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PERTURBATIONS OF EXISTENCE FAMILIES FOR ABSTRACT CAUCHY PROBLEMS

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Abstract. In this paper, we establish Desch-Schappacher type multiplicative and additive perturbation theorems for existence families for arbitrary order abstract Cauchy problems in a Banach space: $u^{(n)}(t) = Au(t)$ $(t \geq 0)$; $u^{(j)}(0) = x_j$ $(0 \leq j \leq n-1)$. As a consequence, we obtain such perturbation results for regularized semigroups and regularized cosine operator functions. An example is also given to illustrate possible applications.

1. Introduction

Notations. N, R, R^+, and $\mathbf{C}$ denote the positive integers, the real numbers, the nonnegative real numbers, and the complex numbers respectively. For Banach spaces X and Y, $\mathbf{L}(X,Y)$ denotes the space of all bounded linear operators from X to Y, and $\mathbf{L}(X) := \mathbf{L}(X,X)$. For a closed linear operator A, we write $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\rho(A)$ for the domain, the range, and the resolvent set of A respectively, and $[\mathcal{D}(A)]$ for the Banach space $\mathcal{D}(A)$ with the graph norm $\|x\|_{[\mathcal{D}(A)]} := \|x\| + \|Ax\|$. For $k \in N \cup \{0\}$, $C^k(R^+, X)$ is the set of all k-times continuously differentiable X-valued functions in R^+, and $C(R^+, X) := C^0(R^+, X)$. M and ω will be positive constants. For a function $F : (\omega, \infty) \to \mathbf{L}(X)$, $F \in LT - \mathbf{L}(X)$ means that there exists a strongly continuous and exponentially bounded function $H(\cdot) : R^+ \to \mathbf{L}(X)$ such that $F(\lambda)x = \int_0^\infty e^{-\lambda t} H(t)x dt$ $(\lambda > \omega,\ x \in X)$.

Let X be a Banach space, and let $C \in \mathbf{L}(X)$ be injective. Based on Lions [15] and his own paper [2], Da Prato [3] introduced the C-regularized semigroup on X in 1966. Since Davies and Pang ([4]) rediscovered it in 1987, this semigroup has been extensively investigated (cf., e.g., [6, 7, 11, 12, 17, 21, 22]), because it can be used to deal with many ill-posed (in the classical sense) abstract Cauchy problems for which the strongly continuous semigroup is not applicable. In 1991, a new type of operator family, called the existence family, for controlling the first order abstract Cauchy problem, which is more general than the C-regularized semigroup, was introduced and discussed by deLaubenfels [5] (see also [6]). It proves to be more flexible in applications, because the existence family does not require commutativity

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among itself, its generator, and the regularizing operator (cf. [5, 6]). The present paper is concerned with this type of operator family. Our focus is to study the Desch-Schappacher type multiplicative and additive perturbations for the existence family. To make the results more meaningful, we will carry out our discussion in a more general setting. Explicitly, we will study the multiplicative and additive perturbations for existence families for arbitrary order abstract Cauchy problems:

$$u^{(n)}(t) = Au(t) \ (t \geq 0), \quad u^{(j)}(0) = x_j \ (0 \leq j \leq n-1), \tag{1.1}$$

where $n \in N$, and A is a closed linear operator in X.

Definition 1.1. The strongly continuous family of operators $\{S(t)\}_{t\geq 0} \subset \mathbf{L}(X)$ is called a C-existence family for (1.1), if for all $x \in X$ and $t \geq 0$ we have $S(\cdot)x \in C^{n-1}(R^+, X)$, $A\int_0^t S(s)xds \in C(R^+, X)$, and

$$S(t)x = \frac{t^{n-1}}{(n-1)!}Cx + A\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}S(s)xds. \tag{1.2}$$

We also say that (1.1) has a C-existence family $\{S(t)\}_{t\geq 0}$.

It is known from [6, Chapter III] that the C-existence family reduces to a C-regularized semigroup when $n = 1$ and $S(t)A \subset AS(t)$ $(t \geq 0)$. Moreover, letting $n = 2$ and $S(t)A \subset AS(t)$ in Definition 1.1 gives the C-regularized cosine operator function $\{S'(t)\}_{t\geq 0}$.

The Desch-Schappacher perturbations were firstly studied in [8] for classical strongly continuous semigroups in 1989. In recent years, this type of perturbations has drawn many researchers' attention, and the related theory has been gaining much development (cf., e.g., Engel and Nagel [10, Section III.3], [1, 7, 9, 13, 18, 19, 20], and references therein). In [9], Diekmann, Gyllenberg and Thieme showed a new viewpoint of perturbations of Desch-Schappacher type, by solving Stieltjes' renewal equations with the basic assumption on the behaviour of the semivariation of the step response function (see also [19]). In [13], Jung investigated how certain properties, like analyticity or norm continuity, of the original semigroup are inherited by the perturbed semigroup. In [7, Section V] by deLaubenfels and Yao, nonlinear additive perturbations of this type for C-regularized semigroups were discussed, and a local existence and uniqueness theorem on the classical solutions of the Cauchy problem for the associated perturbed equation was given. Moreover, in [20, 1, 18], one can see results about such perturbations for classical strongly continuous cosine operator functions, and for solution families or n-times integrated solution families of linear Volterra equations.

