

## MAXIMUM PRINCIPLE ON UNBOUNDED DOMAINS FOR SUB-LAPLACIANS: A POTENTIAL THEORY APPROACH

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ABSTRACT. The maximum principle on a wide class of unbounded domains is proved for solutions to the partial differential inequality  $\Delta_{\mathbb{G}} u + cu \geq 0$ , where  $c \leq 0$  and  $\Delta_{\mathbb{G}}$  is a real sub-Laplacian. A potential theory approach is followed.

### 1. INTRODUCTION AND MAIN RESULTS

In recent years, much attention has been paid to the maximum principle on unbounded domains for solutions to the PDE inequality

$$(1.1) \quad \Delta u + cu \geq 0,$$

where  $\Delta$  is the Laplace operator in  $\mathbb{R}^N$ ,  $N \geq 3$ , and  $c$  is a real non-positive function. As is well known, such a principle plays a crucial rôle in looking for symmetry properties of solutions to semilinear Poisson equations, by using moving planes or sliding methods (see [BHM], [BN], [BCN]). In those settings, one of the most commonly used maximum principles can be stated as follows.

**(MP).** *Let  $\Omega \subseteq \mathbb{R}^N$  be an open set whose complement  $\mathbb{R}^N \setminus \Omega$  contains an infinite open cone. Consider a bounded-above function  $u \in C^2(\Omega)$  satisfying inequality (1.1) in  $\Omega$ , and the boundary condition*

$$(1.2) \quad \limsup_{\Omega \ni y \rightarrow x} u(y) \leq 0, \quad \text{for every } x \in \partial\Omega.$$

*Then  $u(x) \leq 0$  for every  $x \in \Omega$ .*

This rather classical result can be proved by using suitable barrier functions in cones, as in [BCN], Lemma 2.1.

Here we would like to show that it can be easily derived from a 1947 theorem by J. Deny, related to the behaviour at infinity of the bounded-above subharmonic functions in  $\mathbb{R}^N$ ,  $N \geq 3$ . Indeed, if  $u$  is a bounded-above  $C^2(\Omega)$ -function satisfying conditions (1.1) and (1.2), then  $v : \mathbb{R}^N \rightarrow \mathbb{R}$ , defined as

$$v(x) := \begin{cases} u(x) & \text{if } x \in \Omega \text{ and } u(x) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

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is a subharmonic function in  $\mathbb{R}^N$ . Then, since  $v$  is bounded above, there exists a Radon measure  $\mu$  in  $\mathbb{R}^N$  such that

$$(1.3) \quad v(x) = V - \int_{\mathbb{R}^N} \Gamma(x - y) \, d\mu(y),$$

where  $V = \sup_{\mathbb{R}^N} v$  and  $\Gamma$  denotes the fundamental solution of the Laplace operator  $\Delta$  in  $\mathbb{R}^N$  (see, e.g., [HK], Theorem 3.20). Then, by Deny’s theorem ([D], p. 142; see also [HK], Theorem 3.21),

$$v(x) \longrightarrow V$$

as  $|x| \rightarrow \infty$  along almost all fixed rays through any given point. As a consequence, since  $\mathbb{R}^N \setminus \Omega$  contains an infinite open cone,  $v(x)$  goes to  $V$  as  $|x| \rightarrow \infty$  along at least some rays contained in  $\mathbb{R}^N \setminus \Omega$ . Since  $v \equiv 0$  in  $\mathbb{R}^N \setminus \Omega$ , this implies  $V = 0$ , so that  $u \leq 0$  in  $\Omega$ . Thus, (MP) is proved.

The extension of the previous arguments to the general setting of the real sub-Laplacians in  $\mathbb{R}^N$  is our aim in the present paper.

Our starting point will be a representation formula for subharmonic functions related to a sub-Laplacian, generalizing (1.3). This result has been proved in [BL]; for the reader’s convenience, we recall it in Section 2, Theorem 2.2.

In order to show our results, we introduce some definitions and notation, referring to the next section for further details. We will denote by  $\Delta_{\mathbb{G}}$  a real sub-Laplacian in  $\mathbb{R}^N$ , by  $\mathbb{G} := (\mathbb{R}^N, \circ)$  its related homogeneous group and by  $(\delta_\lambda)_{\lambda>0}$  the dilations family of  $\mathbb{G}$ . The homogeneous dimension of  $\mathbb{R}^N$  with respect to  $(\delta_\lambda)_{\lambda>0}$  will be denoted by  $Q$ . We assume  $Q \geq 3$ . The operator  $\Delta_{\mathbb{G}}$  has a fundamental solution  $\Gamma(x, \xi) = \Gamma(\xi^{-1} \circ x)$ , where

