PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 8, Pages 2229–2235
S 0002-9939(02)06560-7
Article electronically published on March 8, 2002

A COMBINATORIAL PROOF OF ANDREWS' PARTITION FUNCTIONS RELATED TO SCHUR'S PARTITION THEOREM

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(Communicated by Dennis A. Hejhal)

ABSTRACT. We construct an involution to show equality between partition functions related to Schur's second partition theorem.

1. INTRODUCTION

In 1926, I. Schur [2] proved the following theorem on partitions.

Theorem 1. *Let $A(n)$ be the number of partitions of n into parts congruent to 1 or 5 $(\bmod 6)$, $B(n)$ the number of partitions of n into distinct nonmultiples of 3, and $D(n)$ the number of partitions of n of the form $b_1 + b_2 + \cdots + b_s$ such that $b_i - b_{i+1} \geq 3$ with strict inequality if $3|b_i$. Then*

$$A(n) = B(n) = D(n).$$

G. E. Andrews [1] found two partition functions equal to the partition functions in Schur's Theorem. One is $C(n)$, the number of partitions of n into odd parts, none appearing more than twice, and the other is $E(n)$, the number of partitions of n in which no part appears more than twice, odd parts appear at most once, the difference between two parts can never be 1, and can be 2 only if both are odd, with weight $(-1)^e$ when partitions have exactly e different parts that appear twice.

In the sequel, we call partitions enumerated by $X(n)$ partitions of type X for $X = A, B, C, D, E$.

The equality of $C(n)$ with one of the partition functions in Schur's Theorem can be easily obtained from their generating functions. However, the case of $E(n)$ is quite obscure and mysterious. Andrews first showed that $e_n(q)$, the generating function of partitions of type E whose parts are less than or equal to n, satisfies the equality

$$\lim_{n\to\infty} \frac{e_{2n-1}(q)}{(q^2;q^2)_n} = \prod_{n=0}^{\infty} \frac{(1+q^{2n+1}+q^{4n+2})}{1-q^{2n+2}},$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. From this it can be shown that the generating function of $E(n)$ is the same as that of $C(n)$.

Received by the editors March 5, 2001.
1991 *Mathematics Subject Classification.* Primary 05A17, 11P81.
This research was supported by the postdoctoral fellowship program from the Korea Science and Engineering Foundation.

At the conclusion of his paper, Andrews [1] asked, "First, is there a purely combinatorial or bijective way of proving that $E(n)$ equals any of the other partition functions in Theorem 1?" In Section 2, we show that $D(n) = E(n)$ by constructing an involution in the set of partitions of type E whose invariant set is the set of partitions of type D.

Furthermore, Andrews [1, Theorem 2] gave a relation between polynomial generating functions $d_n(q)$ and $e_n(q)$, where $d_n(q)$ is the generating function for all partitions of type D whose parts are less than or equal to n.

In the conclusion of his paper, Andrews confessed, "Theorem 2 is still rather a mystery. The proof is purely a verification. Is there an underlying partition-theoretic explanation of Theorem 2?" In Section 3, using the involution we construct in Section 2, we provide a combinatorial proof of his Theorem 2.

2. Relation between $D(n)$ and $E(n)$

In this section, we will prove the following theorem:

Theorem 2. *For all n*

$$D(n) = E(n).$$

We can easily check that the partitions of type D satisfy the condition for partitions of type E. In other words, there is a sign reversing involution v in the set of partitions of type E such that v is the identity map under the set of partitions of type D. Now, let us describe our involution v. In this paper, we assume that the parts of a partition are ordered weakly decreasing.

Proof of Theorem 2. Let a partition $\lambda = \lambda_1\lambda_2\cdots\lambda_l$ of type E be given. Assume that $\lambda_0 = \infty$ and $\lambda_{l+1} = -\infty$ for convention. First, consider the largest pair λ_i, λ_{i+1} among the parts that appear twice and are less than λ_{i-1} by at least four, and consider the largest pair λ_j, λ_{j+1} among the consecutive odd parts. If $\lambda_i > \lambda_j$, then add one to λ_i from λ_{i+1}. Otherwise, add one to λ_{j+1} from λ_j. Then the signs of our new and old partitions are different, and so they are cancelled out in $E(n)$. Since we choose the largest pairs in both cases, our map is reversible. The only remaining partitions are those whose parts differ by at least 3 or which have three consecutive parts $2k+3, 2k, 2k$. We need to map a partition with parts appearing twice to a partition having consecutive multiples of 3 as parts.