In this paper, we will present Desch-Schappacher type multiplicative and additive perturbation theorems for the general existence family given by Definition 1.1, and show the uniqueness of solutions for the corresponding perturbed (1.1) (Theorems 2.1 and 2.2). As a consequence, we obtain Desch-Schappacher type perturbation theorems for regularized semigroups and regularized cosine operator functions (Corollaries 2.3, 2.4 and 2.6), which recover the corresponding results in [5, 6, 8, 19, 20] (see Remarks 2.5 and 2.7). With a new observation of the ranges of perturbation operators, we exhibit in Theorem 2.8 two classes of perturbation operators satisfying the conditions of Theorem 2.1 or Theorem 2.2. Finally, an example (Example 2.9) is given to illustrate possible applications. This example also reflects the features of Theorem 2.8 (see Remark 2.10).

The following characterization and properties of exponentially bounded existence families will be used in the sequel. By a solution of (1.1), we mean a function $u(\cdot) \in C^n(R^+, X) \cap C(R^+, [\mathcal{D}(A)])$ satisfying (1.1).

Proposition 1.2 ([23]). *Let $\lambda^n - A$ be injective for $\lambda > \omega$. Then (1.1) has a C-existence family $\{S(t)\}_{t \geq 0}$ on X with $\left\|S^{(n-1)}(t)\right\| \leq Me^{\omega t}$ $(t \geq 0)$ if and only if $\mathcal{R}(C) \subset \mathcal{R}(\lambda^n - A)$ for $\lambda > \omega$, and the function $\lambda \mapsto \lambda^{n-1}(\lambda^n - A)^{-1} C \in LT - \mathbf{L}(X)$; in this case, for $x_j \in \mathcal{D}(A)$ with $Ax_j \in \mathcal{R}(C)$ $(0 \leq j \leq n-1)$, (1.1) admits a solution $u(\cdot)$ satisfying*

$$\left\|u^{(n)}(t)\right\|, \ \|u(t)\|_{[\mathcal{D}(A)]} \leq Me^{\omega t} \sum_{i=0}^{n-1} \left(\|u_i\| + \left\|C^{-1}Au_i\right\|\right), \quad t \geq 0,$$

and

$$\lambda^{n-1}(\lambda^n - A)^{-1} Cx = \int_0^\infty e^{-\lambda t} S^{(n-1)}(t) x dt, \quad x \in X, \ \lambda > \omega. \tag{1.3}$$

2. Results and proofs

Theorem 2.1. *Let A and $\{S(t)\}_{t \geq 0}$ be as in Proposition 1.2, and let α, $\beta \in \mathbf{C}$. Suppose $B \in \mathbf{L}(X)$ and $\mathcal{R}(B) \subset \mathcal{R}(C)$. If for every $f \in C(R^+, X)$ and $t \geq 0$,*

$$\left\|A \int_0^t S(t-s)C^{-1}Bf(s)ds\right\| \leq M \int_0^t e^{\omega(t-s)} \|f(s)\| ds, \tag{2.1}$$

then

(i) *the Cauchy problem*

$$u^{(n)}(t) = (A(I + \alpha B) + \beta B)u(t) \ (t \geq 0), \quad u^{(j)}(0) = x_j \ (0 \leq j \leq n-1) \tag{2.2}$$

has a C-existence family $\{U(t)\}_{t \geq 0}$ on X and $\left\|U^{(n-1)}(\cdot)\right\|$ is exponentially bounded; and

(ii) *all solutions of (2.2) are unique, provided $CA \subset AC$.*

Proof. Fixing $f \in C(R^+, X)$, by (2.1) and (1.2) we obtain that for $0 \leq t_2 \leq t_1 < \infty$,

$$\begin{aligned}
& \left\|A \int_0^{t_1} S(t_1 - s)C^{-1}Bf(s)ds - A \int_0^{t_2} S(t_2 - s)C^{-1}Bf(s)ds\right\| \\
\leq \ & \left\|A \int_0^{t_1} S(t_1 - s)C^{-1}B\left[f(s) - f(s - t_1 + t_2)\right] ds\right\| \\
& + \left\|A \int_0^{t_1 - t_2} S(t_1 - s)C^{-1}Bf(s - t_1 + t_2)ds\right\| \\
\leq \ & Me^{\omega t_1} \max_{0 \leq s \leq t_1} \|f(s) - f(s - t_1 + t_2)\| + \left\|A \int_{t_2}^{t_1} S(s)C^{-1}Bf(0)ds\right\| \\
\leq \ & Me^{\omega t_1} \max_{0 \leq s \leq t_1} \|f(s) - f(s - t_1 + t_2)\| + \|[S^{(n-1)}(t_1) - S^{(n-1)}(t_2)]C^{-1}Bf(0)\|,
\end{aligned}$$

where $f(-s) := f(0)$ for $s > 0$; this implies that the function

$$t \mapsto A \int_0^t S(t-s)C^{-1}Bf(s)ds \in C(R^+, X)$$

($f \in C(R^+, X)$). We set $W_0(t) = S^{(n-1)}(t)$ ($t \geq 0$) and define $W_n(t)$ inductively by

$$W_n(t)x = (\beta + \alpha A)\int_0^t S(t-s)C^{-1}BW_{n-1}(s)xds, \quad x \in X,\ t \geq 0,\ n \in N. \tag{2.3}$$

