

EQUIVARIANT RESOLUTION OF POINTS OF INDETERMINACY

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ABSTRACT. We prove an equivariant form of Hironaka’s theorem on elimination of points of indeterminacy. Our argument uses canonical resolution of singularities and an extended version of Sumihiro’s equivariant Chow lemma.

1. INTRODUCTION

Throughout this note we shall work over an algebraically closed field k of characteristic 0. All algebraic varieties, schemes, groups, and all maps between them will be defined over k .

Let G be a linear algebraic group (not necessarily connected). The main objects of interest for us will be algebraic varieties with a G -action; we will refer to them as G -varieties. A G -equivariant morphism between two such varieties will be called a morphism of G -varieties. The terms “rational map of G -varieties”, “birational morphism of G -varieties”, “birational isomorphism of G -varieties”, etc., are defined in a similar manner.

Hironaka’s theorem on elimination of points of indeterminacy (see [Hi, §0.5, Question E and Main Theorem II]) asserts that every rational map $f: X \dashrightarrow Y$ can be resolved into a regular map by a sequence of blowups $\pi: X_m \rightarrow \dots \rightarrow X_0 = X$ with smooth centers. In other words, π can be chosen so that the composition $f\pi$ is regular. The purpose of this paper is to prove the following equivariant version of this result.

Theorem 1. *Let $f: X \dashrightarrow Y$ be a rational map of G -varieties, where Y is complete. Then there is a sequence of blowups*

$$\pi: X_m \rightarrow X_{m-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 = X$$

with smooth G -invariant centers such that the composition $f\pi$ is regular.

Our proof will rely on canonical resolution of singularities. Along the way we prove an equivariant form of Chow’s lemma (Proposition 2), generalizing a theorem of Sumihiro ([Su, Theorem 2]). For applications of Theorem 1, see [RY1, Remark 5.4] and [RY2, Remark 6.3].

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2. EQUIVARIANT CHOW LEMMA

In this section we will prove the following generalization of Chow’s lemma.

Proposition 2. *For every G -variety X , there exist a quasiprojective G -variety Z and a proper birational morphism $Z \rightarrow X$. If X is complete, then Z is projective.*

Note that if G is assumed to be connected, this result is a well-known theorem of Sumihiro [Su, Theorem 2]; see also [PV, Theorem 1.3]. The argument below reduces the general case to the case where G is connected.

Proof. The second assertion is an immediate consequence of the first: if X is complete, $Z \rightarrow X$ is proper and Z is quasiprojective, then Z is also complete and, hence, projective.

To prove the first assertion, let G_0 be the connected component of G . Applying Sumihiro’s theorem to X , viewed as a G_0 -variety, yields a quasiprojective G_0 -variety Y and a proper G_0 -equivariant birational morphism $f: Y \rightarrow X$.

For the rest of this proof we shall use set-theoretic notation: by a “point” we will always mean a closed point.

Recall that the homogeneous fiber product $G *_G Y$ is the G -variety defined as the geometric quotient $(G \times Y)/G_0$ for the action of G_0 given by $g_0(g, y) = (gg_0^{-1}, g_0y)$, where $g \in G$, $g_0 \in G_0$ and $y \in Y$; see [PV, Section 4.8]. We shall write $[g, y]$ for the element of $G *_G Y$ represented by $(g, y) \in G \times Y$. Since G_0 has finite index in G , $G *_G Y$ admits a more concrete description as a disjoint union of $|G/G_0|$ copies of Y . More precisely, if we choose a representative a_h for each $h \in G/G_0$, we can explicitly identify $G/G_0 \times Y$ and $G *_G Y$ as abstract varieties, via $(h, y) \mapsto [a_h, y]$. Moreover, if we define a G -action on $G/G_0 \times Y$ by $g(h, y) \mapsto (\bar{g}h, (a_h^{-1}g^{-1}a_{\bar{g}h})y)$, then $(h, y) \mapsto [a_h, y]$ identifies $G/G_0 \times Y$ and $G *_G Y$ as G -varieties. Here \bar{g} is the image of g in G/G_0 , and $(a_h^{-1}g^{-1}a_{\bar{g}h})y$ is well-defined because $a_h^{-1}g^{-1}a_{\bar{g}h}$ is an element of G_0 .

