we will construct some cells of $G_2$. Let $\varphi_i$ be a map of $V^i$ to $G_2$ for $i = 3, 5, 6$ defined respectively by

\[
\varphi_3(x_1, x_2, x_3) = A(x_1, x_2, x_3),
\]

\[
\varphi_5(x_1, x_2, x_3, y_1, y_2) = B(y_1, y_2)A(x_1, x_2, x_3)B(y_1, y_2)^{-1},
\]

\[
\varphi_6(x_1, x_2, x_3, y_1, y_2, z_1) = C(z_1)B(y_1, y_2)A(x_1, x_2, x_3)B(y_1, y_2)^{-1}C(z_1)^{-1}.
\]

We define eight cells $e^j$ for $j = 0, 3, 5, 6, 8, 9, 11, 14$ as follows:

\[
e^0 = \{1\}, \quad e^3 = \text{Im } \varphi_3, \quad e^5 = \text{Im } \varphi_5, \quad e^6 = \text{Im } \varphi_6,
\]

\[
e^8 = e^5e^3, \quad e^9 = e^6e^3, \quad e^{11} = e^6e^5, \quad e^{14} = e^6e^5e^3.
\]

We denote the boundary and the interior of a cell $e^i$ simply by $\hat{e}^i$ and by $\check{e}^i$ respectively.

### 3. A cellular decomposition of $SU(3)$

Yokota [Y1, Y3] constructed a cellular decomposition of $SU(n)$. In this section, we reconstruct a cellular decomposition of $SU(3)$ for our purpose, which is essentially the same as Yokota’s decomposition.

As is known, $SU(2)$ is homeomorphic to $S^3$, and hence $e^0 \cup e^3$ is a cellular decomposition of $SU(2)$.

**Lemma 3.1.** The composite map $p_2\varphi_5 : (V^5, \partial V^5) \to (S^5, \{e_2\})$ is a relative homeomorphism.
Proof. We express the map \((p_2\varphi_5)_{|V^5\setminus gV^5}\) as follows:
\[
\begin{pmatrix}
0 \\
0 \\
1 - a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7
\end{pmatrix}
= p_2\varphi_5(x_1, x_2, x_3, y_1, y_2) =
\begin{pmatrix}
0 \\
1 - 2X^2Y^2 \\
2XY \\
2Y(y_1X^2 - x_1y_2X) \\
-2Y(x_1y_1X + y_2X^2) \\
-2x_2X \\
-2x_3XY
\end{pmatrix},
\]
and hence
\[
\begin{pmatrix}
0 \\
1 - a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7
\end{pmatrix}
= 2XY
\begin{pmatrix}
0 \\
XY \\
x_1Y \\
y_1X - x_1y_2 \\
x_1y_1 - y_2X \\
x_2 \\
x_3
\end{pmatrix}.
\]

Since \(X > 0\), \(Y > 0\) and \(1 - a_2 > 0\), an easy calculation from the second component in the above equation gives the following equation:
\[
XY = \frac{\sqrt{1 - a_2}}{\sqrt{2}},
\]
from which we easily obtain
\[
(3.1) \quad x_2 = \frac{-a_6}{\sqrt{2(1 - a_2)}},
\]
\[
(3.2) \quad x_3 = \frac{-a_7}{\sqrt{2(1 - a_2)}}.
\]

Further we obtain two more equalities from the above equation:
\[
(1 - a_2)^2 + a_3^2 = 4X^2Y^4(x_1^2 + X^2),
\]
\[
a_4^2 + a_5^2 = 4X^2Y^2(y_1^2 + y_2^2)(x_1^2 + X^2) = 4X^2Y^2(1 - Y^2)(x_1^2 + X^2).
\]

Using these two equalities, we obtain
\[
(3.4) \quad Y^2 = \frac{(1 - a_2)^2 + a_3^2}{(1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2}.
\]

It follows from (3.1) and (3.3) that
\[
(3.5) \quad X^2 = \frac{(1 - a_2)((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}{2((1 - a_2)^2 + a_3^2)}.
\]