Clearly, $\{W_n(t)\}_{t\geq 0}$ is a strongly continuous family of bounded linear operators on X, for each $n \in N$. We know by hypothesis that $S(\cdot)$ and $W_0(\cdot)$ are exponentially bounded. So using (2.1) we get by induction that $\|W_n(t)\| \leq M_1^{n+1}e^{\omega_1 t}\frac{t^n}{n!}$ ($t \geq 0$, $n \in N \cup \{0\}$), for certain constants $M_1 > M$, $\omega_1 > \omega$. Define $W(t) := \sum_{n=0}^{\infty} W_n(t)$ ($t \geq 0$). We see by the above arguments that the series converges in the uniform operator topology, uniformly for bounded intervals of R^+, $\|W(t)\| \leq M_1 e^{(\omega_1+M_1)t}$ ($t \geq 0$), and hence $\{W(t)\}_{t\geq 0} \subset \mathbf{L}(X)$ is a strongly continuous family. Thus, by (2.3) we have

$$W(t)x = S^{(n-1)}(t)x + (\beta + \alpha A)\int_0^t S(t-s)C^{-1}BW(s)xds, \quad x \in X,\ t \geq 0.$$

Taking Laplace transforms, we obtain by (1.3) that, for λ large enough and $x \in X$,

$$\begin{aligned}\int_0^\infty & e^{-\lambda t}W(t)xdt \\ &= \lambda^{n-1}(\lambda^n - A)^{-1}Cx + (\beta+\alpha A)(\lambda^n - A)^{-1}B\int_0^\infty e^{-\lambda t}W(t)xdt.\end{aligned}$$

Therefore, for such λ,

$$(\lambda^n - A(I+\alpha B) - \beta B)\int_0^\infty e^{-\lambda t}W(t)xdt = \lambda^{n-1}Cx, \quad x \in X, \tag{2.4}$$

by the equalities

$$\begin{aligned}&(\lambda^n - A)\left[I - (\beta+\alpha A)(\lambda^n - A)^{-1}B\right] \\ &\quad = (\lambda^n - A)\left[I + \alpha B - \alpha\lambda^n(\lambda^n - A)^{-1}B - \beta(\lambda^n - A)^{-1}B\right] \\ &\quad = \lambda^n - A(I+\alpha B) - \beta B.\end{aligned} \tag{2.5}$$

Finally, we show that $I - (\beta+\alpha A)(\lambda^n - A)^{-1}B$ is invertible for large λ. In order to do this, we observe by (2.1) and (1.3) that for each $x \in X$, $t \geq 0$,

$$\left\|(\beta+\alpha A)\int_0^t S(t-s)C^{-1}Bxds\right\| \leq M_2\int_0^t e^{\omega_2(t-s)}\|x\|ds \leq \frac{M_2}{\omega_2}\left(e^{\omega_2 t} - 1\right)\|x\|,$$

$$\begin{aligned}&(\beta+\alpha A)(\lambda^n - A)^{-1}Bx \\ &\quad = \lambda\int_0^\infty e^{-\lambda t}\left[(\beta+\alpha A)\int_0^t S(t-s)C^{-1}Bxds\right]dt, \quad \lambda > \omega_2,\end{aligned}$$

where M_2 and ω_2 are positive constants. So for $\lambda > \omega_2$ and $x \in X$,

$$\left\|(\beta+\alpha A)(\lambda^n - A)^{-1}Bx\right\| \leq \frac{M_2\lambda}{\omega_2}\int_0^\infty e^{-\lambda t}\left(e^{\omega_2 t} - 1\right)\|x\|dt = \frac{M_2\|x\|}{\lambda(\lambda - \omega_2)}.$$

Thus for $\lambda > 2M_2 + \omega_2 + 1$ we have $\left\|(\beta+\alpha A)(\lambda^n - A)^{-1}B\right\| < \frac{1}{2}$, so that

$$I - (\beta+\alpha A)(\lambda^n - A)^{-1}B$$

is invertible. This together with (2.5) yields that for $\lambda > 2M_2 + \omega_2 + \omega + 1$ the operator $\lambda^n - A(I + \alpha B) - \beta B$ is injective, since $\lambda^n - A$ is injective for $\lambda > \omega$. In conclusion, we obtain from (2.4) that for λ sufficiently large,

$$\lambda^{n-1}(\lambda^n - A(I + \alpha B) - \beta B)^{-1}Cx = \int_0^\infty e^{-\lambda t}W(t)xdt \qquad (x \in X).$$

Set, for $t \geq 0$ and $x \in X$,

$$U(t)x = \begin{cases} W(t)x, & \text{if} \quad n = 1, \\ \displaystyle\int_0^t \frac{(t-s)^{n-2}}{(n-2)!}W(s)xds, & \text{if} \quad n \geq 2. \end{cases}$$

Then an application of Proposition 1.2 gives assertion (i).