$$\Gamma(x) = \frac{c_Q}{(d(x))^{Q-2}},$$

$c_Q$  is a suitable positive constant and  $d \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N)$  is a homogeneous norm of degree 1 with respect to  $(\delta_\lambda)_{\lambda>0}$  (see [G]). The application  $(x, \xi) \mapsto d(\xi^{-1} \circ x)$  is a pseudo-distance in  $\mathbb{R}^N$ . We define the  $d$ -ball of center  $x \in \mathbb{R}^N$  and radius  $r > 0$  as

$$D(x, r) := \{y \in \mathbb{R}^N \mid d(x^{-1} \circ y) < r\}.$$

Since  $\Delta_{\mathbb{G}}$  is hypoelliptic, any  $C^\infty(\Omega)$ -function  $u$  such that  $\Delta_{\mathbb{G}}u = 0$  in  $\Omega$  can be called a  $\Delta_{\mathbb{G}}$ -harmonic function in an open set  $\Omega \subseteq \mathbb{R}^N$ . A function  $u : \Omega \rightarrow [-\infty, \infty[$  will be called  $\Delta_{\mathbb{G}}$ -subharmonic if  $u$  is upper semicontinuous,  $u \in L^1_{\text{loc}}(\Omega)$ , and  $\Delta_{\mathbb{G}}u \geq 0$  in the weak sense of distributions.

We can now state our main result, which extends to sub-Laplacians Theorem 3.21 in [HK], a somewhat weaker form of Deny’s result.

**Theorem 1.1.** *Let  $u : \mathbb{R}^N \rightarrow [-\infty, \infty[$  be a bounded-above  $\Delta_{\mathbb{G}}$ -subharmonic function. Then, for every  $q > Q - 2$ , there exists a finite or countable family of closed  $d$ -balls  $\{\overline{D}(x_j, r_j)\}_{j \in J}$  such that*

$$(1.4) \quad \begin{aligned} & \text{(i) } \sum_{j \in J} \left(\frac{r_j}{d(x_j)}\right)^q < \infty; \\ & \text{(ii) setting } \mathcal{D} := \bigcup_{j \in J} \overline{D}_j, \\ & \lim_{d(x) \rightarrow \infty, x \notin \mathcal{D}} u(x) = \sup_{\mathbb{R}^N} u. \end{aligned}$$

In particular,

$$\limsup_{d(x) \rightarrow \infty, x \in \mathbb{R}^N} u(x) = \limsup_{d(x) \rightarrow \infty, x \in F} u(x)$$

for any  $C$ -set for  $\Delta_G$ ,  $F \subseteq \mathbb{R}^N$ .

We agree to define a  $C$ -set for  $\Delta_G$  to be any subset of  $\mathbb{R}^N$  which cannot be covered by a finite or countable family of closed  $d$ -balls  $\overline{D}(x_j, r_j)$  satisfying (i) for some  $q > Q - 2$ .

We will prove this theorem in Section 3. Here we would like to show how it can be applied to obtain maximum principles on unbounded domains. First of all, in light of the second part of Theorem 1.1, we are led to recall a definition which has been largely used in classical and abstract potential theory.

**Definition 1.2.** We say that  $F \subseteq \mathbb{R}^N$  is *not-thin at infinity* for  $\Delta_G$  if

$$(1.5) \quad \limsup_{d(x) \rightarrow \infty, x \in \mathbb{R}^N} u(x) = \limsup_{d(x) \rightarrow \infty, x \in F} u(x)$$

for any bounded-above  $\Delta_G$ -subharmonic function  $u$  in  $\mathbb{R}^N$  (see [B], [CC]).

With Definition 1.2 in hand, the second part of Theorem 1.1 can be stated as follows:

*any  $C$ -set for  $\Delta_G$  is not-thin at infinity for  $\Delta_G$ .*

Not-thin sets at infinity are deeply related to the maximum principle. When we say that the maximum principle for  $\Delta_G$  holds in  $\Omega$ , we mean:

*If  $u : \Omega \rightarrow [-\infty, \infty[$  is a bounded-above  $\Delta_G$ -subharmonic function such that*

$$\limsup_{\Omega \ni y \rightarrow x} u(y) \leq 0, \quad \forall x \in \partial\Omega,$$

*then  $u \leq 0$  in  $\Omega$ .*

Then we have the following result.

