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CONTINUOUS TRANSFORMATION OF BAIRE MEASURES INTO LEBESGUE MEASURE

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Abstract. A recent result by Wulbert on the existence of continuous functions with measure zero level sets is slightly extended and its proof is considerably simplified. As a by–product, a criterion is established for a Baire measure to allow a continuous transformation into Lebesgue measure.

Introduction

As outlined in the survey [5] by Wulbert, the following result is of interest in L_1–approximation theory:

($*$) *Given a nonatomic $\sigma-$finite measure space $(X, \mathfrak{A}, \mu)$, for any finite dimensional subspace $\mathcal{K}$ of $\mathcal{L}_1(X, \mathfrak{A}, \mu)$ there is an extreme element g in the unit ball of the dual space, i.e. satisfying*

$$g(x) = \pm 1 \qquad \text{for } \mu-\text{almost all } x \in X\,, \tag{1}$$

which annihilates $\mathcal{K}$, i.e. satisfies

$$\int_X f g \, d\mu = 0 \qquad \text{for all } f \in \mathcal{K}\,. \tag{2}$$

One approach to this result uses Liapunov's convexity theorem, as has been carried out by Phelps and Dye [4, Theorem 2.5]. An alternative, yielding some additional insight, consists in applying the Borsuk–Ulam theorem on antipodal mappings, as has been done by Hobby and Rice [1] in the classical case $X = [0, 1]$ and $\mu = \lambda$ (:= Lebesgue measure). In fact, this approach has been anticipated in a stochastic context by the present author [2, Satz 3] for arbitrary measure spaces. The additional tool there is a transformation of the underlying measure μ into Lebesgue measure, as is always available under the – obviously necessary – condition that μ is nonatomic and $\sigma-$finite. Actually, all that is needed is an $\mathfrak{A}-$measurable function h with measure zero level sets, i.e.

$$\mu(\{x : h(x) = y\}) = 0 \qquad \text{for all } y \in \mathbf{R}\,. \tag{3}$$

In a recent paper [6] Wulbert proves that in the topological situation the function h may even be chosen to be continuous, which yields for ($*$) above a solution g being the sign of a continuous function. Since his construction is highly involved, a simple proof seemed to be desirable. On the way it turned out that by a slight generalization the question raised by the title of this paper finds a complete answer.

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Preliminaries

In proving the main results, three simple lemmata will be used. First, the notion of moderatedness has to be extended from Borel to Baire measures:

Definition 1. For an arbitrary topological space X denote by $\mathfrak{C}(X)$ the Baire $\sigma-$algebra, generated by the space $\mathcal{C}(X)$ of continuous functions. Then a measure μ on $\mathfrak{C}(X)$ is called "moderated", if $\mathcal{C}(X)$ contains functions $f_n, n \in \mathbf{N}$, satisfying

$$X = \bigcup_{n\in\mathbf{N}} \{x : f_n(x) \neq 0\} \quad \text{and} \quad \mu(\{x : f_n(x) \neq 0\}) < \infty \ \text{ for all } \ n \in \mathbf{N}\,.$$

While any moderated Baire measure is $\sigma-$finite, the converse fails even for nonatomic measures. Indeed, choose an open base $G_n, n \in \mathbf{N}$, for $X = [0,1]$ and measurable functions $f_n \geq 0, n \in \mathbf{N}$, such that

$$\int_{G_n} f_n \, d\lambda = \infty \quad \text{and} \quad \sum_{n\in\mathbf{N}} \lambda(\{x : f_n(x) \neq 0\}) < \infty\,.$$

The last condition implies

$$\lambda(\{x : f_n(x) = 0 \ \text{ for almost all } \ n \in \mathbf{N}\}) = 1\,,$$

hence $f := \sum_{n\in\mathbf{N}} f_n$ is $\lambda-$almost finite and thus $d\mu = f\, d\lambda$ defines a $\sigma-$finite measure. Due to $\mu(G) = \infty$ for all open sets $G \neq \emptyset$, however, it fails to be moderated (this example also solves the final problem in [6]).