In order to verify assertion (ii), we let $v(\cdot)$ be a solution of (2.2) with initial data $x_j = 0$ $(0 \leq j \leq n-1)$. Evidently

$$v(t) = (A(I + \alpha B) + \beta B)\int_0^t \frac{(t-\sigma)^{n-1}}{(n-1)!}v(\sigma)d\sigma, \quad t \geq 0. \tag{2.6}$$

The assumption $CA \subset AC$ implies

$$S(t)C = CS(t)\ (t \geq 0), \quad S(t)Ax = AS(t)x\ (x \in \mathcal{D}(A),\ t \geq 0), \tag{2.7}$$

according to (1.3) and the uniqueness theorem for Laplace transforms. So (1.2) yields

$$S^{(n)}(t)x = S(t)Ax, \quad x \in \mathcal{D}(A),\ t \geq 0. \tag{2.8}$$

Thus, by (2.6) – (2.8) we obtain that for $t \geq s \geq 0$,

$$\begin{aligned} &\frac{d}{ds}\left[\sum_{i=0}^{n-1} S^{(i)}(t-s)(I+\alpha B)\int_0^s \frac{(s-\sigma)^i}{i!}v(\sigma)d\sigma\right] \\ = \;& S(t-s)(I+\alpha B)v(s) + \sum_{i=1}^{n-1} S^{(i)}(t-s)(I+\alpha B)\int_0^s \frac{(s-\sigma)^{i-1}}{(i-1)!}v(\sigma)d\sigma \\ &\qquad - \sum_{i=0}^{n-1} S^{(i+1)}(t-s)(I+\alpha B)\int_0^s \frac{(s-\sigma)^i}{(i-1)!}v(\sigma)d\sigma \\ = \;& S(t-s)(I+\alpha B)v(s) - S^{(n)}(t-s)(I+\alpha B)\int_0^s \frac{(s-\sigma)^{n-1}}{(n-1)!}v(\sigma)d\sigma \\ = \;& CS(t-s)\left[\alpha C^{-1}Bv(s) + \beta C^{-1}B\int_0^s \frac{(s-\sigma)^{n-1}}{(n-1)!}v(\sigma)d\sigma\right]. \end{aligned}$$

Noting that

$$S^{(n-1)}(0) = I, \quad S^{(i)}(0) = 0\ (0 \leq i \leq n-2) \tag{2.9}$$

from (1.2), we then infer that for $t \geq 0$,

$$\begin{aligned} &C(I+\alpha B)\int_0^t \frac{(t-\sigma)^{n-1}}{(n-1)!}v(\sigma)d\sigma \\ = \;& C\int_0^t S(t-s)\left[\alpha C^{-1}Bv(s) + \beta C^{-1}B\int_0^s \frac{(s-\sigma)^{n-1}}{(n-1)!}v(\sigma)d\sigma\right]ds. \end{aligned}$$

Since C is injective, it follows from (2.6) that for $t \geq 0$,

$$\begin{aligned} v(t) &= \beta B \int_0^t \frac{(t-\sigma)^{n-1}}{(n-1)!} v(\sigma)d\sigma + \alpha A \int_0^t S(t-\sigma)C^{-1}Bv(\sigma)d\sigma \\ &\quad + \beta A \int_0^t S(t-\sigma)C^{-1}B \left(\int_0^\sigma \frac{(\sigma-\tau)^{n-1}}{(n-1)!} v(\tau)d\tau \right) d\sigma. \end{aligned}$$

Fix $T > 0$. Then by (2.1) there exists a constant $M_0 > 0$ such that for each $t \in [0,T]$, $\max_{0\leq s\leq t} \|v(s)\| \leq M_0 \int_0^t \max_{0\leq\tau\leq\sigma} \|v(\tau)\| d\sigma$. So the Gronwall-Bellman inequality shows that $v(t) = 0$ for $t \in [0,T]$. Because T was arbitrary, $v(t) \equiv 0$ for $t \geq 0$. This ends the proof.