Now, we consider pairs of parts which are consecutive multiples of 3, and let $\lambda_t = 3m$, $\lambda_{t+1} = 3m-3$ be the largest pair. Also, let $\lambda_s = \lambda_{s+1} = 2k$ be the largest even part appearing twice. Since λ_s is the largest, all parts greater than λ_s differ by at least 3, so we can write parts greater than λ_s as

$$\lambda_{s-i} = \lambda_s + 3i + r_i, \quad r_i \geq 0, \qquad \text{for } i \geq 0. \tag{1}$$

Then $r_0 = r_1 = 0$ since $\lambda_{s-1} = \lambda_s + 3$, and $r_s = \infty$. Let a be the smallest nonnegative integer i satisfying

$$\lambda_s + 3i + k = 3k + 3i < \lambda_{s-i-1} - 3,$$

i.e., $k < r_{i+1}$. Compare $3(k+a)$ and $3m$. We consider two cases: (i) $3(k+a) \geq 3m$ and (ii) $3(k+a) < 3m$.

Case (i): $3(k+a) \geq 3m$. In this case, we define our new partition μ as follows:

$$\mu_{s-i} = \begin{cases} \lambda_{s-i+1}, & \text{if } i \geq -1, \\ \lambda_s + 3i + r_{i+2}, & \text{if } 0 \leq i \leq a-2, \\ \lambda_s + 3i + k, & \text{if } i = a-1, a, \\ \lambda_{s-i}, & \text{otherwise.} \end{cases} \tag{2}$$

Since we removed the part $\lambda_{s+1} = 2k$, our new partition μ has an opposite sign to λ. Also, we can easily check that μ_{s-a} not only is greater than μ_{s-a+1} by exactly 3 but also is a multiple of 3: $\mu_{s-a} = 3(k+a)$ and $\mu_{s-a+1} = 3(k+a) - 3$.

Case (ii): $3(k+a) < 3m$. In this case, as we did above, we first write parts less than λ_t as

$$\lambda_{t+i} = \lambda_t - 3i - \tilde{r}_i, \;\; \tilde{r}_i \geq 0, \qquad \text{for } i \geq 0. \tag{3}$$

Then $\tilde{r}_0 = \tilde{r}_1 = 0$ since $\lambda_{t+1} = \lambda_t - 3$, and $\tilde{r}_{l+1-t} = \infty$. Let us compare $2(m-i)$ and λ_{t+i+1} for the nonnegative integer i, and let b be the smallest i satisfying $2(m-i) \geq \lambda_{t+i+1} + 3$, i.e., $\tilde{r}_{i+1} \geq m - i$. We define μ as follows:

$$\mu_{t+i} = \begin{cases} \lambda_{t+i-1}, & \text{if } i \geq b+2, \\ 2(m-b), & \text{if } i = b+1, \\ \lambda_t - 3i - (m-i), & \text{if } i = b-1, b, \\ \lambda_t - 3i - \tilde{r}_{i+2}, & \text{if } 0 \leq i \leq b-2, \\ \lambda_{t+i}, & \text{otherwise.} \end{cases} \tag{4}$$

Since $\mu_{t+b-1} = 2(m-b)+3$ and $\mu_{t+b} = \mu_{t+b+1} = 2(m-b)$, our new partition μ has exactly one more part appearing twice than λ does.