**Proposition 1.3.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^N$ . Then, the maximum principle for  $\Delta_G$  holds in  $\Omega$  if and only if  $\mathbb{R}^N \setminus \Omega$  is not-thin at infinity for  $\Delta_G$ .*

*Proof.* Since non-constant  $\Delta_G$ -subharmonic functions in  $\mathbb{R}^N$  do not attain local maxima, (1.5) is equivalent to

$$(1.6) \quad \sup_{\mathbb{R}^N} u = \sup_{\mathbb{R}^N \setminus \Omega} u,$$

for all bounded-above  $\Delta_G$ -subharmonic functions  $u$  in  $\mathbb{R}^N$ . Suppose the maximum principle for  $\Delta_G$  holds in  $\Omega$ . Let  $u$  be a bounded-above  $\Delta_G$ -subharmonic function in  $\mathbb{R}^N$ . Define  $C := \sup_{\mathbb{R}^N \setminus \Omega} u$  and  $v := u - C$ . Then,  $v$  is a bounded-above  $\Delta_G$ -subharmonic function in  $\Omega$  and

$$\limsup_{\Omega \ni y \rightarrow x} v(y) \leq v(x) \leq 0 \quad \forall x \in \partial\Omega.$$

Hence, by the maximum principle,  $v \leq 0$  in  $\Omega$  and (1.6) follows. Vice versa, let  $u$  be a bounded-above  $\Delta_G$ -subharmonic function in  $\Omega$  such that  $\limsup_{y \rightarrow x} u(y) \leq 0$  for every  $x \in \partial\Omega$ . Define

$$v := \begin{cases} \max\{u, 0\} & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

It is a standard matter to show that  $v$  is a bounded-above  $\Delta_{\mathbb{G}}$ -subharmonic function in  $\mathbb{R}^N$  (see, e.g., [B], Satz 1.3.10). Hence, if (1.6) holds, we have

$$\sup_{\mathbb{R}^N} v = \sup_{\mathbb{R}^N \setminus \Omega} v = 0.$$

This yields  $v \leq 0$ , i.e.,  $u \leq 0$  in  $\Omega$ . □

From Theorem 1.1 and the previous proposition, we immediately get a geometric criterion for the maximum principle.

**Corollary 1.4.** *The maximum principle for  $\Delta_{\mathbb{G}}$  holds in the open set  $\Omega$  if  $\mathbb{R}^N \setminus \Omega$  is a C-set for  $\Delta_{\mathbb{G}}$ .*

An easy consequence of this corollary is the following extension to real sub-Laplacians of the maximum principle stated at the beginning of the Introduction.

**Theorem 1.5.** *Let  $\Omega \subseteq \mathbb{R}^N$  be an open set whose complement  $\mathbb{R}^N \setminus \Omega$  is a C-set for  $\Delta_{\mathbb{G}}$ . Let  $c : \Omega \rightarrow \mathbb{R}$ ,  $c \leq 0$ , and  $u$  a bounded-above  $C^2(\Omega)$ -function satisfying*

$$(1.7) \quad \begin{cases} \Delta_{\mathbb{G}}u + cu \geq 0 & \text{in } \Omega, \\ \limsup_{y \rightarrow x} u(y) \leq 0 & \text{for every } x \in \partial\Omega. \end{cases}$$

*Then  $u \leq 0$  in  $\Omega$ .*

*Proof.* Define  $v := \max\{u, 0\}$ . Since  $c \leq 0$ , if  $x \in \Omega$  and  $u(x) > 0$  we have  $\Delta_{\mathbb{G}}v \geq 0$  in a suitable neighborhood of  $x$ . Then  $v$  is  $\Delta_{\mathbb{G}}$ -subharmonic in  $\Omega$  (see [BL]). Moreover, from the boundary condition in (1.7), we have  $\limsup_{y \rightarrow x} v(y) \leq 0$  for every  $x \in \partial\Omega$ . Since  $\mathbb{R}^N \setminus \Omega$  is a C-set for  $\Delta_{\mathbb{G}}$  and  $v$  is bounded above, by Corollary 1.4,  $v \leq 0$  in  $\Omega$ , and the same holds for  $u$ , because  $u \leq v$ . □

In order to give a criterion for a subset of  $\mathbb{R}^N$  to be a C-set for  $\Delta_{\mathbb{G}}$ , we fix the notion of  $\mathbb{G}$ -cone. A subset  $\mathcal{C}$  of  $\mathbb{R}^N$  is called a  $\mathbb{G}$ -cone with vertex at the origin if

$$\delta_{\lambda}\xi \in \mathcal{C}, \quad \forall \xi \in \mathcal{C}, \forall \lambda > 0.$$

If  $\mathcal{C}$  is such a cone, we will call  $\xi_o \circ \mathcal{C} := \{\xi_o \circ \xi \mid \xi \in \mathcal{C}\}$  a  $\mathbb{G}$ -cone with vertex at  $\xi_o$ . By Proposition 4.1 in Section 4, any non-empty open  $\mathbb{G}$ -cone is a C-set for  $\Delta_{\mathbb{G}}$ .