Theorem 2.2. *Let A and $\{S(t)\}_{t\geq 0}$ be as in Proposition* 1.2. *Suppose $\mathbf{B}$ is a closed linear operator in X such that $\mathcal{D}(\mathbf{B}) \supset \mathcal{D}(A)$ and $\mathcal{R}(\mathbf{B}) \subset \mathcal{R}(C)$. If for each $f \in C(R^+, [\mathcal{D}(A)])$ and $t \geq 0$ we have*

$$\left\| A \int_0^t S(t-s)C^{-1}\mathbf{B}f(s)ds \right\| \leq M \int_0^t e^{\omega(t-s)} \|f(s)\|_{[\mathcal{D}(A)]} ds, \tag{2.10}$$

then

(i) *the Cauchy problem*

$$u^{(n)}(t) = (A+\mathbf{B})u(t) \ (t \geq 0), \quad u^{(j)}(0) = x_j \ (0 \leq j \leq n-1) \tag{2.11}$$

has a C-existence family $\{V(t)\}_{t\geq 0}$ on $[\mathcal{D}(A)]$, and $\left\|V^{(n-1)}(\cdot)\right\|_{\mathbf{L}([\mathcal{D}(A)])}$ is exponentially bounded;

(ii) *for any $x_j \in C(\mathcal{D}(A))$ $(0 \leq j \leq n-1)$, the function $\sum_{j=0}^{n-1} V^{(n-1-j)}(\cdot)C^{-1}x_j$ is a solution of* (2.11)*; and*

(iii) *all solutions of* (2.11) *are unique, provided $CA \subset AC$.*

Proof. Define $Y(t) = \sum_{n=0}^\infty Y_n(t)$ $(t \geq 0)$, where $Y_0(t) = S^{(n-1)}(t)$ and $Y_n(t)x = \int_0^t S(t-s)C^{-1}\mathbf{B}Y_{n-1}(s)xds$ $(t \geq 0,\ x \in \mathcal{D}(A),\ n \in N)$. Then, arguing similarly as in the proof of Theorem 2.1, we obtain that $\{Y(t)\}_{t\geq 0}$ is an exponentially bounded, strongly continuous family of bounded linear operators on $[\mathcal{D}(A)]$, and for λ large enough, $(\lambda^n - (A+\mathbf{B})) \int_0^\infty e^{-\lambda t} Y(t)xdt = \lambda^{n-1}Cx$ $(x \in \mathcal{D}(A))$, and

$$\left\|(\lambda^n - A)^{-1}\mathbf{B}\right\|_{\mathbf{L}([\mathcal{D}(A)])} < \frac{1}{2}, \tag{2.12}$$

so that $\lambda^n - (A+\mathbf{B})$ is injective and

$$\lambda^{n-1}(\lambda^n - (A+\mathbf{B}))^{-1}Cx = \int_0^\infty e^{-\lambda t} Y(t)xdt, \quad x \in \mathcal{D}(A). \tag{2.13}$$

Therefore, (2.11) has a C-existence family $\{V(t)\}_{t\geq 0}$ on $[\mathcal{D}(A)]$, given by

$$V(t)x := \begin{cases} Y(t)x, & \text{if} \quad n = 1, \\ \displaystyle\int_0^t \frac{(t-s)^{n-2}}{(n-2)!} Y(s)xds, & \text{if} \quad n \geq 2, \end{cases}$$

in view of Proposition 1.2. This completes the proof of part (i).

Next, from the above conclusion we have

$$\begin{aligned} V(t)x - \frac{t^{n-1}}{(n-1)!}Cx &= (A+\mathbf{B}) \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} V(s)xds \\ &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} (A+\mathbf{B})V(s)xds, \quad x \in \mathcal{D}(A),\ t \geq 0. \end{aligned}$$

This leads to part (ii) immediately.

To prove part (iii) we let $w(\cdot)$ be a solution of (2.11) with $x_j = 0$ for all $0 \leq j \leq n-1$. By (2.7) and (2.8) we deduce that $\frac{d}{ds}\left[\sum_{i=0}^{n-1} S^{(i)}(t-s)w^{(n-1-i)}(s)\right] = CS(t-s)C^{-1}\mathbf{B}w(s)$ $(t \geq s \geq 0)$, so that $w(t) = \int_0^t S(t-s)C^{-1}\mathbf{B}w(s)ds$ $(t \geq 0)$ by (2.9). Thus from (2.10) we get $\|e^{-\omega t}w(t)\|_{[\mathcal{D}(A)]} \leq M' \int_0^t \|e^{-\omega t}w(t)\|_{[\mathcal{D}(A)]}\, ds$ $(t \geq 0)$, for some constant $M' > 0$. It follows that $w(t) \equiv 0$ for $t \geq 0$ by using the Gronwall-Bellman inequality. The proof is complete.

In what follows, we give multiplicative and additive perturbation theorems with regard to exponentially bounded regularized semigroups and regularized cosine operator functions, as consequences of Theorems 2.1 and 2.2. Let A and $\{S(t)\}_{t\geq 0}$ be as in Proposition 1.2. If $n = 1$ (resp. $n = 2$) and $CA \subset AC$, then $S(\cdot)$ (resp. $C(\cdot) := S'(\cdot)$) is an exponentially bounded C-regularized semigroup (resp. cosine operator function), with $C^{-1}AC$ as its generator. In this case, A is called a subgenerator of $S(\cdot)$ (resp. $C(\cdot)$); in other words, A subgenerates $S(\cdot)$ (resp. $C(\cdot)$). For more information on regularized semigroups and regularized cosine operator functions, we refer to, e.g., [6, 16, 22] and references therein.

