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A FEJÉR TYPE THEOREM TO DETERMINE JUMPS IN TERMS OF THE ABEL-POISSON MEAN OF DOUBLE FOURIER SERIES

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Abstract. We extend from single to double Fourier series a theorem of Zygmund to determine the generalized jumps of a periodic integrable function at a simple discontinuity point. As a by-product of the proof, we obtain an estimate of the fourth mixed partial derivative of the Abel-Poisson mean of any integrable function $F(x, y)$ at such a point where F is smooth. We also consider the extension of the Zygmund classes λ_* and Λ_* to the two-dimensional torus $\mathcal{T}^2$.

1. Known results for single Fourier series

Let $f \in L^1(\mathcal{T})$ be a periodic function with Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{j=1}^{\infty}(a_j \cos jx + b_j \sin jx), \tag{1.1}$$

where $\mathcal{T} := [-\pi, \pi)$ is the one-dimensional torus and the Fourier coefficients a_j, b_j are defined by

$$a_j := \frac{1}{\pi}\int_{\mathcal{T}} f(x) \cos jx dx \quad \text{and} \quad b_j := \frac{1}{\pi}\int_{\mathcal{T}} f(x) \sin jx dx.$$

Differentiate formally the series in (1.1):

$$\sum_{j=1}^{\infty} j(b_j \cos jx - a_j \sin jx);$$

then consider the first arithmetic mean of the sequence $\{j(b_j \cos jx - a_j \sin jx) : j = 1, 2, \dots\}$:

$$s_n(f; x) := \frac{1}{n}\sum_{j=1}^{n} j(b_j \cos jx - a_j \sin jx), \quad n = 1, 2, \dots .$$

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Fejér [1] (see also [3, p. 107]) proved the following

Theorem 1. *If $f \in L^1(\mathcal{T})$ and f has a simple discontinuity at some point $x_0 \in \mathcal{T}$, then*

$$\lim_{n\to\infty} s_n(f;x_0) = \frac{1}{\pi} d(f;x_0),$$

where

$$d(f;x_0) := \lim_{h\to 0+} \{f(x_0+h) - f(x_0-h)\}$$

is the jump of f at x_0.

Instead of the first arithmetic mean, one may consider the Abel-Poisson mean of the sequence $\{j(b_j \cos jx - a_j \sin jx) : j = 1, 2, \dots\}$, that is, the mean

$$(1-r)\sum_{j=1}^{\infty} j(b_j \cos jx - a_j \sin jx) r^j = (1-r)\frac{\partial f(r;x)}{\partial x},$$

where

$$f(r;x) := \frac{1}{2}a_0 + \sum_{j=1}^{\infty}(a_j \cos jx + b_j \sin jx) r^j, \quad 0 \le r < 1,$$

is the Abel-Poisson mean of the series in (1.1).

Zygmund [3, p. 108] proved the following

Theorem 2. *If $f \in L^1(\mathcal{T})$ and the finite limit*

$$\lim_{h\to 0+} \frac{1}{h}\int_0^h \{f(x_0+t) - f(x_0-t)\} dt =: \delta(f;x_0)$$

exists at some point $x_0 \in \mathcal{T}$, then

$$\lim_{r\to 1-} (1-r)\frac{\partial f(r,x_0)}{\partial x} = \frac{1}{\pi}\delta(f;x_0).$$

The limit $\delta(f;x_0)$ is called the generalized jump of f at x_0. Clearly, the existence of $d(f;x_0)$ implies that of $\delta(f;x_0)$ and both numbers are equal.

We recall that a function $F(x)$ is said to be smooth at some inner point x_0 of the domain of F if

$$\Delta(F;x_0;h) := \frac{1}{h}\{F(x_0+h) + F(x_0-h) - 2F(x_0)\} \to 0 \quad \text{as} \quad h \to 0. \tag{1.2}$$

The function class $\lambda_*(\mathcal{T})$ consists of all periodic continuous functions F such that (1.2) holds uniformly in $x_0 \in \mathcal{T}$. If the ratio $\Delta(F;x_0;h)$ is only uniformly bounded in $x_0 \in \mathcal{T}$ and $h > 0$, then we write $F \in \Lambda_*(\mathcal{T})$.