As a consequence, if  $F \subseteq \mathbb{R}^N$  definitively contains a non-empty open  $\mathbb{G}$ -cone, then  $F$  is a C-set for  $\Delta_{\mathbb{G}}$  (we say that  $F$  definitively contains  $\mathcal{C}$  if  $F \supseteq \mathcal{C} \setminus D(0, R)$  for a suitable  $R > 0$ ). By using this criterion, it is easy to show that any half-space of  $\mathbb{R}^N$  is a C-set for  $\Delta_{\mathbb{G}}$  (see Corollary 4.2).

Then, the maximum principle for  $\Delta_{\mathbb{G}}$  and in particular Theorem 1.5 hold on every half-space of  $\mathbb{R}^N$ .

In the special case  $\Delta_{\mathbb{G}} = \Delta_{\mathbb{H}^N}$  (the Kohn Laplacian), this last result has been very recently proved by I. Birindelli and J. Prajapat in [BP], with a completely different technique. This latter is based on a very delicate construction of suitable barrier functions in cones, whose extension to our general setting seems non-trivial.<sup>1</sup>

We would like to stress that maximum principles in half spaces are crucial tools in looking for monotonicity and symmetry properties of solutions to the semilinear equation

$$\Delta_{\mathbb{G}}u + f(u) = 0 \quad \text{in } \mathbb{R}^N$$

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<sup>1</sup>We would like to warmly thank I. Birindelli for making available to us her joint paper with J. Prajapat [BP].

(see [BHM] and [BP] for some recent noteworthy results in the cases  $\Delta$  and  $\Delta_{\mathbb{H}^N}$ , respectively).

In closing, we would like to note that Theorem 1.1 can be used as a starting point in studying the asymptotic behaviour of  $\Delta_{\mathbb{G}}$ -subharmonic functions *not* bounded above in  $\mathbb{R}^N$ . Analogously to the classical Laplacian case, these behaviours seem to have important links with eigenvalue problems for the operator  $\Delta_{\mathbb{G}}$  restricted to the boundary of the ball  $D(0, 1)$  (see [FH]).

### 2. NOTATION AND KNOWN FACTS

The second order differential operator  $\Delta_{\mathbb{G}} = \sum_{k=1}^p X_k^2$  is a *real sub-Laplacian* in  $\mathbb{R}^N$  if  $X_k$  is a smooth-vector field, left-translation-invariant on a homogeneous group  $\mathbb{G} = (\mathbb{R}^N, \circ)$  whose Lie algebra  $\mathfrak{g}$  is nilpotent, stratified and everywhere  $N$ -dimensional. Furthermore, if  $\bigoplus_{k=1}^{\nu} \mathfrak{g}_k$  is the stratification of  $\mathfrak{g}$ , then  $\{X_1, \dots, X_p\}$  is a basis of  $\mathfrak{g}_1$  (see [F]).

Let us denote by  $(\delta_{\lambda})_{\lambda>0}$  the dilations of  $\mathbb{G}$ , defined by

$$(x^{(1)}, x^{(2)}, \dots, x^{(\nu)}) \mapsto \delta_{\lambda}(x) := (\lambda x^{(1)}, \lambda^2 x^{(2)} \dots, \lambda^{\nu} x^{(\nu)}),$$

where  $x^{(j)} \in \mathbb{R}^{N_j}$ , and  $N_j$  are positive integers such that  $N_1 = p$  and  $\sum_{k=1}^{\nu} N_k = N$ . The natural number  $Q := \sum_{k=1}^{\nu} k N_k$  is called the *homogeneous dimension* of  $\mathbb{G}$ . There exists a homogeneous norm  $|\cdot|$  on  $\mathbb{G}$  such that  $\Gamma(x, \xi) = c_Q |x^{-1} \circ \xi|^{2-Q}$  is a fundamental solution for  $\Delta_{\mathbb{G}}$  ( $c_Q$  is a suitable positive constant, see [G]). Hence,  $\Gamma(x, \xi)$  is of class  $C^{\infty}$  for  $x \neq \xi$ , and  $\Delta_{\mathbb{G}}$  is hypoelliptic. We shall also write  $d(\cdot)$  instead of  $|\cdot|$ . The Lebesgue measure on  $\mathbb{R}^N$  is invariant with respect to left and right translations of  $\mathbb{G}$ . If  $K := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N \mid d(x) + d(y) = 1\}$  and  $\mathbf{c} := \max_{(x,y) \in K} d(x \circ y)$ , the following *pseudo-triangular inequality* can be readily proved:

$$(2.1) \quad d(x \circ y) \leq \mathbf{c} (d(x) + d(y)).$$

Let  $\mu$  be a Radon measure in  $\mathbb{R}^N$ . We denote by  $\Gamma_{\mu}$  the  $\Gamma$ -potential of  $\mu$ :

$$\Gamma_{\mu}(x) = \int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x) d\mu(y), \quad x \in \mathbb{R}^N.$$

The following result, proved in [BL], extends a classical result of potential theory to the general setting of real sub-Laplacians.