Corollary 2.3. *Assume that A subgenerates an exponentially bounded C-regularized semigroup $\{S(t)\}_{t\geq 0}$ (resp. cosine operator function $\{C(t)\}_{t\geq 0}$) on X. Let α, $\beta \in \mathbf{C}$, and $B \in \mathbf{L}(X)$ with $\mathcal{R}(B) \subset \mathcal{R}(C)$, and let $C_1 \in \mathbf{L}(X)$ be injective such that $\mathcal{R}(C_1) \subset \mathcal{R}(C)$ and $C_1[A(I+\alpha B)+\beta B] \subset [A(I+\alpha B)+\beta B]C_1$. If (2.1) holds (in the case of the cosine operator function, $S(t)x := \int_0^t C(s)xds$), then $A(I+\alpha B)+\beta B$ subgenerates an exponentially bounded C_1-regularized semigroup (resp. cosine operator function) on X.*

Proof. Apply Theorem 2.1. $U(t)C^{-1}C_1$ (resp. $U'(t)C^{-1}C_1$) is the C_1-regularized semigroup (resp. cosine operator function), as claimed.

Corollary 2.4. *Assume that A subgenerates an exponentially bounded C-regularized semigroup $\{S(t)\}_{t\geq 0}$ (resp. cosine operator function $\{C(t)\}_{t\geq 0}$) on X. Let $\mathbf{B}$ be a closed linear operator in X such that $\mathcal{D}(\mathbf{B}) \supset \mathcal{D}(A)$ and $\mathcal{R}(\mathbf{B}) \subset \mathcal{R}(C)$. Let $C_1 \in \mathbf{L}(X)$ be injective such that $\mathcal{R}(C_1) \subset \mathcal{R}(C)$, $C^{-1}C_1 : \mathcal{D}(A) \to \mathcal{D}(A)$, and $C_1(A+\mathbf{B}) \subset (A+\mathbf{B})C_1$. If (2.10) holds (in the case of the cosine operator function, $S(t)x := \int_0^t C(s)xds$), then $A+\mathbf{B}$ subgenerates an exponentially bounded C_1-regularized semigroup (resp. cosine operator function) on X, provided that $\rho(A)$ contains a sequence of real numbers, tending to $+\infty$.*

Proof. From (2.12) we see that there exists a $\mu_0 \in \rho(A)$ such that

$$\|(\mu_0 - A)^{-1}\mathbf{B}\|_{\mathbf{L}([\mathcal{D}(A)])} < \frac{1}{2},$$

and therefore $\mu_0 - (A+\mathbf{B}) = (\mu_0 - A)\left(I - (\mu_0 - A)^{-1}\mathbf{B}\right)$ is invertible on X. Letting $Y(\cdot)$ be as in (2.13) with $n = 1$ (resp. $n = 2$), we put

$$\widetilde{Y}(t) := [\mu_0 - (A+\mathbf{B})]\, Y(t)C^{-1}C_1\, [\mu_0 - (A+\mathbf{B})]^{-1} \quad (t \geq 0).$$

Then $\{\widetilde{Y}(t)\}_{t\geq 0}$ is a strongly continuous family of operators in $\mathbf{L}(X)$, and for λ large enough,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\widetilde{Y}(t)xdt &= \lambda^{n-1}\left[\mu_0-(A+\mathbf{B})\right]\left[\lambda^n-(A+\mathbf{B})\right]^{-1}C_1\left[\mu_0-(A+\mathbf{B})\right]^{-1} \\ &= \lambda^{n-1}\left[\lambda^n-(A+\mathbf{B})\right]^{-1}C_1x, \quad t\geq 0,\ x\in X,\end{aligned}$$

with $n=1$ (resp. $n=2$).

Remark 2.5. For the case when $C=C_1=I$, $\alpha=1$, $\beta=0$, and A generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ (resp. strongly continuous cosine operator function $\{C(t)\}_{t\geq 0}$) on X, Corollaries 2.3 and 2.4 can be found in [8, 19, 20]. In this case, $\{S(t)\}_{t\geq 0}$ (resp. $\{C(t)\}_{t\geq 0}$) is exponentially bounded and $\rho(A)$ contains a right half plane, automatically.

It is plain to see that (2.1) holds for

$$B\in \mathbf{L}(X) \quad \text{with} \ \ \mathcal{R}(B)\subset \mathcal{D}(AC^{-1}), \tag{2.14}$$

and that (2.10) holds for any closed linear operator $\mathbf{B}$ in X with $\mathcal{D}(\mathbf{B})\supset\mathcal{D}(A)$ and $\mathcal{R}(\mathbf{B})\subset\mathcal{D}(AC^{-1})$. Specifically, we have the following result.