Now, we show that v is reversible. Assume that we get a new partition μ from Case (i). As we noted, $\mu_{s-a} = 3(k+a)$ and $\mu_{s-a+1} = 3(k+a) - 3$ are a pair of consecutive multiples of 3. Let us show that μ_{s-a} and μ_{s-a+1} are the largest pair of consecutive multiples of 3. Assume that there is a pair μ_i and μ_{i+1} of consecutive multiples of 3 for an $i < s - a$. This implies that $\mu_i > \mu_{s-a}$. However, this is a contradiction, since $\mu_i = \lambda_i$ and $\mu_{s-a} = 3(k+a) \geq \lambda_t = 3m$, i.e., the larger one of the largest consecutive multiples of 3 in λ. So, μ_{s-a} and μ_{s-a+1} are the largest pair of consecutive multiples of 3.

We need to investigate the largest triple with even parts appearing twice in μ; let $\mu_{s'-1} = 2k'+3$ and $\mu_{s'} = \mu_{s'+1} = 2k'$ be the largest triple. Let us write the parts larger than $\mu_{s'}$ as

$$\mu_{s'-i} = \mu_{s'} + 3i + r'_i \tag{5}$$

as we did in (1) and let a' be the smallest integer i satisfying $k' < r'_{i+1}$. What we want to show is that $3(k+a) > 3(k'+a')$. Then we can apply Case (ii).

From the maximality of λ_s and the definition of v, $\lambda_s > \lambda_{s'+1} = \mu_{s'}$, i.e., $2k > 2k'$. In fact, we can see that $2k - 2k' > 3(s'-s)$ since $\mu_{s'} = \lambda_{s'+1}$ and all parts between λ_{s+1} and $\lambda_{s'+1}$ differ by at least 3. From (5),

$$\mu_{s-a} = \mu_{s'} + 3(s'-s+a) + r'_{s'-s+a}.$$

Since $\mu_{s-a} = 3(k+a)$, $\mu_{s'} = 2k'$ and $2k - 2k' \geq 3(s'-s)$, we get

$$\begin{aligned} 3(k+a) &= \mu_{s'} + 3(s'-s) + 3a + r'_{s'-s+a} \\ &\leq \mu_{s'} + 2k - 2k' \; + 3a + r'_{s'-s+a} = 2k + 3a + r'_{s'-s+a}. \end{aligned}$$

This implies that $k \le r'_{s'-s+a}$, and since $k' < k$, we get $k' < r'_{s'-s+a}$. From the definition of a', $a' \le s' - s + a - 1$. So,

$$\mu_{s-a} = \mu_{s'} + 3(s' - s + a) + r'_{s'-s+a} > 2k' + 3a' + k',$$

i.e., $3(k + a) > 3(k' + a')$. Thus the partition μ satisfies the condition for applying Case (ii).

Now, we write parts of μ less than μ_{s-a} as in (3):

$$\mu_{s-a+j} = \mu_{s-a} - 3j - \tilde{r}_j. \tag{6}$$

To determine b, we need to compare $\mu_{s-a+j+1}$ and $2(k + a - j)$. Let us consider the case when $j = 0$. Since $\mu_{s-a+1} = 3(k + a) > 2(k + a)$, b must be greater than 0. Let us suppose that there is an h for $1 \le h < a$ satisfying

$$\mu_{s-a+h+1} + 3 \le 2(k + a - h). \tag{7}$$

From (2) and (6), we get that $\tilde{r}_j = k - r_{a-j+2}$ for $2 \le j \le a$, so (7) becomes $-r_{a-h-1} \ge a - h$; it is impossible since $r_{a-h-1} \ge 0$ and $a > h$. On the other hand, $\mu_{s+1} = \lambda_{s+2}$. From this, $\mu_{s+1} + 3 \le \lambda_s = 2k$. Hence, b is replaced by a. Using (4) we have produced the required partition λ.

Similarly, we can show that λ is also restored when μ is obtained from λ under Case (ii). □

Example. Let us consider some partitions of 38:

$$\begin{aligned} 21+8+4+4+1 &\overset{v}{\longleftrightarrow} 21+8+5+3+1, \\ 15+9+6+6+2 &\overset{v}{\longleftrightarrow} 15+12+9+2. \end{aligned}$$

In the first example, the partition $21 + 8 + 4 + 4 + 1$ has an even part appearing twice and the part 8 is greater than 4+3, so we get $21 + 8 + 5 + 3 + 1$ by adding 1 to the third part 4 from the following part 4. Also, the partition $21 + 8 + 5 + 3 + 1$ has the consecutive odd parts $5, 3$, so by adding 1 to the part 3 from the part 5, we get the partition $21+8+4+4+1$. However, these two partitions $21+8+4+4+1$ and $21 + 8 + 5 + 3 + 1$ have opposite signs. In other words, they are cancelled in $E(38)$.