It follows from (3.2), (3.3) and (3.5) that
\[
(3.6) \quad x_1^2 = \frac{a_3^2((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}{2((1 - a_2)^2 + a_3^2)}.
\]

Since \(Y > 0\), (3.4) implies that
\[
(3.7) \quad Y = \frac{\sqrt{(1 - a_2)^2 + a_3^2}}{\sqrt{(1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2}}.
\]
Since $X > 0$, (3.5) implies that

$$X = \frac{1}{2} \frac{(1 - a_2)((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}{2((1 - a_2)^2 + a_5^2)}.$$  

Since the signs of $x_1$ and $a_3$ are the same, (3.6) implies that

$$x_1 = \frac{a_3(1 - a_2)^2 + a_4^2 + a_5^2}{\sqrt{2(1 - a_2)((1 - a_2)^2 + a_5^2)}}.$$  

Now we determine $y_2$; we have

$$a_4x_1 + a_5X = -2XY(x_1^2 + X^2)y_2.$$  

Substituting the equations (3.1), (3.8) and (3.9) in the above equation, we obtain

$$y_2 = \frac{-a_3a_4 - (1 - a_2)a_5}{\sqrt{((1 - a_2)^2 + a_3^2)((1 - a_2)^2 + a_4^2 + a_5^2)}}.$$  

Finally we determine $y_1$; we have

$$a_4X - a_5x_1 = 2XY(x_1^2 + X^2)y_1.$$  

Substituting the equations (3.1), (3.8) and (3.9) in the above equation, we obtain

$$y_1 = \frac{(1 - a_2)a_4 - a_3a_5}{\sqrt{((1 - a_2)^2 + a_3^2)((1 - a_2)^2 + a_4^2 + a_5^2)}}.$$  

Thus we have expressed $x_1, x_2, x_3, y_1, y_2$ in terms of $a_2, \cdots, a_7$, that is, the inverse map has been constructed, which completes the proof.

**Proposition 3.2.** $e^0 \cup e^3 \cup e^5 \cup e^8$ is a cellular decomposition of $SU(3)$.

**Proof.** First we will show that $\hat{e}^i \cap \hat{e}^j = \emptyset$ if $i \neq j$. We consider the following three cases:

1. For the case where $i = 0$ and $j = 3$, it is obvious that $\hat{e}^0 \cap \hat{e}^3 = \emptyset$ since $e^0 \cup e^3$ is a cellular decomposition of $SU(2)$.
2. For the case where $i \in \{0, 3\}$ and $j \in \{5, 8\}$, we have $p_2(\hat{e}^i) = e_2$ and $p_2(\hat{e}^j) = S^5 \setminus \{e_2\}$. Then we have $\hat{e}^i \cap \hat{e}^j = \emptyset$.
3. For the case where $i = 5$ and $j = 8$, suppose that $A \in \hat{e}^5 \cap \hat{e}^8$. Since $\hat{e}^8 = \hat{e}^5 \hat{e}^3$, we can put $A = A_1A_2$ where $A_1 \in \hat{e}^5$ and $A_2 \in \hat{e}^3$. We have $A = A_1$ since $p_2(A) = p_2(A_1A_2) = p_2(A_1)$ and $p_2|_{\hat{e}^5}$ is monic. Then we have $A_2 = 1 \in \hat{e}^3$, which is a contradiction. Thus $\hat{e}^5 \cap \hat{e}^8 = \emptyset$.

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. It is obvious that the boundary $\hat{e}^3$ is included in $e^0$. Observe that the boundary $\hat{e}^5$ is a union of the following two sets:

$$\{BAB^{-1} \mid A \in A(\hat{D}^3), B \in B(D^2)\},$$

$$\{BAB^{-1} \mid A \in A(D^3), B \in B(\hat{D}^2)\}.$$  

The first set contains only the identity element since $A$ is the identity element. Lemma 2.1 implies that the second set is contained in $SU(2)$ since $B$ is contained in $H$. Thus we have $\hat{e}^5 \subset e^3$. Further we have $\hat{e}^8 = \hat{e}^5 \hat{e}^3 \subset e^3 \cup e^3 \cup e^3 \cup e^3 = e^3 \cup e^3$.