The following criterion will be required in the sequel:

Lemma 1. *A Baire measure $\mu \,|\, \mathfrak{C}(X)$ is moderated if and only if $\mathcal{C}(X)$ contains a $\mu-$integrable strictly positive function f.*

Proof. Given functions $f_n, n \in \mathbf{N}$, according to Definition 1, choose constants $\gamma_n > 0$ such that

$$\sum_{n\in\mathbf{N}} \gamma_n < \infty \quad \text{and} \quad \sum_{n\in\mathbf{N}} \gamma_n \, \mu(\{x : f_n(x) \neq 0\}) < \infty\,.$$

Then the function

$$f := \sum_{n\in\mathbf{N}} (|f_n| \wedge \gamma_n)$$

meets all requirements. Conversely, if f is strictly positive and $\mu-$integrable, then the functions

$$f_n := \left(f - \frac{1}{n}\right)^+, n \in \mathbf{N},$$

satisfy the conditions of Definition 1. □

Next, some notation has to be introduced:

Definition 2. For a Hausdorff topological space Y denote by $\mathfrak{B}(Y)$ the Borel $\sigma-$algebra and by $M(Y)$ the space of all finite measures ν on $\mathfrak{B}(Y)$ endowed with the weak (narrow) topology. Then

$$\delta(\nu) := \max_{y\in Y} \nu(\{y\}) \qquad \text{for } \ \nu \in M(Y)\,.$$

The following fact will be used only for $Y = \mathbf{R}$, but is of independent interest:

Lemma 2. *If Y is a second countable metrizable space, then the functional δ is upper semicontinuous.*

Proof. Fix $\nu_0 \in M(Y)$ and $\delta_0 > 0$ with $\delta(\nu_0) < \delta_0$. Since Y is second countable, ν_0-atoms are singletons, which is easily seen to imply that Y may be partitioned into a finite number of sets $B_m \in \mathfrak{B}(Y)$, $1 \leq m \leq n$, with $\nu_0(B_m) < \delta_0$. Since ν_0 is outer regular, this yields a cover of Y by open sets G_m, $1 \leq m \leq n$, with $\nu_0(G_m) < \delta_0$. Now in a metrizable space each open set G can be approximated from inside by a closed set F with measure zero boundary ∂F (choose a metric d and consider the closed sets $F_t := \{y : d(y, z) \geq t \text{ for } z \notin G\}$, having disjoint boundaries for different values $t > 0$). Thus there are closed sets $F_m \subset G_m$ with

$$\nu_0(F_m) > \nu_0(G_m) - \frac{1}{n}\,\delta_0 \quad \text{and} \quad \nu_0(\partial F_m) = 0\,.$$

Adjoining to $F_1, \dots, F_n$ the set

$$F_0 := Y \setminus \bigcup_{1 \leq m \leq n} (F_m \setminus \partial F_m)$$

yields a cover of Y by closed sets F_m, $0 \leq m \leq n$, with $\nu_0(F_m) < \delta_0$ for all m, because also

$$\begin{aligned} \nu_0(F_0) &= \nu_0\big(\bigcup_{1 \leq m \leq n} G_m\big) - \nu_0\big(\bigcup_{1 \leq m \leq n} F_m\big) \\ &\leq \sum_{1 \leq m \leq n} \nu_0(G_m \setminus F_m) < \delta_0\,. \end{aligned}$$

Thus the weakly open set

$$M_0 := \bigcap_{0 \leq m \leq n} \{\nu \in M(Y) : \nu(F_m) < \delta_0\}$$

contains ν_0 and satisfies $\delta(\nu) < \delta_0$ for all $\nu \in M_0$, as had to be shown. □

The final auxiliary result is the essential content of [6, Lemma 4.1] (for a similar argument see the proof of [3, Lemma 1]):