Let $\alpha: G *_G Y \rightarrow X$ and $\beta: G *_G Y \rightarrow G/G_0$ be the maps of G -varieties given by $\alpha: [g, y] \mapsto gf(y)$ and $\beta: [g, y] \mapsto \bar{g}$. (Here G acts on G/G_0 by left multiplication.) These maps are shown in the diagram below:

$$\begin{array}{ccc}
 & G *_G Y \simeq G/G_0 \times Y & \\
 \alpha \swarrow & & \searrow \beta \\
 X & & G/G_0
 \end{array}$$

Let S be the set of all sections s of β . Note that if we identify $G *_G Y$ with $G/G_0 \times Y$ as above, then $\beta: G/G_0 \times Y \rightarrow G/G_0$ is the projection to the first factor. Thus $S \simeq Y^{|G/G_0|}$ as an abstract variety. Moreover, since β is G -equivariant, G acts on this variety by $g: s \mapsto t$, where $s, t \in S$ and $t(h) = g \cdot s(\bar{g}^{-1}h)$ for any $h \in G/G_0$.

Let Z be the closed G -invariant subvariety of S consisting of those sections $s: G/G_0 \rightarrow G *_G Y \simeq G/G_0 \times Y$ with the property that $\alpha \circ s(G/G_0)$ is a single point of X ; we shall denote this point by x_s .

We claim that the morphism $\phi: Z \rightarrow X$ given by $s \rightarrow x_s$ has the properties asserted in the proposition. Indeed, since Y is quasiprojective, and Z is a closed subvariety of $S \simeq Y^{|G/G_0|}$, Z is quasiprojective as well.

To show that ϕ is a birational morphism, assume the birational morphism $f: Y \rightarrow X$ is an isomorphism over a dense open subset $U \subset X$. Then $V =$

$\bigcap_{h \in G/G_0} a_h U$ is also a dense open subset of X , and for every $x \in V$, $\alpha^{-1}(x) = \{[a_h, f^{-1}(a_{h^{-1}}x)] : h \in G/G_0\}$; it is the image of the unique section $s_x \in S$ satisfying $x_{s_x} = x$. This section is given by $s_x(h) = [a_h, f^{-1}(a_{h^{-1}}x)]$, and the morphism $V \rightarrow Z, x \mapsto s_x$, is a two-sided rational inverse to ϕ . \square

3. BIRATIONAL MORPHISMS AS BLOWUPS

Our proof of Theorem 1 in the next section will rely on the following equivariant analogue of [Ha, Theorem 7.17].

Proposition 3. *Let $f: X' \rightarrow X$ be a birational proper morphism of G -varieties, where X is smooth and X' is quasiprojective. Then there exists a G -invariant sheaf of ideals \mathcal{I} on X such that X' is the blowup of \mathcal{I} .*

Proof. Let $\sigma: G \times X \rightarrow X$ be the given action of G on X and $\text{pr}_2: G \times X \rightarrow X$ the projection onto the second factor.

By a theorem of Kambayashi [Ka], there exists an action of G on the projective space \mathbb{P}^n (via a representation $G \rightarrow \text{PGL}_{n+1}$) and a G -equivariant embedding $X' \hookrightarrow \mathbb{P}^n$; this yields a G -equivariant embedding $i: X' \hookrightarrow \mathbb{P}^n \times X$.

Here $\mathbb{P}^n \times X$ is a projective space over X ; set $\mathcal{L} = i^* \mathcal{O}_{\mathbb{P}^n \times X}(1)$ and $\mathcal{T} = \bigoplus_{d=0}^{\infty} f_*(\mathcal{L}^d)$, where $\mathcal{L}^0 = \mathcal{O}_{\mathbb{P}^n \times X}$. Let $\mathcal{T}_1 = f_* \mathcal{L}$ be the component of \mathcal{T} of degree one. The action of G on $\mathbb{P}^n \times X$ yields a G -linearization of the sheaf \mathcal{T}_1 , i.e., an isomorphism $\sigma^* \mathcal{T}_1 \simeq \text{pr}_2^* \mathcal{T}_1$ which satisfies the same cocycle condition as in the definition of G -linearization of an invertible sheaf (see, e.g., [MFK, Definition 1.6]); informally speaking, G acts on the pair (X, \mathcal{T}_1) .