Finally, we will show that the inclusion map $e^0 \cup e^3 \cup e^5 \cup e^8 \to SU(3)$ is epic. Let $g \in SU(3)$. If $p_2(g) = e_2$, then $g$ is contained in $SU(2) = e^0 \cup e^3$. Suppose that $p_2(g) \neq e_2$. There is an element $h \in e^5$ such that $p_2(h) = p_2(g)$. Then we have
$h^{-1}g \in SU(2) = e^0 \cup e^3$, since $p_2(h^{-1}g) = e_2$. Therefore we have $g \in h(e^0 \cup e^3) \subset e^0 \cup e^3 \cup e^5 \cup e^8$.

4. A cellular decomposition of $G_2$

First we need to show

**Lemma 4.1.** The composite map $p_1\varphi_6 : (V^6, \partial V^6) \to (S^6, \{e_1\})$ is a relative homeomorphism.

**Proof.** We express the map $(p_1\varphi_6)|_{V^6 \setminus \partial V^6}$ as follows:

$$
\begin{pmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6 \\
  a_7
\end{pmatrix} = p_1\varphi_6(x_1, x_2, x_3, y_1, y_2, z_1) =
\begin{pmatrix}
  1 - 2X^2Y^2Z^2 & 2x_1XY^2Z & 2z_1X^2Y^2Z \\
  -2XYZ(x_1y_1 + y_2X) & -2XYZ(y_1z_1X - x_1y_2z_1 + x_2Z) & -2XYZx_3 \\
  -2XYZ^2(y_1XZ - x_1y_2Z - x_2z_1) &
\end{pmatrix},
$$

and hence

$$
\begin{pmatrix}
  1 - a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6 \\
  a_7
\end{pmatrix} = 2XYZ
\begin{pmatrix}
  XYZ \\
  x_1Y \\
  z_1X \\
  -x_1y_1 - y_2X \\
  -y_1z_1X + x_1y_2z_1 - x_2Z \\
  -x_3 \\
  -y_1XZ + x_1y_2Z + x_2z_1
\end{pmatrix}.
$$

We set for simplicity

$$
\alpha_1 = (1 - a_1)^2 + a_3^2, \quad \alpha_2 = (1 - a_1)^2 + a_2^2 + a_3^2, \quad \alpha_3 = (1 - a_1)^2 + a_2^2 + a_3^2 + a_4^2, \\
\beta_1 = a_3a_5 + (1 - a_1)a_7, \quad \beta_2 = a_2a_4 + a_3a_5 + (1 - a_1)a_7.
$$

Since $1 - a_1 > 0$, we have $\alpha_i > 0$ for $i = 1, 2, 3$. By an easy calculation one can obtain the following three equations:

$$
(4.1) \quad Z^2 = \frac{(1 - a_1)^2}{\alpha_1},
$$
$$
(4.2) \quad X^2Y^2 = \frac{1 - a_1}{2Z^2} = \frac{\alpha_1}{2(1 - a_1)},
$$
$$
(4.3) \quad z_1^2 = 1 - Z^2 = \frac{\alpha_2^2}{\alpha_1}.
$$

Since $Z \geq 0$ and $1 - a_1 \geq 0$, (4.1) implies that

$$
(4.4) \quad Z = \frac{1 - a_1}{\sqrt{\alpha_1}}.
$$

Since the signs of $z_1$ and $a_3$ are the same, (4.3) implies that

$$
(4.5) \quad z_1 = \frac{a_3}{\sqrt{\alpha_1}}.
$$
We easily have
\[ x_3 = \frac{-a_6}{2YZ} = \frac{-a_6}{\sqrt{2(1 - a_1)}}. \]