Lemma 3. *If Y is a Hausdorff topological vector space, then for any $\nu \in M(Y)$ there are only countably many one–dimensional affine subspaces S of Y satisfying the inequality $\nu(S) > \delta(\nu)$.*

Proof. Let $\mathfrak{S}$ denote the class of all one–dimensional affine (closed, hence Borel measurable) subspaces S of Y and A be the set of points $y \in Y$ with $\nu(\{y\}) > 0$. Let, moreover, the measure ν_0 be defined by $d\nu_0 = 1_{X \setminus A}\, d\nu$. Then $S \in \mathfrak{S}$ satisfies $\nu(S) \leq \delta(\nu)$, whenever (a) S intersects A in at most one point and (b) S is a ν_0-null set. But there are only countably many exceptions of (a), because A is countable, and this holds as well for (b), because two sets $S \in \mathfrak{S}$ meet in at most one point and thus are ν_0-almost disjoint. □

Main results

No further auxiliary results on nonatomic measures (as in Section 3 of [6]) are needed to establish the following extension of Wulbert's main result:

Proposition 1. *Let μ be a nonatomic $\sigma-$finite Baire measure on some topological space X and define the topology in $\mathcal{C}(X)$ by the (finite or infinite) uniform norm. Then*

$$\mathcal{C}_0 := \{h \in \mathcal{C}(X) : \mu(\{x : h(x) = y\}) = 0 \text{ for all } y \in \mathbf{R}\}$$

is a dense subset of $\mathcal{C}(X)$.

Proof. 1. Since μ may be replaced by any dominating measure, it is sufficient to consider the case of a probability measure. Since, moreover, μ is nonatomic, there exists a Baire measurable function $h : X \to [0,1]$ with

$$\mu(\{x : h(x) = y\}) = 0 \qquad \text{for } \ 0 \le y \le 1 \,.$$

Let $h_n \in \mathcal{C}(X), n \in \mathbf{N}$, be an L_1–approximation of h, i.e. satisfying

$$\lim{}_{n\to\infty} \int_X |h_n - h| \, d\mu = 0 \,,$$

where $0 \le h_n \le 1$ may and will be assumed. Then $h_n \to h$ in $\mu-$measure and thus the distributions $\mu \circ h_n^{-1}$ converge weakly to $\mu \circ h^{-1}$. Since this distribution is nonatomic, Lemma 2 yields

$$\limsup{}_{n\to\infty} \delta(\mu \circ h_n^{-1}) \le \delta(\mu \circ h^{-1}) = 0 \,.$$

2. Now, with $f \in \mathcal{C}(X)$ and $n \in \mathbf{N}$ being fixed, define

$$g_t := f + t\, h_n \, \in \, \mathcal{C}(X) \qquad \text{for } \ t \in \mathbf{R} \,.$$

Then an application of Lemma 3 to $Y = \mathbf{R}^2$ and the joint distribution ν of (f, h_n) shows the inequality $\delta(\mu \circ g_t^{-1}) \le \delta(\nu)$ to hold, except for countably many values of t. Combined with the trivial inequality $\delta(\nu) \le \delta(\mu \circ h_n^{-1})$ this provides a dense subset T of $\mathbf{R}$ such that

$$\delta(\mu \circ g_t^{-1}) \le \delta(\mu \circ h_n^{-1}) \qquad \text{for } \ t \in T \,.$$

3. Next, consider the sets

$$\mathcal{G}_n := \Big\{ g \in \mathcal{C}(X) : \delta(\mu \circ g^{-1}) < \delta(\mu \circ h_n^{-1}) + \frac{1}{n} \Big\} \qquad \text{for } \ n \in \mathbf{N} \,.$$

Since the mapping $\mathcal{C}(X) \ni g \mapsto \mu \circ g^{-1} \in M(\mathbf{R})$ is continuous, they are open subsets of $\mathcal{C}(X)$ by Lemma 2. Since, moreover, they are dense in $\mathcal{C}(X)$ by part 2 of the proof and $\mathcal{C}(X)$ is a Baire space, the set $\bigcap_{n\in\mathbf{N}} \mathcal{G}_n$ is as well dense in $\mathcal{C}(X)$. Since, finally, this intersection equals $\mathcal{C}_0$ by part 1 of the proof, the assertion is established.* □