The following theorem was also observed by Zygmund [3, p. 109].

Theorem 3. *If $F \in \lambda_*(\mathcal{T})$, then we have*

$$\frac{\partial^2 F(r;x_0)}{\partial x^2} = o\{(1-r)^{-1}\} \quad \textit{as} \quad r \to 1-, \tag{1.3}$$

uniformly in $x_0 \in \mathcal{T}$. If $F \in \Lambda_*(\mathcal{T})$, then (1.3) holds with 'O' in place of 'o', uniformly in $x_0 \in \mathcal{T}$ and $0 \le r < 1$.

2. New results for double Fourier series

The second-named author [2] extended Theorem 1 from single to double Fourier series. Now, our goal is to extend Theorems 2 and 3 to double Fourier series, as well.

For the sake of brevity in writing, we shall use complex notation for double Fourier series. Let $f \in L^1(\mathcal{T}^2)$ be a function periodic in each variable and with Fourier series

$$f(x,y) \sim \sum_{(j,k)\in\mathcal{Z}^2} \sum c_{jk} e^{i(jx+ky)}, \tag{2.1}$$

where $\mathcal{T}^2 := \mathcal{T} \times \mathcal{T}$ is the two-dimensional torus, $\mathcal{Z}^2 := \mathcal{Z} \times \mathcal{Z}$, $\mathcal{Z} := \{\dots, -1, 0, 1, 2, \dots\}$, and the Fourier coefficients c_{jk} are defined by

$$c_{jk} := \frac{1}{(2\pi)^2} \int\int_{\mathcal{T}^2} f(x,y) e^{-i(jx+ky)} dx dy.$$

The Abel-Poisson mean of the series in (2.1) is defined by

$$f(r,s;x,y) := \sum_{(j,k)\in\mathcal{Z}^2} \sum c_{jk} e^{i(jx+ky)} r^{|j|} s^{|k|}, \quad 0 \le r, s < 1. \tag{2.2}$$

This time the ordinary jump of f at some point $(x_0, y_0) \in \mathcal{T}^2$ may be defined by

$$\begin{aligned} d(f;x_0,y_0) := \lim_{h\to 0+ \text{ and } k\to 0+} \{ & f(x_0+h, y_0+k) \\ & - f(x_0-h, y_0+k) - f(x_0+h, y_0-k) + f(x_0-h, y_0-k)\}, \end{aligned}$$

provided the finite limit exists. Instead, we shall consider the generalized jump of f at (x_0, y_0) defined by

$$\delta(f;x_0,y_0) := \lim_{h\to 0+ \text{ and } k\to 0+} \delta(f;x_0,y_0;h,k), \tag{2.3}$$

where

$$\delta(f;x_0,y_0;h,k) := \frac{1}{hk} \int_0^h \int_0^k \{f(x_0+u, y_0+v) \tag{2.4}$$

$$-f(x_0-u, y_0+v) - f(x_0+u, y_0-v) + f(x_0-u, y_0-v)\} du dv, \quad h > 0, k > 0,$$

provided that the finite limit exists in (2.3). Again, the existence of $d(f;x_0,y_0)$ implies that of $\delta(f;x_0,y_0)$ and both numbers are equal.

Our main result is the following

Theorem 4. *If $f \in L^1(\mathcal{T}^2)$ is such that $\delta(f;x_0,y_0)$ exists at some point $(x_0,y_0) \in \mathcal{T}^2$ and the ratio $\delta(f;x_0,y_0;h,k)$ is bounded in $h, k > 0$, then*

$$\lim_{r\to 1- \text{ and } s\to 1-} (1-r)(1-s) \frac{\partial^2 f(r,s;x_0,y_0)}{\partial x \partial y} = \frac{1}{\pi^2} \delta(f;x_0,y_0). \tag{2.5}$$

A function $F(x,y)$ may be said to be smooth at some inner point (x_0,y_0) of the domain of F if

$$\Delta(F;x_0,y_0;h,k) := \frac{1}{hk} \{F(x_0+h, y_0+k) + F(x_0-h, y_0+k) \tag{2.6}$$

$$+F(x_0+h, y_0-k) + F(x_0-h, y_0-k) - 4F(x_0,y_0)\} \to 0 \quad \text{as} \quad h, k \to 0.$$