The partition $15 + 9 + 6 + 6 + 2$ in the second row has even part 6 appearing twice, but the part 9 is greater than 6 by exactly 3, i.e, $s = 3$. Also the parts 9 and 6 are consecutive multiples of 3, i.e., $t = 2$, but $9 + 3 \ge 9$. So, we can apply Case (i); $s = 3$ and $k = 3$. Let us write the part 15 as

$$15 = 6 + 3 \times 2 + 3.$$

Since $r_2(= 3)$ is not greater than $k(= 3)$, a is 2. We add $r_2(= 3)$ to the part 6 from 15, and then $k(= 3)$ to $12(= 15-r_2)$ and 9. Thus we get the partition $15+12+9+2$. On the other hand, the partition $15 + 12 + 9 + 2$ has two parts $15, 12$ which are consecutive multiples of 3; $t = 1$ and $m = 5$. Let us write parts 9 and 2 as

$$\begin{aligned} 9 &= 15 - 2 \times 3 - 0, \\ 2 &= 15 - 3 \times 3 - 4, \end{aligned}$$

i.e., $\tilde{r}_2 = 0$ and $\tilde{r}_3 = 4$. From this, $\tilde{r}_2(= 0) < m - 1 = 4$, but $\tilde{r}_3(= 4) \ge m - 2 = 3$. Thus 2 becomes b. We get the partition $15 + 9 + 6 + 6 + 2$ from (4). They are also cancelled in $E(38)$ due to their signs.

Example. There are 7 partitions of 9 of type E, and they are mapped by the involution v as follows:

$$\begin{aligned} 9 &\longleftrightarrow 9, \\ 8+1 &\longleftrightarrow 8+1, \\ 7+2 &\longleftrightarrow 7+2, \\ 6+3 &\longleftrightarrow 5+2+2, \\ 5+3+1 &\longleftrightarrow 4+4+1. \end{aligned}$$

The first three partitions 9, $8+1$, and $7+2$ have no parts which are even appearing twice, consecutive odds, or consecutive multiples of 3. The set of partitions of 9 of type D consists of the partitions 9, $8+1$, and $7+2$. Hence, $D(9) = E(9)$.

3. Relation between $d_n(q)$ and $e_n(q)$

Now, let us restrict the size of parts. Then some partitions of type E do not have their images under the involution v we constructed in Section 2. We investigate what partitions that are not of type D still remain under v.

Recall the definitions of $d_n(q)$ and $e_n(q)$: the generating functions for all partitions of type D and E whose parts are less than or equal to n, respectively. For convention, we let $d_n = e_n = 1$ for $n \leq 0$.

Theorem 3 (Andrews [1]). *If $d_m(q)$ is defined for all m, then*

$$(8)\quad d_{2n-1}(q) = \sum_{j\geq 0} q^{6nj-6j^2+3j-6\lfloor \frac{n}{3}\rfloor j} \begin{bmatrix} \lfloor \frac{n}{3} \rfloor \\ j \end{bmatrix}_{q^6} e_{2n-6j-1}(q),$$

$$(9)\quad d_{2n}(q) = \sum_{j\geq 0} q^{6nj-6j^2+3j-6\lfloor \frac{n-1}{3}\rfloor j} \begin{bmatrix} \lfloor \frac{n-1}{3} \rfloor \\ j \end{bmatrix}_{q^6} (e_{2n-6j-1}(q) + q^{2n-6j} e_{2n-6j-3}(q)),$$

where

$$\begin{bmatrix} x \\ y \end{bmatrix}_q = \begin{cases} \frac{(q;q)_x}{(q;q)_y (q;q)_{x-y}}, & \text{if } 0 \leq x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

