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A GEOMETRIC PROOF OF THE EXISTENCE OF THE GREEN BUNDLES

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Abstract. We give a new proof of the existence of the Green bundles.

Let (M, g) be a compact Riemannian manifold, and denote by g_t its geodesic flow on the tangent bundle TM. Let $\pi : TM \to M$ be the canonical projection and for all $\theta \in TM$ let $V(\theta)$ be the kernel of $d\pi_\theta$. Two points θ_1, θ_2 are said to be conjugate if $\theta_2 = g_t\theta_1$ and $dg_tV(\theta_1) \cap V(\theta_2) \neq 0$.

It was proved by Hopf [5] that a two-dimensional torus without conjugate points is flat. Afterwards, Green [8] proved that the integral of the scalar curvature of a manifold without conjugate points is nonpositive and it vanishes if the metric is locally flat. A main ingredient was the existence, under the condition of no conjugate points, of the following bundles:

$$E^s(\theta) = \lim_{t\to\infty} dg_{-t}V(g_t(\theta)), \tag{1}$$

$$E^u(\theta) = \lim_{t\to\infty} dg_{t}V(g_{-t}(\theta)). \tag{2}$$

Hopf's result was generalized to higher dimensions in [2], but there are still new rigidity type results using these bundles; see for example [1].

These bundles have other applications: among other ideas they where used by Freire and Mañé [7] to obtain estimates of the topological entropy. Foulon [6] generalized this result to the case of Finsler metrics. The bundles were also used by Eberlain [4] who proved that these are transverse if and only if the geodesic flow is Anosov. This result was also generalized to the case of convex Hamiltonians without conjugate points; see [3]. The purpose of this note is to give a new proof of the following

Theorem. *If the geodesic flow g_t of a compact manifold does not have conjugate points, then for every θ in TM the limits* (1) *and* (2) *exist.*

We recall from [4] the definition of the connection map $K : T_\theta TM \to T_{\pi(\theta)}M$. For ξ on $T_\theta TM$ let $Z : (-\epsilon, \epsilon) \to TM$ be a curve with initial velocity ξ. Define $K(\xi) = Z'(0)$ to be the covariant derivative of Z along the curve $\pi \circ Z$. The definition does not depend on the curve Z.

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Denote $E_T = dg_{-T}V(g_T(\theta))$. Since $d\pi : E_T(\theta) \to T_{\pi(\theta))}M$ is an isomorphism, we may use the connection map to define a linear map $S_T : T_{\pi(\theta)}M \to T_{\pi(\theta)}M$ by $S_T(Y) = K(\xi)$ where ξ is in E_T and $Y = d\pi(\xi)$.

Let ω be the canonical symplectic structure on TM, i.e. $\omega(\xi,\eta) = \langle d\pi(\xi), K(\eta)\rangle - \langle K(\xi), d\pi(\eta)\rangle$. Clearly the vertical space is Lagrangian, meaning that $\omega(\xi,\eta) = 0$ for all ξ,η on $V(\theta)$. Since the geodesic flow preserves the symplectic form, it follows that E_T is a Lagrangian subspace. This immediately implies that S_T is a symmetric linear map.

Recall that it is possible to give a partial order in the set of symmetric linear transformations defining $A \succ B$ if $A - B$ is positive definite. To prove the theorem it is enough to see that the sequence S_T is monotone and bounded. The original proof of this fact is a slightly clumsy calculation.

For all $t < s$ the linear transformation $S_s - S_t$ is symmetric. We claim that the signature does not change in the region contained in $\mathbb{R}^2$ such that $0 < t < s$. Otherwise we can find $t_0 < s_0$ such that $S_{s_0} - S_{t_0}$ has 0 as an eigenvalue; this implies that θ_{s_0} and θ_{t_0} are conjugate.

On the other hand let $Y_\xi(t)$ be the Jacobi field along the geodesic $\gamma = \pi g_t$ with initial conditions $Y(0) = d\pi(\xi)$ and $Y'(0) = K(\xi)$. Denote by $P_{-t} : T_{\gamma(t)}M \to T_{\pi(\theta)}M$ the parallel translation. Then for ξ in $E_s(\theta)$ we have that

$$0 = P_{-s}(Y_\xi(s)) = Y_\xi(0) + sY'_\xi(0) + O(s^2) = (Id + sS_s)(Y_\xi(0)),$$

where Id is the identity transformation. Hence $S_s = -\frac{1}{s}Id + O(s^2)$. Consequently $S_s - S_t$ is positive definite and S_s is monotone increasing as s tends to infinity.

Similarly for all $t < 0 < s$ the linear transformation $S_s - S_t$ has constant signature. Moreover by the same estimate the linear transformation $S_s - S_t$ is negative definite, for $|s|, |t| \ll 1$. So S_s is bounded by S_t for any $t < 0$. Similar arguments apply for the unstable bundle E^u.

Finally let us remark that the same argument, the constancy of the signature of $S_s - S_t$ and a local computation applies for convex Hamiltonians without conjugate points. See [3] to define the order of symmetric matrices,

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