We refer to the proof of [Ha, Theorem 7.17] for the following facts:

- (1) After replacing the embedding i by its e -fold embedding for some positive integer e (thus replacing \mathcal{L} by \mathcal{L}^e), we may assume that the graded \mathcal{O}_X -algebra \mathcal{T} is generated by \mathcal{T}_1 .
- (2) $X' \simeq \text{Proj } \mathcal{T}$.
- (3) Assume \mathcal{T} is generated by \mathcal{T}_1 as in (1). If there are an invertible sheaf \mathcal{M} on X and a sheaf of ideals \mathcal{I} on X such that $\mathcal{I} \simeq \mathcal{T}_1 \otimes \mathcal{M}$, then X' is isomorphic to the blowup of \mathcal{I} .

The variety X is smooth, and hence, for any sheaf of ideals \mathcal{F} on X of rank one without torsion, its dual $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is an invertible sheaf. (To see this, note that locally at any point $x \in X$, the generator of \mathcal{F}_x^* is given by the homomorphism $\mathcal{F}_x \rightarrow \mathcal{O}_{X,x}$ which maps the generators of \mathcal{F}_x , as an $\mathcal{O}_{X,x}$ -module of rank one, into elements of $\mathcal{O}_{X,x}$ not having a nontrivial common multiple; such a homomorphism is unique up to an invertible multiple, since the local ring $\mathcal{O}_{X,x}$ is regular, and hence, factorial.)

Thus the second dual \mathcal{T}_1^{**} is an invertible sheaf, and we have an embedding $\mathcal{T}_1 \hookrightarrow \mathcal{T}_1^{**}$. The G -linearization of \mathcal{T}_1 yields a G -linearization of \mathcal{T}_1^{**} , and the above embedding is, in fact, an embedding of G -linearized sheaves. Taking $\mathcal{M} = (\mathcal{T}_1^{**})^{-1}$, we see that X' is isomorphic to the blowup of the sheaf of ideals $\mathcal{I} = \mathcal{T}_1 \otimes (\mathcal{T}_1^{**})^{-1}$.

The G -linearizations of \mathcal{T}_1 and \mathcal{T}_1^{**} yield a G -linearization of \mathcal{I} and a G -linearized embedding $\mathcal{I} = \mathcal{T}_1 \otimes (\mathcal{T}_1^{**})^{-1} \hookrightarrow \mathcal{T}_1^{**} \otimes (\mathcal{T}_1^{**})^{-1} = \mathcal{O}_X$. This shows that \mathcal{I} is a G -invariant sheaf of ideals on X . \square

4. PROOF OF THEOREM 1

First of all, we may assume without loss of generality that X is smooth. Indeed, let

$$(1) \quad X_l \xrightarrow{\pi_l} \dots \xrightarrow{\pi_1} X_0 = X$$

be the canonical resolution of singularities of X , as in [V, Theorem 7.6.1] or [BM, Theorem 13.2]. Here X_i is smooth, the centers $C_i \subset X_i$ are smooth and G -invariant, and the action of G lifts to the entire resolution sequence (1). Replacing X by X_l , we may assume that X is smooth.

Now let Γ be the closure of the graph of $f: X \dashrightarrow Y$ in $X \times Y$. By Proposition 2 there exists a proper birational morphism $\phi: Z \rightarrow \Gamma$ of G -varieties, where Z is quasi-projective. Thus we have the following diagram of G -equivariant maps:

$$\begin{array}{ccc} & Z & \\ & \downarrow \phi & \\ & \Gamma & \\ \alpha \swarrow & & \searrow \beta \\ X & \xrightarrow{f} & Y, \end{array}$$

where the maps $\alpha: \Gamma \rightarrow X$ and $\beta: \Gamma \rightarrow Y$ are the natural projections to the first and the second factors. Since Y is complete, α is a proper birational morphism, and consequently so is $\alpha\phi: Z \rightarrow X$. Proposition 3 now says that $\alpha\phi$ is the blowup of a G -invariant ideal sheaf \mathcal{I} on X . By a theorem of Hironaka [Hi, Main Theorem II] there is a sequence

$$\pi: X_m \xrightarrow{\pi_n} X_{n-1} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X$$

of blowups with smooth centers such that $\pi^*\mathcal{I}$ is locally principal. Moreover, by a theorem of Bierstone and Milman [BM, Theorem 1.10] the sequence π can be chosen in a canonical way; in particular, the center of each π_i can be chosen to be smooth and G -invariant; see [BM, Remark 1.5]. Now by the universal property of blowing up (see [Ha, Proposition II.7.14]), the composition $(\alpha\phi)^{-1}\pi: X_m \rightarrow Z$ is a regular map. Hence, $f\pi = \beta\phi(\alpha\phi)^{-1}\pi: X_m \rightarrow Y$ is also regular, as claimed. \square

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