In preparation for the proof of Theorem 3, let $\mathcal{P}(E; 2n-1)$ be a set of partitions of type E with parts less than or equal to $2n-1$. From the definition of v, partitions with consecutive odd parts or equal even parts less than previous parts by at least 4 are cancelled out in $e_{2n-1}(q)$. Hence, we only consider the subset of $\mathcal{P}(E; 2n-1)$ which includes only partitions whose parts differ by at least 3 or which have three consecutive parts $2k+3, 2k, 2k$. In the sequel, we regard $\mathcal{P}(E; 2n-1)$ as the very subset. Among partitions in $\mathcal{P}(E; 2n-1)$, if, under the mapping v, the number of even parts appearing twice increases, then the partitions have their images in $\mathcal{P}(E; 2n-1)$ since their parts would not exceed $2n-1$ under v. Hence, they are cancelled in $e_{2n-1}(q)$; we only consider partitions with even parts appearing twice whose number decreases under v.

Proof of Theorem 3. We only show the case $2n-1$. Let $\mathcal{P}(D; 2n-1)$ be a set of partitions of type D with parts less than or equal to $2n-1$. Let us consider three cases: $n = 3m, 3m+1$ and $3m+2$.

Case (i): $n = 3m$. In this case, equation (8) becomes

$$d_{6m-1} = e_{6m-1} + \sum_{j\geq 1} q^{12mj-6j^2+3j} \begin{bmatrix} m \\ j \end{bmatrix}_{q^6} e_{6m-6j-1}(q). \tag{10}$$

First, let us consider what partitions do not have their images in $\mathcal{P}(E; 6m-1)$. Suppose that a partition $\lambda = \lambda_1\lambda_2\cdots\lambda_l$ is in $\mathcal{P}(E; 6m-1)$ with three consecutive parts $2k+3, 2k, 2k$, and $v(\lambda) = \mu$ such that $\mu_1 = 3k_1+3$ and $\mu_2 = 3k_1$. Then from the definition of v, either $\mu_3 = \lambda_3$ or $\mu_3 = \lambda_1 - 6$. Hence the partition $\mu_3\mu_4\cdots\mu_{l-1}$ is in $\mathcal{P}(E; 6m-7)$. When $k_1 \geq 2m-1$, we can see that $3k_1+3 \geq 6m$. In other words, partitions with these three consecutive parts have no images in $\mathcal{P}(E; 6m-1)$. Also, since $2k_1+3 \leq 6m-1$, $2m-1 \leq k_1 \leq 3m-2$. Let $\mathcal{P}_1$ be the set of all partitions μ satisfying $\mu_1 = 3k_1+3, \mu_2 = 3k_1$ for $2m-1 \leq k_1 \leq 3m-2$ and $\mu_3\cdots\mu_{l-1} \in \mathcal{P}(E; 6m-7)$. Then the generating function for partitions in $\mathcal{P}_1$ is

$$q^{12m-3} \begin{bmatrix} m \\ 1 \end{bmatrix}_{q^6} e_{6m-7}(q).$$

Unfortunately, the set of images of that kind of partition in $\mathcal{P}(E; 6m-1)$ is a subset of $\mathcal{P}_1$. To determine those partitions in $\mathcal{P}_1$ which do not have their image under v in $\mathcal{P}(E; 6m-1)$, let us apply v to a partition $\nu \in \mathcal{P}_1$, where $\nu_1 = 3k_1+3, \nu_2 = 3k_1$.