Corollary 2.6. *Suppose that A subgenerates an exponentially bounded C-regularized semigroup on X, and that $B_1\in\mathbf{L}(X)$ and $\mathcal{R}(B_1)\subset\mathcal{R}(C)$. Then:*

(i) *The Cauchy problem $(*)$: $u'(t)=(A+B_1)u(t)$ $(t\geq 0)$, $u(0)=x$ has an exponentially bounded C-existence family on X.*
(ii) *All solutions of $(*)$ are unique, provided $CA\subset AC$.*
(iii) *$A+B_1$ subgenerates an exponentially bounded C_1-regularized semigroup on X, whenever $C_1\in\mathbf{L}(X)$ is injective, $\mathcal{R}(C_1)\subset\mathcal{R}(C)$ and $C_1(A+B_1)\subset(A+B_1)C_1$.*

Proof. Take $\alpha=0$ and $\beta=1$ in Theorem 2.1 and Corollary 2.3.

Remark 2.7. Conclusion (i) of Corollary 2.6 appeared in [5, 6]. Generally speaking, a C-existence family for a first order Cauchy problem ensures uniqueness of the exponentially bounded solutions, but not all solutions (see [6, Proposition 2.9]). This indicates the significance of assertion (ii). Conclusion (iii) is due to [21].

Let A_0 be a linear operator in X satisfying

$$(\omega,\infty)\subset\rho(A_0), \quad \sup_{\lambda>\omega}\left\|\lambda(\lambda-A_0)^{-1}\right\|<\infty. \tag{2.15}$$

We set $F_{A_0}:=\left\{x\in X;\ \overline{\lim}_{\lambda\to+\infty}\left\|\lambda A_0(\lambda-A_0)^{-1}x\right\|<\infty\right\}$. It is easy to verify that F_{A_0}, endowed with the norm $\|x\|_{F_{A_0}}:=\|x\|+\overline{\lim}_{\lambda\to+\infty}\left\|\lambda A_0(\lambda-A_0)^{-1}x\right\|$, is a Banach space. When A_0 is the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, F_{A_0} coincides with the Favard class of $T(t)$, cf. [10, p.130, Proposition 5.12].

Theorem 2.8. *Let A and $\{S(t)\}_{t\geq 0}$ be as in Proposition* 1.2*, and let $C_0\in\mathbf{L}(X)$ with $CC_0=C_0C$ and $C_0A\subset AC_0$. Suppose A_0 is a densely defined linear operator in X satisfying* (2.15)*, such that $\mathcal{D}(A_0)\subset\mathcal{D}(AC_0)$, $CA_0\subset A_0C$, and $(\lambda-A_0)^{-1}A\subset A(\lambda-A_0)^{-1}$ for $\lambda>\omega$. Then:*

(i) (2.1) *is valid for $B=CC_0B_0$, if $B_0\in\mathbf{L}(X)$ and $\mathcal{R}(B_0)\subset F_{A_0}$.*
(ii) (2.10) *is valid for $\mathbf{B}=CC_0\mathbf{B}_0$, if $\mathbf{B}_0$ is a closed linear operator in X, $\mathcal{D}(\mathbf{B}_0)\supset\mathcal{D}(A)$, and $\mathcal{R}(\mathbf{B}_0)\subset F_{A_0}$.*

Proof. Using the density of $\mathcal{D}(A_0)$ and (2.15), we have

$$\lim_{\lambda\to+\infty}\lambda(\lambda-A_0)^{-1}x = x, \quad x\in X. \tag{2.16}$$

Moreover, by hypothesis, $A_0(\lambda-A_0)^{-1}(\mu^n-A)^{-1}C = (\mu^n-A)^{-1}CA_0(\lambda-A_0)^{-1}$ (λ, $\mu>\omega$). In combination with (1.3), this shows that for $\lambda>\omega$, $t\geq 0$,

$$A_0(\lambda-A_0)^{-1}S(t) = S(t)A_0(\lambda-A_0)^{-1}, \tag{2.17}$$

by the uniqueness theorem for Laplace transforms. Likewise,

$$C_0S(t) = S(t)C_0, \quad t\geq 0. \tag{2.18}$$

Let B_0 and B be as in (i). For each $f\in C(R^+,X)$ and $t>0$, we take a sequence $\{f_m\}_{m\in N}\subset C^1([0,t],X)$ such that

$$\max_{s\in[0,t]}\|f_m(s)-f(s)\|\to 0, \quad \text{as } m\to\infty. \tag{2.19}$$