Next we determine \(X\) and \(Y\); we have
\[
\begin{align*}
(a_1^2 + (a_5z_1 + a_7Z)^2)Y^2 &= 4X^2Y^4Z^2(y_1^2 + y_2^2)(x_1^2 + X^2) \\
&= 4X^2Y^4Z^2(1 - Y^2)(x_1^2 + X^2), \\
a_5^2(1 - Y^2) &= 4X^2Y^4Z^2(1 - Y^2)x_1^2.
\end{align*}
\]
It follows from these two equalities that
\[
(a_1^2 + (a_5z_1 + a_7Z)^2)Y^2 - a_5^2(1 - Y^2) = 4X^4Y^4Z^2(1 - Y^2).
\]

Substituting the equations (4.2), (4.4) and (4.5) in the above equation, we obtain
\[
a_4^2Y^2 + \frac{(a_3a_5 + (1 - a_1)a_7)^2}{a_1}Y^2 + a_3^2Y^2 - a_2^2 = \alpha_1(1 - Y^2),
\]
whence we have
\[ Y^2 = \frac{\alpha_1\alpha_2}{\alpha_1\alpha_3 + \beta_1^2}. \]

Since \(Y \geq 0\), we have
\[ Y = \frac{\sqrt{\alpha_1\alpha_2}}{\sqrt{\alpha_1\alpha_3 + \beta_1^2}}. \]

Since \(X \geq 0\), (4.2) and (4.7) imply that
\[ X = \frac{\sqrt{\alpha_1\alpha_3 + \beta_1^2}}{\sqrt{2(1 - a_1)a_2}}. \]

We easily have
\[ x_1 = \frac{a_2}{2X^2Z} = \frac{a_2}{\sqrt{2(1 - a_1)a_2}}. \]

Now we determine \(y_1\) and \(y_2\); we have
\[
a_4x_1 + a_5z_1X + a_7XZ = -2XYZ(x_1^2 + X^2)y_1,
\]
into which substituting (4.4), (4.5), (4.7), (4.8) and (4.9) we obtain
\[ y_1 = \frac{-\beta_1\sqrt{a_1}}{\sqrt{\alpha_2(\alpha_1\alpha_3 + \beta_1^2)}}. \]

Quite similarly the equation
\[
a_4X - a_5x_1z_1 - a_7x_1Z = -2XYZ(x_1^2 + X^2)y_2
\]
gives rise to
\[ y_2 = \frac{a_2\beta_1 - a_4a_1}{\sqrt{\alpha_2(\alpha_1\alpha_3 + \beta_1^2)}}. \]

Finally we determine \(x_2\); we have
\[ a_5Z - a_7z_1 = -2XYX_2, \]
which gives
\[
(4.12) \quad x_2 = \frac{a_3a_7 - (1 - a_1)a_5}{\sqrt{2(1 - a_1)a_1}}.
\]

Thus we have expressed \(x_1, x_2, x_3, y_1, y_2, z_1\) in terms of \(a_1, \ldots, a_7\), that is, the inverse map has been constructed and this completes the proof. \(\square\)

The following is our main result.

**Theorem 4.2.** The cell complex \(e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^11 \cup e^{14}\) thus constructed gives a cellular decomposition of \(G_2\).