Let k' be a positive integer such that ν has three consecutive parts $2k'+3, 2k', 2k'$. From the definition of v, if $3(k'+a') \geq 3k_1$, where a' is the same as a in the definition of v, then the image of ν is not in $\mathcal{P}(E; 6m-1)$. In this case, we have to apply Case (i) of v. In other words, if we let σ be the image of ν, then the σ_i are multiples of 3 for $i = 1, 2, 3, 4$, and $\sigma_5\cdots\sigma_{\ell(\nu)-1}$ is in $\mathcal{P}(E; 6m-13)$, where $\ell(\nu)$ is the number of parts of ν. Since $\nu_1 = 3k_1+3$, $\nu_2 = 3k_1$ and $\nu_2 \neq \nu_3$, the part $2k'+3 \leq \nu_3$, so a' must be greater than 1. Let us replace $k'+a'-2$ by k_2. This implies that if $k_1-2 \leq k_2 \leq 3m-5$ and $2m-1 \leq k_1 \leq 3m-2$, then these partitions do not have their images in $\mathcal{P}(E; 6m-1)$. Let us denote the set of all those partitions by $\mathcal{P}_2$, and then calculate the generating function. By substituting k_1 for k_1-2, we find that $2m-3 \leq k_1 \leq k_2 \leq 3m-5$. Using Gaussian coefficients, we deduce that

$$\sum_{k_1,k_2} q^{6(k_1+2)+3+6k_2+3} = q^{12m-3+12m-15} \begin{bmatrix} m \\ 2 \end{bmatrix}_{q^6}.$$

Therefore, the generating function for partitions in $\mathcal{P}_2$ is

$$q^{12m-3+12m-15} \begin{bmatrix} m \\ 2 \end{bmatrix}_{q^6} e_{6m-13}(q),$$

which is the same as the third term on the right side of (10). However, again, some partitions in $\mathcal{P}_2$ do not have their images in $\mathcal{P}_1$. Let us consider the images of that kind of partition $\sigma \in \mathcal{P}_2$. In similar way, we find that if σ has three consecutive parts $2k''+3, 2k'', 2k''$ for $k'' \leq 3m-8$ such that $3(k''+a'') \geq 3k_2$, where a'' is the same as a in the definition of v, then we have to apply Case (i) of v. In other words, the image of σ is not in $\mathcal{P}_2$. Also, since $a'' > 1$, let us replace $k''+a''-2$ by k_3. From this, we deduce that

$$k_2 - 2 \leq k_3 \leq 3m-8 \quad \text{and} \quad 2m-5 \leq k_1 - 4 \leq k_2 - 2 \leq 3m-7. \tag{11}$$

Let us substitute k_1, k_2 for k_1-4, k_2-2, respectively. Then we obtain

$$\sum_{k_1,k_2,k_3} q^{6(k_1+4)+3+6(k_2+2)+3+6k_3+3} = q^{12m-3+12m-15+12m-27} \begin{bmatrix} m \\ 3 \end{bmatrix}_{q^6}.$$

Let $\mathcal{P}_3$ be the set of all partitions ρ such that $\rho_{2i-1} = 3k_i + 3$ and $\rho_{2i} = 3k_i$, where k_i's satisfy (11) for $i = 1, 2, 3$, and $\rho_7 \cdots \rho_{l-3} \in \mathcal{P}(E; 6m-19)$. We obtain the correct generating function for partitions in $\mathcal{P}_3$

$$q^{12m-3+12m-15+12m-27} \begin{bmatrix} m \\ 3 \end{bmatrix}_{q^6} e_{6m-19}(q).$$

Iterating this process, we consider $\mathcal{P}_j$, the set of all partitions such that its $(2i-1)$st part is $3k_i+3$ and $2i$th part is $3k_i$ for $2m-2j+1 \leq k_1 \leq k_2 \cdots \leq k_j \leq 3m-3j+1$, and the remainder of the parts satisfy the condition for $\mathcal{P}(E; 6(m-j)-1)$. The generating function for partitions in $\mathcal{P}_j$ is

$$q^{12mj-6j^2+3j} \begin{bmatrix} m \\ j \end{bmatrix}_{q^6} e_{6m-6j-1}(q).$$

This coincides with the jth term in the sum on the right side of (10). At last, only the partitions of type D with parts less than $6m-1$ remain after cancellation. We have thus completed the proof of (10). Similarly, we can prove the other cases: $n = 3m+1$ and $3m+2$. □

Acknowledgment

The author thanks Bruce C. Berndt for his comments and advice, and also thanks George E. Andrews for his encouragement.

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