**Theorem 2.1.** *Let  $u$  be a bounded-above  $\Delta_{\mathbb{G}}$ -subharmonic function in  $\mathbb{R}^N$  with least upper bound  $U$ . Then there exists a Radon measure  $\mu$  on  $\mathbb{R}^N$  such that*

$$(2.2) \quad u(x) = U - \Gamma_{\mu}(x), \quad x \in \mathbb{R}^N.$$

Moreover, if  $n(t) := \mu(D(0, t))$ , we have

$$(2.3) \quad \int_1^{\infty} \frac{n(t)}{t^{Q-1}} dt < \infty.$$

### 3. PROOF OF THEOREM 1.1

The proof is an adaptation of the arguments in [HK], pp. 131-134, to our setting. We shall provide all details, for convenience. From now on,  $\mathbf{c}$  will denote the constant appearing in the pseudo-triangular inequality (2.1).

**Lemma 3.1.** *Let  $\mu$  be a Radon measure in  $\mathbb{R}^N$  such that  $\mu_o := \mu(\mathbb{R}^N) < \infty$ . Let  $q > Q - 2$ . Then, if  $h > 0$ , the set*

$$\{x \in \mathbb{R}^N \mid \Gamma_\mu(x) \geq h\}$$

*can be covered by a finite or countable family of closed balls  $\overline{D}(x_n, r_n)$  such that*

$$\sum_n r_n^q < A(\mu_o/h)^{q/(Q-2)}.$$

*The constant  $A$  depends only on  $Q, q$  and  $\mathbf{c}$ .*

*Proof.* We fix  $\nu \in \mathbb{N}$  and set

$$r_\nu := (\mu_o/h)^{1/(Q-2)} 2^{-\frac{2\nu}{q+Q-2}}.$$

Let

$$\mathcal{D}_\nu := \{\overline{D}_{k,\nu} = \overline{D}(x_{k,\nu}, r_\nu/2) \mid k = 1, \dots, k_\nu\}$$

be a maximal family of disjoint closed balls such that

$$\mu(\overline{D}_{k,\nu}) \geq \frac{\mu_o}{2^\nu}, \quad \forall k = 1, \dots, k_\nu.$$

Since  $\mu_o < \infty$  and the balls are disjoint, then  $k_\nu \leq 2^\nu$ . Define

$$\mathcal{D} := \bigcup_{\nu \in \mathbb{N}} \bigcup_{k \leq k_\nu} \overline{D}(x_{k,\nu}, \mathbf{c} r_\nu).$$

$\overline{D}(x, r_\nu/2)$  does not intersect any ball of the maximal family  $\mathcal{D}_\nu$ . Then, if  $x \notin \mathcal{D}$ ,

$$(3.1) \quad \mu(\overline{D}(x, r_\nu/2)) < \frac{\mu_o}{2^\nu}, \quad \forall \nu \in \mathbb{N}.$$

In particular, we have

$$(3.2) \quad \mu(\{x\}) = \lim_{\nu \rightarrow \infty} \mu(\overline{D}(x, r_\nu/2)) = 0.$$

Hence, if  $x \notin \mathcal{D}$ , by (3.2) we have

$$\begin{aligned} \Gamma_\mu(x) &= \int_{\mathbb{R}^N \setminus \{x\}} \Gamma(x^{-1} \circ y) \, d\mu(y) \\ &= \left( \int_{|x^{-1} \circ y| \geq \frac{r_1}{2}} + \sum_{\nu=1}^{\infty} \int_{\frac{r_{\nu+1}}{2} \leq |x^{-1} \circ y| < \frac{r_\nu}{2}} \right) \Gamma(x^{-1} \circ y) \, d\mu(y) \\ &\leq c_Q (r_1/2)^{2-Q} \mu_o + \sum_{\nu=1}^{\infty} c_Q (r_{\nu+1}/2)^{2-Q} \int_{\frac{r_{\nu+1}}{2} \leq |x^{-1} \circ y| < \frac{r_\nu}{2}} d\mu(y) \\ \text{(by (3.1))} &\leq c_Q 2^{Q-2} \mu_o \sum_{\nu=1}^{\infty} 2^{1-\nu} r_\nu^{2-Q} = c_Q 2^{Q-1} h \sum_{\nu=1}^{\infty} 2^{\frac{Q-2-q}{Q-2+q} \nu} =: A_1 h, \end{aligned}$$