Now the concluding criterion is an easy consequence:

Proposition 2. *A Baire measure μ on a topological space X allows a continuous transformation φ into Lebesgue measure on the interval $[0, \mu(X)]$ resp. $\mathbf{R}_+$ if and only if it is nonatomic and moderated.*

Proof. The condition is necessary, as is clear for "nonatomic" and follows for "moderated" from Lemma 1. To prove sufficiency choose a function $f \in \mathcal{C}(X)$ according to Lemma 1 and a function $h \in \mathcal{C}_0$ with $\|h - (\frac{1}{f} + 1)\| < 1$ according to Proposition 1. Then $h \ge 0$, and it follows from

$$\mu(\{x : h(x) \le y\}) \, \le \, \mu\Big(\Big\{x : f(x) \ge \frac{1}{y}\Big\}\Big) < \infty \qquad \text{for } \ y > 0 \,,$$

that the "distribution function"

$$\psi(y) := \mu(\{x : h(x) \le y\}) \qquad \text{for } \ y \ge 0$$

* The original proof of Proposition 1 used arguments from measure theory, resulting in

$$\delta(\mu \circ (f + \textstyle\sum_{n\in\mathbf{N}} \frac{t_n}{2^n} \, h_n)^{-1}) = 0 \qquad \text{for } \ \lambda^{\mathbf{N}} \text{–almost all } \ (t_n, n \in \mathbf{N}) \in [0,1]^{\mathbf{N}} \,.$$

Following a suggestion by the referee, it was slightly simplified by using topological arguments.

is continuous and finite–valued. Using the "inverse function"

$$\psi^{-1}(z) := \max\{y : \psi(y) \leq z\} \qquad \text{for } 0 \leq z < \mu(X)\,,$$

it is now easily checked that

$$\mu(\{x : \psi \circ h\,(x) \leq y\}) = y \qquad \text{for } 0 \leq y < \mu(X)\,,$$

i.e. the continuous function $\varphi := \psi \circ h$ is appropriate. □

Finally, it should be mentioned that for infinite μ a continuous transformation into Lebesgue measure on the whole real line is as well possible. Indeed, to replace $X = \mathbf{R}_+$ by $Y = \mathbf{R}$, it suffices to choose for the first step a continuous function h with the properties

$$\lambda(\{x : |h(x)| = y\}) = 0 \quad \text{and} \quad \lambda(\{x : |h(x)| \leq y\}) < \infty \qquad \text{for all } y \geq 0\,,$$

satisfying in addition

$$\lambda(\{x : h(x) \leq 0\}) = \infty = \lambda(\{x : h(x) \geq 0\})$$

(take for instance $h(x) = x^2 \sin x$). The second step then uses the "two–sided" distribution function ψ, being defined by $\lambda(\{x : 0 \leq h(x) \leq y\})$ for $y \geq 0$ and by $-\lambda(\{x : y \leq h(x) \leq 0\})$ for $y \leq 0$, respectively.

Added in proof

As the referee asked for a direct solution, here is a continuous transformation φ of Lebesgue measure on $\mathbf{R}_+$ into Lebesgue measure on $\mathbf{R}$: define mappings $\varphi_n : [0,1] \mapsto \mathbf{R}_+, n \in \mathbf{N}$, by affine interpolation of the points

$$(0\,,\,0)\,,\quad \left(\frac{1}{2(n+1)}\,,\,n\right)\,,\quad \left(\frac{1}{2}\,,\,n+1\right)\,,\quad \left(1-\frac{1}{2(n+1)}\,,\,n\right)\,,\quad (1\,,\,0)$$

and let φ be composed of the translates of φ_n resp. $-\varphi_n$ to the intervals $[2n-2, 2n-1]$ resp. $[2n-1, 2n]$.

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