

CONTINUOUS TRANSFORMATION OF BAIRE MEASURES INTO LEBESGUE MEASURE

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ABSTRACT. A recent result by Wulbert on the existence of continuous functions with measure zero level sets is slightly extended and its proof is considerably simplified. As a by-product, a criterion is established for a Baire measure to allow a continuous transformation into Lebesgue measure.

INTRODUCTION

As outlined in the survey [5] by Wulbert, the following result is of interest in L_1 -approximation theory:

(*) *Given a nonatomic σ -finite measure space (X, \mathfrak{A}, μ) , for any finite dimensional subspace \mathcal{K} of $\mathcal{L}_1(X, \mathfrak{A}, \mu)$ there is an extreme element g in the unit ball of the dual space, i.e. satisfying*

$$(1) \quad g(x) = \pm 1 \quad \text{for } \mu\text{-almost all } x \in X,$$

which annihilates \mathcal{K} , i.e. satisfies

$$(2) \quad \int_X fg d\mu = 0 \quad \text{for all } f \in \mathcal{K}.$$

One approach to this result uses Liapunov's convexity theorem, as has been carried out by Phelps and Dye [4, Theorem 2.5]. An alternative, yielding some additional insight, consists in applying the Borsuk–Ulam theorem on antipodal mappings, as has been done by Hobby and Rice [1] in the classical case $X = [0, 1]$ and $\mu = \lambda$ ($:=$ Lebesgue measure). In fact, this approach has been anticipated in a stochastic context by the present author [2, Satz 3] for arbitrary measure spaces. The additional tool there is a transformation of the underlying measure μ into Lebesgue measure, as is always available under the – obviously necessary – condition that μ is nonatomic and σ -finite. Actually, all that is needed is an \mathfrak{A} -measurable function h with measure zero level sets, i.e.

$$(3) \quad \mu(\{x : h(x) = y\}) = 0 \quad \text{for all } y \in \mathbf{R}.$$

In a recent paper [6] Wulbert proves that in the topological situation the function h may even be chosen to be continuous, which yields for (*) above a solution g being the sign of a continuous function. Since his construction is highly involved, a simple proof seemed to be desirable. On the way it turned out that by a slight generalization the question raised by the title of this paper finds a complete answer.

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PRELIMINARIES

In proving the main results, three simple lemmata will be used. First, the notion of moderatedness has to be extended from Borel to Baire measures:

Definition 1. For an arbitrary topological space X denote by $\mathfrak{C}(X)$ the Baire σ -algebra, generated by the space $\mathcal{C}(X)$ of continuous functions. Then a measure μ on $\mathfrak{C}(X)$ is called “moderated”, if $\mathcal{C}(X)$ contains functions $f_n, n \in \mathbf{N}$, satisfying

$$X = \bigcup_{n \in \mathbf{N}} \{x : f_n(x) \neq 0\} \quad \text{and} \quad \mu(\{x : f_n(x) \neq 0\}) < \infty \quad \text{for all } n \in \mathbf{N}.$$

While any moderated Baire measure is σ -finite, the converse fails even for nonatomic measures. Indeed, choose an open base $G_n, n \in \mathbf{N}$, for $X = [0, 1]$ and measurable functions $f_n \geq 0, n \in \mathbf{N}$, such that

$$\int_{G_n} f_n d\lambda = \infty \quad \text{and} \quad \sum_{n \in \mathbf{N}} \lambda(\{x : f_n(x) \neq 0\}) < \infty.$$

The last condition implies

$$\lambda(\{x : f_n(x) = 0 \text{ for almost all } n \in \mathbf{N}\}) = 1,$$

hence $f := \sum_{n \in \mathbf{N}} f_n$ is λ -almost finite and thus $d\mu = f d\lambda$ defines a σ -finite measure. Due to $\mu(G) = \infty$ for all open sets $G \neq \emptyset$, however, it fails to be moderated (this example also solves the final problem in [6]).

The following criterion will be required in the sequel:

Lemma 1. *A Baire measure $\mu \upharpoonright \mathfrak{C}(X)$ is moderated if and only if $\mathcal{C}(X)$ contains a μ -integrable strictly positive function f .*

Proof. Given functions $f_n, n \in \mathbf{N}$, according to Definition 1, choose constants $\gamma_n > 0$ such that

$$\sum_{n \in \mathbf{N}} \gamma_n < \infty \quad \text{and} \quad \sum_{n \in \mathbf{N}} \gamma_n \mu(\{x : f_n(x) \neq 0\}) < \infty.$$

Then the function

$$f := \sum_{n \in \mathbf{N}} (|f_n| \wedge \gamma_n)$$

meets all requirements. Conversely, if f is strictly positive and μ -integrable, then the functions

$$f_n := \left(f - \frac{1}{n}\right)^+, n \in \mathbf{N},$$

satisfy the conditions of Definition 1. □

Next, some notation has to be introduced:

Definition 2. For a Hausdorff topological space Y denote by $\mathfrak{B}(Y)$ the Borel σ -algebra and by $M(Y)$ the space of all finite measures ν on $\mathfrak{B}(Y)$ endowed with the weak (narrow) topology. Then

$$\delta(\nu) := \max_{y \in Y} \nu(\{y\}) \quad \text{for } \nu \in M(Y).$$

The following fact will be used only for $Y = \mathbf{R}$, but is of independent interest:

Lemma 2. *If Y is a second countable metrizable space, then the functional δ is upper semicontinuous.*

Proof. Fix $\nu_0 \in M(Y)$ and $\delta_0 > 0$ with $\delta(\nu_0) < \delta_0$. Since Y is second countable, ν_0 -atoms are singletons, which is easily seen to imply that Y may be partitioned into a finite number of sets $B_m \in \mathfrak{B}(Y)$, $1 \leq m \leq n$, with $\nu_0(B_m) < \delta_0$. Since ν_0 is outer regular, this yields a cover of Y by open sets G_m , $1 \leq m \leq n$, with $\nu_0(G_m) < \delta_0$. Now in a metrizable space each open set G can be approximated from inside by a closed set F with measure zero boundary ∂F (choose a metric d and consider the closed sets $F_t := \{y : d(y, z) \geq t \text{ for } z \notin G\}$, having disjoint boundaries for different values $t > 0$). Thus there are closed sets $F_m \subset G_m$ with

$$\nu_0(F_m) > \nu_0(G_m) - \frac{1}{n} \delta_0 \quad \text{and} \quad \nu_0(\partial F_m) = 0.$$

Adjoining to F_1, \dots, F_n the set

$$F_0 := Y \setminus \bigcup_{1 \leq m \leq n} (F_m \setminus \partial F_m)$$

yields a cover of Y by closed sets F_m , $0 \leq m \leq n$, with $\nu_0(F_m) < \delta_0$ for all m , because also

$$\begin{aligned} \nu_0(F_0) &= \nu_0\left(\bigcup_{1 \leq m \leq n} G_m\right) - \nu_0\left(\bigcup_{1 \leq m \leq n} F_m\right) \\ &\leq \sum_{1 \leq m \leq n} \nu_0(G_m \setminus F_m) < \delta_0. \end{aligned}$$

Thus the weakly open set

$$M_0 := \bigcap_{0 \leq m \leq n} \{\nu \in M(Y) : \nu(F_m) < \delta_0\}$$

contains ν_0 and satisfies $\delta(\nu) < \delta_0$ for all $\nu \in M_0$, as had to be shown. □

The final auxiliary result is the essential content of [6, Lemma 4.1] (for a similar argument see the proof of [3, Lemma 1]):

Lemma 3. *If Y is a Hausdorff topological vector space, then for any $\nu \in M(Y)$ there are only countably many one-dimensional affine subspaces S of Y satisfying the inequality $\nu(S) > \delta(\nu)$.*

Proof. Let \mathfrak{S} denote the class of all one-dimensional affine (closed, hence Borel measurable) subspaces S of Y and A be the set of points $y \in Y$ with $\nu(\{y\}) > 0$. Let, moreover, the measure ν_0 be defined by $d\nu_0 = 1_{X \setminus A} d\nu$. Then $S \in \mathfrak{S}$ satisfies $\nu(S) \leq \delta(\nu)$, whenever (a) S intersects A in at most one point and (b) S is a ν_0 -null set. But there are only countably many exceptions of (a), because A is countable, and this holds as well for (b), because two sets $S \in \mathfrak{S}$ meet in at most one point and thus are ν_0 -almost disjoint. □

MAIN RESULTS

No further auxiliary results on nonatomic measures (as in Section 3 of [6]) are needed to establish the following extension of Wulbert's main result:

Proposition 1. *Let μ be a nonatomic σ -finite Baire measure on some topological space X and define the topology in $\mathcal{C}(X)$ by the (finite or infinite) uniform norm. Then*

$$\mathcal{C}_0 := \{h \in \mathcal{C}(X) : \mu(\{x : h(x) = y\}) = 0 \text{ for all } y \in \mathbf{R}\}$$

is a dense subset of $\mathcal{C}(X)$.

Proof. 1. Since μ may be replaced by any dominating measure, it is sufficient to consider the case of a probability measure. Since, moreover, μ is nonatomic, there exists a Baire measurable function $h : X \rightarrow [0, 1]$ with

$$\mu(\{x : h(x) = y\}) = 0 \quad \text{for } 0 \leq y \leq 1.$$

Let $h_n \in \mathcal{C}(X), n \in \mathbf{N}$, be an L_1 -approximation of h , i.e. satisfying

$$\lim_{n \rightarrow \infty} \int_X |h_n - h| d\mu = 0,$$

where $0 \leq h_n \leq 1$ may and will be assumed. Then $h_n \rightarrow h$ in μ -measure and thus the distributions $\mu \circ h_n^{-1}$ converge weakly to $\mu \circ h^{-1}$. Since this distribution is nonatomic, Lemma 2 yields

$$\limsup_{n \rightarrow \infty} \delta(\mu \circ h_n^{-1}) \leq \delta(\mu \circ h^{-1}) = 0.$$