The two-dimensional analogues of the function classes λ_* and Λ_* may be defined as follows. We write $F \in \lambda_*(\mathcal{T}^2)$ if F is continuous and periodic in each variable, and (2.6) holds uniformly in $(x_0, y_0) \in \mathcal{T}^2$. If the ratio $\Delta(F; x_0, y_0; h, k)$ is only uniformly bounded in $(x_0, y_0) \in \mathcal{T}^2$ and $h, k > 0$, then we write $F \in \Lambda_*(\mathcal{T}^2)$.

The proof of Theorem 4 essentially contains the proof of the following

Theorem 5. *If $F \in \lambda_*(\mathcal{T}^2)$, then we have*

$$\frac{\partial^4 F(r, s; x_0, y_0)}{\partial x^2 \partial y^2} = o\{(1-r)^{-1}(1-s)^{-1}\} \quad as \quad r \to 1- \quad and \quad s \to 1-, \tag{2.7}$$

uniformly in $(x_0, y_0) \in \mathcal{T}^2$. If $F \in \Lambda_*(\mathcal{T}^2)$, then (2.7) holds with 'O' in place of 'o', uniformly in $(x_0, y_0) \in \mathcal{T}^2$ and $0 \le r, s < 1$.

3. Proof of Theorem 4

Let F be the integral of f:

$$F(x, y) := \int_0^x \int_0^y f(u, v) du dv, \quad (x, y) \in \mathcal{R}^2. \tag{3.1}$$

(i) First, we assume that

$$\int_{\mathcal{T}} f(x, v) dv = 0, \ \ x \in \mathcal{T}; \quad \text{and} \quad \int_{\mathcal{T}} f(u, y) du = 0, \ y \in \mathcal{T}. \tag{3.2}$$

These are equivalent to the assumption that the Fourier coefficients c_{jk} of f with $\min\{|j|, |k|\} = 0$ vanish: $c_{j0} = c_{0k} = 0$, $j, k \in \mathcal{Z}$. Due to (3.2), F is periodic in each variable:

$$F(x + 2\pi, y) = F(x, y + 2\pi) = F(x, y), \quad (x, y) \in \mathcal{R}^2.$$

Let C_{jk} be the Fourier coefficients of F. By integration by parts, we obtain

$$C_{jk} = -\frac{c_{jk}}{jk}, \quad j \ne 0, \ k \ne 0.$$

This means that if we denote by $F(r, s; x, y)$ the Abel-Poisson mean of F, then we have

$$\frac{\partial^2 F(r, s; x, y)}{\partial x \partial y} = f(r, s; x, y), \quad 0 \le r, s < 1, \tag{3.3}$$

where $f(r, s; x, y)$ is the Abel-Poisson mean of f defined in (2.2).

In the sequel, we make use of the kernel representation

$$F(r, s; x, y) = \frac{1}{\pi^2} \iint_{\mathcal{T}^2} F(u, v) P(r, x-u) P(s, y-v) du dv, \tag{3.4}$$

where

$$P(r, t) := \frac{1}{2} + \sum_{j=1}^{\infty} r^j \cos jt = \frac{1-r^2}{2(1 - 2r\cos t + r^2)}, \quad 0 \le r < 1,$$

is the Poisson kernel. It follows that

$$\frac{\partial^4 F(r, s; x, y)}{\partial x^2 \partial y^2} = \frac{1}{\pi^2} \iint_{\mathcal{T}^2} F(x-u, y-v) P''(r, u) P''(s, v) du dv, \tag{3.5}$$

where

$$P''(r, t) = (1-r^2) r \frac{2r(1 + \sin^2 t) - (1 + r^2)\cos t}{(1 - 2r\cos t + r^2)^3}.$$