where  $A_1$  depends only on  $Q$  and  $q$ . Then

$$\{x \in \mathbb{R}^N \mid \Gamma_\mu(x) > A_1 h\} \subseteq \bigcup_{\nu \in \mathbb{N}} \bigcup_{k \leq k_\nu} \overline{D}(x_{k,\nu}, \mathbf{c} r_\nu).$$

Moreover, since  $k_\nu \leq 2^\nu$ , we have

$$\sum_{\nu \in \mathbb{N}} \sum_{k \leq k_\nu} (\mathbf{c} r_\nu)^q \leq A_2 \left(\frac{\mu_o}{h}\right)^{q/(Q-2)},$$

where  $A_2$  depends only on  $q, Q$  and  $\mathbf{c}$ . □

*Proof of Theorem 1.1.* Let  $\nu \in \mathbb{N}$  and define

$$C_\nu := \{x \in \mathbb{R}^N \mid (2\mathbf{c})^\nu < |x| \leq (2\mathbf{c})^{\nu+1}\}.$$

By Theorem 2.1, if  $\mu$  is the  $\Delta_{\mathbb{G}}$ -Riesz measure of  $u$  and  $U := \sup_{\mathbb{R}^N} u$ , we have

$$U - u(x) = \Gamma_\mu(x) = I_1(x) + I_2(x) + I_3(x),$$

where

$$\begin{aligned} I_1(x) &:= \int_{|y| \leq (2\mathbf{c})^{\nu-1}} c_Q |x^{-1} \circ y|^{2-Q} \, d\mu(y), \\ I_2(x) &:= \int_{(2\mathbf{c})^{\nu-1} < |y| < (2\mathbf{c})^{\nu+2}} c_Q |x^{-1} \circ y|^{2-Q} \, d\mu(y), \\ I_3(x) &:= \int_{|y| \geq (2\mathbf{c})^{\nu+2}} c_Q |x^{-1} \circ y|^{2-Q} \, d\mu(y). \end{aligned}$$

From now on, we shall denote by  $C, C', \dots$  positive constants depending only on  $Q, \mathbf{c}$  and  $q$ . If  $x \in C_\nu$ , we have

$$\begin{aligned} I_1(x) &\leq c_Q (2\mathbf{c})^{(\nu-1)(2-Q)} \int_{|y| \leq (2\mathbf{c})^{\nu-1}} d\mu(y) \leq C \cdot [(2\mathbf{c})^\nu]^{2-Q} n((2\mathbf{c})^\nu) \\ &\leq \text{(because } n(t) \text{ is increasing)} \quad C' \int_{(2\mathbf{c})^\nu}^\infty \frac{n(t)}{t^{Q-1}} \, dt. \end{aligned}$$

Analogously,

$$\begin{aligned} I_3(x) &\leq c_Q \int_{|y| \geq (2\mathbf{c})^{\nu+2}} \left(\frac{|y|}{2\mathbf{c}}\right)^{2-Q} \, d\mu(y) \leq C \int_{(2\mathbf{c})^\nu}^\infty t^{2-Q} \, dn(t) \\ &= C' \left[\frac{n(t)}{t^{Q-2}}\right]_{(2\mathbf{c})^\nu}^\infty + C'' \int_{(2\mathbf{c})^\nu}^\infty \frac{n(t)}{t^{Q-1}} \, dt \leq C''' \int_{(2\mathbf{c})^\nu}^\infty \frac{n(t)}{t^{Q-1}} \, dt. \end{aligned}$$

The estimate of  $I_2(x)$ ,  $x \in C_\nu$ , is the crucial step of the proof. Let  $q > Q - 2$  be fixed. Define

$$\mu_\nu := \mu(\{y \in \mathbb{R}^N \mid (2\mathbf{c})^{\nu-1} < |y| < (2\mathbf{c})^{\nu+2}\}),$$

$\eta_\nu := \mu_\nu (2\mathbf{c})^{(2-Q)\nu}$  and  $\varepsilon_\nu := \eta_\nu^{1-(Q-2)/q}$ . Then,

$$(3.3) \quad \sum_{\nu=1}^\infty \eta_\nu \leq C \sum_{\nu=1}^\infty \int_{(2\mathbf{c})^{\nu-1}}^{(2\mathbf{c})^{\nu+2}} \frac{dn(t)}{t^{Q-2}} \leq C' \int_1^\infty \frac{dn(t)}{t^{Q-2}} < \infty.$$