2. Now, with $f \in \mathcal{C}(X)$ and $n \in \mathbf{N}$ being fixed, define

$$g_t := f + t h_n \in \mathcal{C}(X) \quad \text{for } t \in \mathbf{R}.$$

Then an application of Lemma 3 to $Y = \mathbf{R}^2$ and the joint distribution ν of (f, h_n) shows the inequality $\delta(\mu \circ g_t^{-1}) \leq \delta(\nu)$ to hold, except for countably many values of t . Combined with the trivial inequality $\delta(\nu) \leq \delta(\mu \circ h_n^{-1})$ this provides a dense subset T of \mathbf{R} such that

$$\delta(\mu \circ g_t^{-1}) \leq \delta(\mu \circ h_n^{-1}) \quad \text{for } t \in T.$$

3. Next, consider the sets

$$\mathcal{G}_n := \left\{ g \in \mathcal{C}(X) : \delta(\mu \circ g^{-1}) < \delta(\mu \circ h_n^{-1}) + \frac{1}{n} \right\} \quad \text{for } n \in \mathbf{N}.$$

Since the mapping $\mathcal{C}(X) \ni g \mapsto \mu \circ g^{-1} \in M(\mathbf{R})$ is continuous, they are open subsets of $\mathcal{C}(X)$ by Lemma 2. Since, moreover, they are dense in $\mathcal{C}(X)$ by part 2 of the proof and $\mathcal{C}(X)$ is a Baire space, the set $\bigcap_{n \in \mathbf{N}} \mathcal{G}_n$ is as well dense in $\mathcal{C}(X)$. Since, finally, this intersection equals \mathcal{C}_0 by part 1 of the proof, the assertion is established.* □

Now the concluding criterion is an easy consequence:

Proposition 2. *A Baire measure μ on a topological space X allows a continuous transformation φ into Lebesgue measure on the interval $[0, \mu(X)]$ resp. \mathbf{R}_+ if and only if it is nonatomic and moderated.*

Proof. The condition is necessary, as is clear for “nonatomic” and follows for “moderated” from Lemma 1. To prove sufficiency choose a function $f \in \mathcal{C}(X)$ according to Lemma 1 and a function $h \in \mathcal{C}_0$ with $\|h - (\frac{1}{f} + 1)\| < 1$ according to Proposition 1. Then $h \geq 0$, and it follows from

$$\mu(\{x : h(x) \leq y\}) \leq \mu\left(\left\{x : f(x) \geq \frac{1}{y}\right\}\right) < \infty \quad \text{for } y > 0,$$

that the “distribution function”

$$\psi(y) := \mu(\{x : h(x) \leq y\}) \quad \text{for } y \geq 0$$

* The original proof of Proposition 1 used arguments from measure theory, resulting in

$$\delta(\mu \circ (f + \sum_{n \in \mathbf{N}} \frac{t_n}{2^n} h_n)^{-1}) = 0 \quad \text{for } \lambda^{\mathbf{N}}\text{-almost all } (t_n, n \in \mathbf{N}) \in [0, 1]^{\mathbf{N}}.$$

Following a suggestion by the referee, it was slightly simplified by using topological arguments.

is continuous and finite-valued. Using the “inverse function”

$$\psi^{-1}(z) := \max \{y : \psi(y) \leq z\} \quad \text{for } 0 \leq z < \mu(X),$$

it is now easily checked that

$$\mu(\{x : \psi \circ h(x) \leq y\}) = y \quad \text{for } 0 \leq y < \mu(X),$$

i.e. the continuous function $\varphi := \psi \circ h$ is appropriate. \square

Finally, it should be mentioned that for infinite μ a continuous transformation into Lebesgue measure on the whole real line is as well possible. Indeed, to replace $X = \mathbf{R}_+$ by $Y = \mathbf{R}$, it suffices to choose for the first step a continuous function h with the properties

$$\lambda(\{x : |h(x)| = y\}) = 0 \quad \text{and} \quad \lambda(\{x : |h(x)| \leq y\}) < \infty \quad \text{for all } y \geq 0,$$

satisfying in addition

$$\lambda(\{x : h(x) \leq 0\}) = \infty = \lambda(\{x : h(x) \geq 0\})$$

(take for instance $h(x) = x^2 \sin x$). The second step then uses the “two-sided” distribution function ψ , being defined by $\lambda(\{x : 0 \leq h(x) \leq y\})$ for $y \geq 0$ and by $-\lambda(\{x : y \leq h(x) \leq 0\})$ for $y \leq 0$, respectively.

ADDED IN PROOF

As the referee asked for a direct solution, here is a continuous transformation φ of Lebesgue measure on \mathbf{R}_+ into Lebesgue measure on \mathbf{R} : define mappings $\varphi_n : [0, 1] \mapsto \mathbf{R}_+$, $n \in \mathbf{N}$, by affine interpolation of the points

$$(0, 0), \left(\frac{1}{2(n+1)}, n\right), \left(\frac{1}{2}, n+1\right), \left(1 - \frac{1}{2(n+1)}, n\right), (1, 0)$$

and let φ be composed of the translates of φ_n resp. $-\varphi_n$ to the intervals $[2n - 2, 2n - 1]$ resp. $[2n - 1, 2n]$.

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