Clearly, $P^{''}(r,t)$ is even in t and

$$\int_0^\pi P^{''}(r,t)dt = P'(r,\pi) - P'(r,0) = 0. \tag{3.6}$$

Due to (3.5) and the evenness of $P^{''}(r,t)$ in t, we may write that

$$\begin{aligned}\frac{\partial^4 F(r,s;x,y)}{\partial x^2 \partial y^2} &= \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \{F(x+u,y+v)+F(x-u,y+v) \\ &+F(x+u,y-v)+F(x-u,y-v)\}P^{''}(r,u)P^{''}(s,v)dudv.\end{aligned} \tag{3.7}$$

By (2.4) and (3.1), it is easy to check that

$$\begin{aligned}hk\delta(f;x_0,y_0;h,k) &= F(x_0+h,y_0+k)+F(x_0-h,y_0+k) \\ &+F(x_0+h,y_0-k)+F(x_0-h,y_0-k)-2\{F(x_0+h,y_0) \\ &+F(x_0-h,y_0)+F(x_0,y_0+k)+F(x_0,y_0-k)\}+4F(x_0,y_0).\end{aligned}$$

Combining this with (3.7), while keeping (3.3) and (3.6) in mind, gives

$$\begin{aligned}&(1-r)(1-s)\frac{\partial^2 f(r,s;x_0,y_0)}{\partial x \partial y} \\ &= \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \delta(f;x_0,y_0;u,v)K(r,u)K(s,v)dudv,\end{aligned} \tag{3.8}$$

where

$$K(r,t) := (1-r)tP^{''}(r,t), \quad 0 \le r < 1. \tag{3.9}$$

In his proof, Zygmund [3, p. 109] shows that the kernel $K(r,t)$ is quasi-positive, that is, there exists an absolute constant B such that

$$\int_0^\pi |K(r,t)|dt \le B, \quad 0 \le r < 1. \tag{3.10}$$

Furthermore, for any $\eta > 0$, we have

$$\mu(r;\eta) := \max_{\eta \le t \le \pi} |K(r,t)| \to 0 \quad \text{as} \quad r \to 1-. \tag{3.11}$$

(ii) Now, in the particular case when

$$\delta(f;x_0,y_0) = 0, \tag{3.12}$$

we can prove (2.5) in a routine way. In fact, given any $\varepsilon > 0$, by (3.12) we can choose $\eta > 0$ so that

$$|\delta(f;x_0,y_0;h,k)| < \varepsilon \quad \text{if} \quad 0 < h,k < \eta. \tag{3.13}$$

By assumption in Theorem 4, we have

$$|\delta(f;x_0,y_0;h,k)| \le B_1, \quad 0 < h,k \le \pi, \tag{3.14}$$

with an appropriate bound $B_1 < \infty$.

Taking into account (3.8), (3.10), (3.11), (3.13), and (3.14), we estimate as follows:

$$\begin{aligned}
&|\pi^2(1-r)(1-s)\frac{\partial^2 f(r,s;x_0,y_0)}{\partial x\partial y}| \\
&\quad\le \int_0^\eta\int_0^\eta |\delta(f;x_0,y_0;u,v)K(r,u)K(s,v)|dudv \\
&\qquad + \mu(r;\eta)\int_\eta^\pi\int_0^\eta |\delta(f;x_0,y_0;u,v)K(s,v)|dudv \\
&\qquad + \mu(s;\eta)\int_0^\eta\int_\eta^\pi |\delta(f;x_0,y_0;u,v)K(r,u)|dudv \\
&\qquad + \mu(r;\eta)\mu(s;\eta)\int_\eta^\pi\int_\eta^\pi |\delta(f;x_0,y_0;u,v)|dudv \\
&\quad\le B^2\varepsilon + \pi B_1 B\mu(r;\eta) + \pi B_1 B\mu(s;\eta) + \pi^2 B_1\mu(r;\eta)\mu(s;\eta) \\
&\quad\le (B^2+1)\varepsilon,
\end{aligned}$$

provided that r and s are close enough to 1 (from the left). This proves (2.5) in the particular case (3.12).