This last inequality follows from (2.3). On the other hand, by Lemma 3.1, there exists a countable family of closed balls  $\{\overline{D}(x_{k,\nu}, r_{k,\nu})\}_{k \in J_\nu}$  such that

$$\{x \in C_\nu \mid I_2(x) < \varepsilon_\nu\} \supseteq C_\nu \setminus \bigcup_{k \in J_\nu} \overline{D}(x_{k,\nu}, r_{k,\nu}),$$

and

$$(3.4) \quad \sum_{k \in J_\nu} (r_{k,\nu})^q < A \left(\frac{\mu_\nu}{\varepsilon_\nu}\right)^{q/(Q-2)}.$$

As a consequence,

$$(r_{k,\nu})^q \leq A \left(\frac{\mu_\nu}{\varepsilon_\nu}\right)^{q/(Q-2)} = A \eta_\nu (2\mathbf{c})^{q\nu},$$

so that

$$(3.5) \quad r_{k,\nu} \leq (A\eta_\nu)^{1/q}(2\mathbf{c})^\nu.$$

We may also suppose  $\overline{D}(x_{k,\nu}, r_{k,\nu}) \cap C_\nu \neq \emptyset$  for every  $k \in J_\nu$ . This implies  $|x_{k,\nu}| \geq (2\mathbf{c})^{\nu-2}$ . Indeed,

$$\begin{aligned} |x_{k,\nu}| &\geq \text{(see (3.5)) } \frac{1}{\mathbf{c}}(2\mathbf{c})^\nu - (A\eta_\nu)^{1/q}(2\mathbf{c})^\nu \\ &= (2\mathbf{c})^\nu \left( \frac{1}{\mathbf{c}} - (A\eta_\nu)^{1/q} \right), \end{aligned}$$

and, since  $\eta_\nu \rightarrow 0$  as  $\nu \rightarrow \infty$  (see (3.3)), the claim follows. As a consequence, from (3.3), (3.4), the choice of  $\varepsilon_\nu$  and the bound for  $|x_{k,\nu}|$ , we get

$$(3.6) \quad \sum_{\nu=1}^\infty \sum_{k \in J_\nu} \left( \frac{r_{k,\nu}}{|x_{k,\nu}|} \right)^q \leq A \sum_{\nu=1}^\infty \frac{(2\mathbf{c})^{\nu q}}{(2\mathbf{c})^{(\nu-2)q}} \eta_\nu = C \sum_{\nu=1}^\infty \eta_\nu < \infty.$$

Collecting the estimates for  $I_1, I_2$  and  $I_3$ , we finally obtain

$$(3.7) \quad U - u(x) < C \int_{(2\mathbf{c})^\nu}^\infty \frac{n(t)}{t^{Q-1}} dt + \eta_\nu^{1-(Q-2)/q},$$

for every  $x \in C_\nu \setminus \bigcup_{k \in J_\nu} \overline{D}(x_{k,\nu}, r_{k,\nu})$  and for every  $\nu \in \mathbb{N}$ . By (2.3) and the positivity of the exponent of  $\eta_\nu$ , the right-hand side of (3.7) goes to zero as  $\nu \rightarrow \infty$ . Together with (3.6), this proves the first part of the theorem. Finally, let  $F \subseteq \mathbb{R}^N$  be a C-set for  $\Delta_{\mathbb{G}}$ . Then, with the previous notation, for a suitable  $q > Q - 2$ , there exists a sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $F$  such that  $d(x_n) \rightarrow_n \infty$  and  $x_n \notin \mathcal{D}$ , for all  $n$ . Consequently, we have

$$U \geq \limsup_{x \rightarrow \infty} u(x) \geq \limsup_{F \ni x \rightarrow \infty} u(x) \geq \lim_{n \rightarrow \infty} u(x_n) = U,$$

which completes the proof. □

#### 4. SOME EXAMPLES OF A C-SET FOR $\Delta_{\mathbb{G}}$

**Corollary 4.1.** *Any non-empty open  $\mathbb{G}$ -cone is a C-set for  $\Delta_{\mathbb{G}}$ .*

*Proof.* Let  $\mathcal{C}$  be a non-empty open  $\mathbb{G}$ -cone with vertex at the origin (this is not restrictive). We set  $h := 2(\mathbf{c}^4 + 1)$  and define

$$A_k := \{ \xi \in \mathcal{C} \mid h^k < d(\xi) < 2h^k \}, \quad k \in \mathbb{N}.$$