From [22], we know that

$$\int_0^t S(t-s)C_0B_0f_m(s)ds\in\mathcal{D}(A), \quad m\in N.$$

Therefore, noting that $\mathcal{D}(AC_0)\supset\mathcal{D}(A_0)$ and using (2.16) – (2.18), we obtain

$$\begin{aligned}
&\left\|A\int_0^t S(t-s)C^{-1}Bf_m(s)ds\right\| = \left\|AC_0\int_0^t S(t-s)B_0f_m(s)ds\right\| \\
\leq\ & \left\|AC_0(\omega+1-A_0)^{-1}\right\| \\
&\quad\times\lim_{\lambda\to+\infty}\left\|\int_0^t S(t-s)\left[\lambda(\omega+1-A_0)(\lambda-A_0)^{-1}\right]B_0f_m(s)ds\right\| \\
\leq\ & \widetilde{M}\int_0^t e^{\omega(t-s)}\|B_0f_m(s)\|_{F_{A_0}}ds \\
\leq\ & \widetilde{M}\|B_0\|_{\mathbf{L}(X,F_{A_0})}\int_0^t e^{\omega(t-s)}\|f_m(s)\|ds, \quad m\in N,
\end{aligned}$$

where $\widetilde{M}$ is a constant independent of m and t. This proves part (i), by (2.19) and the closedness of A. The same type of argument gives part (ii).

Example 2.9. Let $X=UC_b(R)$ be the space of uniformly continuous and bounded functions, and let

$$\begin{aligned}
A &= i\frac{d^2}{d\xi^2} \quad \text{with } \mathcal{D}(A)=\{f\in C^2(R);\ f \text{ is bounded and } f''\in X\}, \\
A_0 &= \frac{d^2}{d\xi^2} \quad \text{with } \mathcal{D}(A_0)=\mathcal{D}(A).
\end{aligned}$$

It is known that the operator $D:=\frac{d^2}{d\xi^2}$ with domain $W^{2,1}(R)$ generates a strongly continuous semigroup on $L^1(R)$, and A_0 is the generator of its sun dual semigroup on X. Thus F_{A_0} coincides with the domain of the adjoint operator of D (cf. [10, p.135, Proposition 5.19]). Hence it is not hard to see that $F_{A_0}=\{f\in C^1(R);\ f \text{ is bounded and } f' \text{ is Lipschitz continuous}\}$. From [14], we see that A generates an exponentially bounded once integrated semigroup, and so generates an exponentially bounded C-regularized semigroup (cf. [6, Theorem 18.3]) for $C:=(1-A_0)^{-1}$. Moreover, define B_0 by $(B_0f)(\xi)=ig(\xi)\int_a^b f(\sigma)d\sigma$ $(f\in X)$, where $g(\xi) \in F_{A_0}$ and $a,\ b \in R$. Then $B_0 \in \mathbf{L}(X)$ and $\mathcal{R}(B_0) \subset F_{A_0}$. Taking $\alpha = 1$, $\beta = -i$, $C_0 = I$, $B = CC_0B_0$, we have $A(\alpha I + B) + \beta B = A - iB_0$. Applying Theorem 2.1, Theorem 2.8 and Proposition 1.2, we conclude that for each $\phi \in UC_b(R) \cap C^4(R)$ with $\phi^{(4)} \in UC_b(R)$, the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} = i\dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + g(\xi)\displaystyle\int_a^b u(t,\sigma)d\sigma, \quad t \geq 0,\ \xi \in R, \\ u(0,\xi) = \phi(\xi), \quad \xi \in R, \end{cases}$$

has a unique solution in $C^1(R, UC_b(R))$.

Remark 2.10. Given an exponentially bounded C-regularized semigroup $\{S(t)\}_{t\geq 0}$, one can define the Favard class of $\{S(t)\}_{t\geq 0}$, similarly as in the case of strongly continuous semigroups (see, e.g., [8], [10, Section III.3]), by

$$Fav(S(t)) := \left\{ x \in X;\ \sup_{t>0} \left\| \frac{1}{t}(S(t)x - x) \right\| < \infty \right\},$$

and prove that (2.1) holds for $B \in \mathbf{L}(X)$ with

$$\mathcal{R}(B) \subset C^2(Fav(S(t))), \tag{2.20}$$

and that (2.10) holds for $\mathbf{B}$ a closed linear operator in X with $\mathcal{D}(\mathbf{B}) \supset \mathcal{D}(A)$ and $\mathcal{R}(\mathbf{B}) \subset C^2(Fav(S(t))$. When $\{S(t)\}_{t\geq 0}$ is a strong continuous semigroup, this result is essentially Theorem 2.8.

In Example 2.9, the permissible space for $\mathcal{R}(B)$ can be so large as

$$C(F_{A_0}) = \{f \in C^3(R);\ f \text{ is bounded and } f''' \text{ is Lipschitz continuous}\}.$$

However, if either (2.20) or (2.14) were used, the range of B would be restricted to a set which is smaller than or equal to

$$\mathcal{R}(C^2) = \{f \in C^4(R);\ f \text{ is bounded and } f^{(4)} \in UC_b(R)\}.$$

It is clear that here $C(F_{A_0})$ strictly contains $\mathcal{R}(C^2)$. This reflects the feature of Theorem 2.8, on which Example 2.9 was based.

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