(iii) Now, we can get rid of assumption (3.12) by making use of an approach due to Fejér [1]. In case $\delta(f;x_0,y_0)\neq 0$, we introduce a new function g as follows:

$$g(\xi,\eta) := f(\xi,\eta) - \frac{\delta(f;x_0,y_0)}{\pi^2}\phi(\xi-x_0)\phi(\eta-y_0), \tag{3.15}$$

where

$$\phi(t) := (\pi - t)/2 \quad \text{for} \quad 0 < t < 2\pi,$$

$$\phi(0) = \phi(2\pi) := 0,$$

and ϕ is continued periodically. Since the ordinary jump of the function $\phi(\cdot - x_0)$ at the point x_0 equals π, we have (3.12) with g in place of f. So, by (ii) just proved, we have (2.5) with g in place of f. Keeping (3.15) in mind, it remains to take into account that the Fourier series of $\phi(\cdot - x_0)$ is

$$\phi(\xi - x_0) \sim {\sum_{j\in\mathcal{Z}}}' \frac{e^{ij(\xi-x_0)}}{2ij} = \sum_{j=1}^\infty \frac{\sin j(\xi-x_0)}{j},$$

where $\sum'$ indicates that $j=0$ is excluded from the summation, whence we see immediately that

$$(1-r)\frac{\partial\phi(r,\xi-x_0)}{\partial\xi}\Big|_{\xi:=x_0} = (1-r)\sum_{k=1}^\infty r^k \to 1 \quad \text{as} \quad r\to 1-.$$

This completes the proof of (2.5) under assumption (3.2).

(iv) Finally, we have to deal with the general case when the conditions in (3.2) are not satisfied. To this effect, we introduce another function g_1 as follows:

$$g_1(x,y) := f(x,y) - \frac{1}{2\pi}\int_{\mathcal{T}} f(x,v)dv - \frac{1}{2\pi}\int_{\mathcal{T}} f(u,y)du$$

$$+\frac{1}{(2\pi)^2}\int\int_{\mathcal{T}^2} f(u,v)dudv$$

(the last term on the right-hand side equals the Fourier coefficient c_{00}). Clearly, the conditions in (3.2) are satisfied with g_1 in place of f. It is also obvious that

$$\frac{\partial^2 g_1(r,s;x,y)}{\partial x \partial y} \equiv \frac{\partial^2 f(r,s;x,y)}{\partial x \partial y}, \quad 0 \le r, s < 1.$$

Consequently, it suffices to refer to (iii) in order to conclude (2.5) in the general case. □

We note that the Fourier series of g_1 in Step (iv) is of the form

$$g_1(x,y) \sim \sum_{j\in\mathcal{Z}}{}' \sum_{k\in\mathcal{Z}}{}' c_{jk} e^{i(jx+ky)}$$

(cf. (2.1)), where those pairs (j,k) of integers are excluded from the double summation for which either $j=0$ or $k=0$.

4. Proof of Theorem 5

We start with (3.4), which is valid for any function $F \in L^1(\mathcal{T}^2)$, periodic in each variable. Differentiating (3.4) with respect to x and y, twice each time, we arrive at (3.7), which we may rewrite into the following form:

$$\begin{aligned}
&(1-r)(1-s)\frac{\partial^4 F(r,s;x,y)}{\partial x^2 \partial y^2} \\
&\quad = (1-r)(1-s)\frac{1}{\pi^2}\int_0^\pi \int_0^\pi \frac{1}{uv}\{F(x+u,y+v)+F(x-u,y+v) \\
&\qquad + F(x+u,y-v)+F(x-u,y-v)-4F(x,y)\}uP''(r,u)vP''(s,v)dudv \\
&\quad = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \Delta(F;x,y;u,v)K(r,u)K(s,v)dudv,
\end{aligned}$$

due to (3.6), where we used the notations in (2.6) and (3.9). Now, the same reasoning as in Part (ii) of the proof of Theorem 4 shows that for any function $F \in L^1(\mathcal{T}^2)$, which is periodic in each variable and smooth at some point $(x_0,y_0) \in \mathcal{T}^2$, we have (2.7). With the obvious extension to uniformity, we can easily complete the proof of Theorem 5 in the case when $F \in \lambda_*(\mathcal{T}^2)$. The proof in the case when $F \in \Lambda_*(\mathcal{T}^2)$ is even simpler. □

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