**Proof.** First we show that \(e^i \cap e^j = \emptyset\) if \(i \neq j\). We consider the following three cases:

1. For the case where \(i, j \in \{0, 3, 5, 8\}\), both cells \(e^i\) and \(e^j\) are in \(SU(3)\) and \(e^0 \cup e^3 \cup e^5 \cup e^8\) is a cellular decomposition of \(SU(3)\). Then we have \(e^i \cap e^j = \emptyset\) if \(i \neq j\).
2. For the case where \(i \in \{0, 3, 5, 8\}\) and \(j \in \{6, 9, 11, 14\}\), we have \(p_1(e^i) = \{e_1\}\) and \(p_1(e^j) = S^0\{e_1\}\). Then we have \(e^i \cap e^j = \emptyset\).
3. For the case where \(i, j \in \{6, 9, 11, 14\}\), suppose that \(A \in e^i \cap e^j\). Since \(e^i = e^{6,\bar{i}}\) and \(e^j = e^{6,\bar{j}}\), we can put \(A = A_1A_2 = A'_1A'_2\) where \(A_1, A'_1 \in e^6\), \(A_2, A'_2 \in e^{6,\bar{6}}\). We have \(A_1 = A'_1\), since \(p_1(A_1) = p_1(A'_1A_2) = p_1(A'_1)\) and \(p_1(x)\) is monic. Then we have \(A_2 = A'_2\) and the first case shows that \(i - 6 = j - 6\), that is, \(i = j\). Thus \(e^i \cap e^j = \emptyset\) if \(i \neq j\).

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. In the proof of Proposition 3.2 it is proved that the boundaries \(e^3, e^5\) and \(e^8\) are included in the lower dimensional cells. Observe that the boundary \(e^6\) is a union of the following three sets:

\[
\{CBAB^{-1}C^{-1} \mid A \in A(D^3), B \in B(D^2), C \in C(D^1)\},
\]
\[
\{CBAB^{-1}C^{-1} \mid A \in A(D^3), B \in B(D^2), C \in C(D^1)\},
\]
\[
\{CBAB^{-1}C^{-1} \mid A \in A(D^3), B \in B(D^2), C \in C(D^1)\}.
\]

The first set contains only the identity element, since \(A\) is the identity element. Lemma 2.1 implies that the second set is contained in \(SU(2)\), since \(B\) and \(C\) are contained in the subgroup \(H\). We consider the third set. If \(C = C(1) = 1\), it is obvious that \(CBAB^{-1}C^{-1} = BAB^{-1} \in e^5\). Suppose that \(C = C(-1)\). It is easy to check that

\[
CB(y_1, y_2)C^{-1} = B(y_1, -y_2),
\]
\[
CA(x_1, x_2, x_3)C^{-1} = A(-x_1, x_2, -x_3).
\]

Thus the third set is contained in \(e^5\), since we have

\[
CB(y_1, y_2)A(x_1, x_2, x_3)B(y_1, y_2)^{-1}C^{-1} = (CB(y_1, y_2)C^{-1})(CA(x_1, x_2, x_3)C^{-1})(CB(y_1, y_2)^{-1}C^{-1})^{-1}
\]
\[
= B(y_1, -y_2)A(-x_1, x_2, -x_3)B(y_1, -y_2)^{-1}C^{-1}.
\]

We have \(e^3 = e^{6,\bar{3}} \cup e^{6,\bar{3}} \subset e^6 \cup e^3 = e^6 \cup e^9\), \(e^5 = e^{6,\bar{5}} \cup e^{6,\bar{5}} \subset e^6 \cup e^5 \subset e^9 \cup e^5\), and \(e^{14} = e^{6,\bar{14}} \cup e^{6,\bar{14}} \subset e^6 \cup e^{14}\). We also have \(e^{11} = e^{6,\bar{11}} \cup e^{6,\bar{11}} \subset e^9 \cup e^{11}\).
Finally, we will show that the inclusion map $e_0 \cup e_3 \cup e_5 \cup e_6 \cup e_8 \cup e_9 \cup e_{11} \cup e_{14} \to G_2$ is epic. Let $g \in G_2$. If $p_1(g) = e_1$, then $g$ is contained in $SU(3) = e_0 \cup e_3 \cup e_5 \cup e_8$. Suppose that $p_1(g) \neq e_1$. There is an element $h \in e_0$ such that $p_1(h) = p_1(g)$. Thus we have $h^{-1} g \in SU(3) = e_0 \cup e_3 \cup e_5 \cup e_8$ since $p_1(h^{-1} g) = e_1$. Therefore we have $g \in h(e_0 \cup e_3 \cup e_5 \cup e_8) \subset e_0 \cup e_3 \cup e_5 \cup e_8 \cup e_9 \cup e_{11} \cup e_{14}$.

**References**


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