We shall prove that  $\mathcal{E} := \bigcup_k A_k$  is a C-set for  $\Delta_{\mathbb{G}}$ . Arguing by contradiction, we assume the existence of a countable family  $\{D_j := \overline{D}(x_j, r_j)\}_j$  of closed balls satisfying (i) of Theorem 1.1, with  $q = Q$  and such that  $\mathcal{E} \subseteq \bigcup_j D_j$ . Given  $\varepsilon > 0$ , there exists  $j^* = j^*(\varepsilon)$  such that  $r_j < \varepsilon \cdot d(x_j)$ , for all  $j \geq j^*$ . We may choose  $\varepsilon$  so small that  $J_k \neq J_h$  for every  $k \neq h$ . Here we have set  $J_k := \{j \geq j^* \mid A_k \cap D_j \neq \emptyset\}$ . Finally, let  $k^* = k(j^*)$  be such that  $\bigcup_{k \geq k^*} A_k \subseteq \bigcup_{j \geq j^*} D_j$ . Then, since  $A_k = \delta_{h^k}(A_0)$ , denoting the Lebesgue measure by  $m$ , we have

$$h^{kQ} \cdot m(A_0) = m(A_k) \leq \sum_{j \in J_k} m(D_j) = C \sum_{j \in J_k} (r_j)^Q, \quad \forall k \geq k^*.$$



On the other hand, if  $D_j \cap A_k \neq \emptyset$ , then  $d(x_j) \leq Ch^k$  for a suitable  $C > 0$  independent of  $j$  and  $k$ . It follows that

$$m(A_0) \leq C \sum_{j \in J_k} \left( \frac{r_j}{d(x_j)} \right)^Q, \quad \forall k \geq k^*.$$

Since  $A_0$  is open and non-empty,  $m(A_0) > 0$ . Then

$$\sum_{j \in J} \left( \frac{r_j}{d(x_j)} \right)^Q \geq \sum_{k \geq k^*} \sum_{j \in J_k} \left( \frac{r_j}{d(x_j)} \right)^Q = \infty,$$

in contradiction with the assumption  $\sum_{j \in J} (r_j/d(x_j))^Q < \infty$ . □

**Corollary 4.2.** *Any half-space  $\pi$  of  $\mathbb{R}^N$  is a  $C$ -set for  $\Delta_{\mathbb{G}}$ .*

*Proof.* It suffices to prove that any half-space  $\pi$  of  $\mathbb{R}^N$  definitively contains an open  $\mathbb{G}$ -cone. Let

$$\pi = \{x \in \mathbb{R}^N \mid f(x) := \sum_{i=1}^N a_i \cdot x_i \geq \alpha \},$$

where  $a = (a_1, \dots, a_N) \in \mathbb{R}^N \setminus \{0\}$  and  $\alpha \in \mathbb{R}$ . We consider the set

$$Q_\varepsilon := \{a + \xi \mid -\varepsilon < \xi_i < \varepsilon, \forall i = 1, \dots, N \}.$$

The assertion follows if we prove the existence of  $\varepsilon > 0$  and  $M > 0$  such that

$$\delta_\lambda \zeta \in \pi, \quad \forall \zeta \in Q_\varepsilon, \quad \forall \lambda \geq M.$$

With the notation of Section 2, for all  $\zeta \in Q_\varepsilon$  we have

$$f(\delta_\lambda \zeta) = \sum_{i=1}^\nu \sum_{j=1}^{N_i} a_j^{(i)} \cdot \lambda^i (a_j^{(i)} + \xi_j^{(i)}) \geq \sum_{i=1}^\nu \lambda^i \sum_{j=1}^{N_i} (a_j^{(i)})^2 - \sum_{i=1}^\nu \lambda^i \sum_{j=1}^{N_i} |a_j^{(i)}| \cdot |\xi_j^{(i)}|.$$

We now set  $\varepsilon := \min_{i,j} \{ |a_j^{(i)}|/2 : a_j^{(i)} \neq 0 \}$ . If  $|\xi_j^{(i)}| < \varepsilon$  for every  $i$  and  $j$ , we have

$$f(\delta_\lambda \zeta) \geq \sum_{i=1}^\nu \lambda^i \sum_{j=1}^{N_i} |a_j^{(i)}|^2/2.$$

Finally, if  $\lambda \geq M := \max\{1, 2\alpha/\sum_i a_i^2\}$ , we have  $f(\delta_\lambda \zeta) \geq \lambda/2 \cdot \sum_i a_i^2 \geq \alpha$ , and the assertion is proved. □

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