PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 9, Pages 2679–2685
S 0002-9939(02)06348-7
Article electronically published on February 4, 2002

ANTI-WICK QUANTIZATION WITH SYMBOLS IN L^p SPACES

PAOLO BOGGIATTO AND ELENA CORDERO

(Communicated by David Tartakoff)

Abstract. We give a classification of pseudo-differential operators with anti-Wick symbols belonging to L^p spaces: if $p = 1$ the corresponding operator belongs to trace classes; if $1 \leq p \leq 2$ we get Hilbert-Schmidt operators; finally, if $p < \infty$, the operator is compact. This classification cannot be improved, as shown by some examples.

Introduction: Weyl and anti-Wick quantization

A common feature of all types of quantization proposed in the literature is that they establish a correspondence between self-adjoint operators and classical observables, i.e. real functions on the phase space. In general in the frame of pseudo-differential calculus the correspondence between symbols and operators does not fulfill this requirement, that is, the operator fails to be self-adjoint when the symbol is real valued. However this is true for the the Weyl and the anti-Wick symbols, as is shown for instance in [10]. If on the phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi = \mathbb{R}^{2n}_z$, $z = (x, \xi)$, we consider the Shubin classes

$$\Gamma^m_\rho = \{a(z) \in C^\infty(\mathbb{R}^{2n}) : |\partial^\gamma_z a(z)| \leq C_\gamma \langle z\rangle^{m-\rho|\gamma|}\} \tag{0.1}$$

with $m \in \mathbb{R}$, $\rho \in (0, 1]$, $\langle z\rangle = \sqrt{1 + |z|^2}$, then a satisfactory pseudo-differential calculus, both in the case of Weyl and anti-Wick operators, has been developed, see for instance [10].

M.W. Wong analysed in [13] the case of Weyl quantization with symbols in $L^p(\mathbb{R}^{2n})$ and gave conditions for the operators to be continuous, compact and Hilbert-Schmidt. In this work we also consider symbols belonging to $L^p(\mathbb{R}^{2n})$ and study the behaviour of the corresponding anti-Wick operators. We obtain a classification of the operators according to the $L^p(\mathbb{R}^{2n})$ space of the symbols.

To be more precise, given a function $a(z)$, $z = (x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi = \mathbb{R}^{2n}_z$, consider the Weyl operator:

$$Op_w[a]u(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\xi} a\Big(\frac{x+y}{2}, \xi\Big) u(y) dy đ\xi, \qquad đ\xi = (2\pi)^{-n} d\xi,$$

as well as the anti-Wick operator:

$$Op_{aw}[a]u(x) = \int_{\mathbb{R}^{2n}} a(z) P_z u \; dz,$$

Received by the editors April 12, 2001.
2000 *Mathematics Subject Classification.* Primary 47G30, 35S05.
Key words and phrases. Anti-Wick, pseudo-differential operators, Hilbert-Schmidt, trace class.

where $P_z u(x) = (u, \Phi_z)\Phi_z(x)$ are orthogonal projections in $L^2(\mathbb{R}^n)$ on the functions $\Phi_z(x) = \pi^{-\frac{n}{4}} e^{ix\eta} e^{-\frac{|x-y|^2}{2}}$ depending on the parameter $z = (y, \eta) \in \mathbb{R}^{2n}$. If $a(z)$ belongs to a Shubin class Γ_ρ^m, then both operators are continuous maps from $S(\mathbb{R}^n)$ to itself, extending to continuous maps on the duals $S'(\mathbb{R}^n)$. The adjoint operator is in both cases the operator associated with $\overline{a}(z)$, the complex conjugate of the symbol $a(z)$, so that, as we mentioned before, real symbols correspond to self-adjoint operators. Moreover the anti-Wick operator satisfies the additional property: $a(z) \geq 0$ ($a(z) > 0$) for all $z \in \mathbb{R}^{2n}$ implies $Op_{aw}[a] \geq 0$ ($Op_{aw}[a] > 0$) and, if $a(z) > 0$ and elliptic, the operator is a bijection of $S(\mathbb{R}^n)$ extending to a bijection of $S'(\mathbb{R}^n)$; see for reference [2], [3]. The relation between Weyl and anti-Wick operators is given by the equality $Op_{aw}[a] = Op_w[(2\pi)^{-n} a * \sigma]$; that means every anti-Wick operator is also a Weyl operator with Weyl symbol given by the convolution between its anti-Wick symbol and the Gaussian function $\sigma(z) = 2^n e^{-|z|^2}$. The converse is true only modulo regularizing operators; that is, every Weyl operator can be written as the sum of an anti-Wick operator plus a regularizing operator, for details see [10], [2]. If $a(z)$ is in the Shubin class Γ_ρ^m, then the convolution $(a * \sigma)(z)$ is also in the same class, so that in this case the properties of Weyl and anti-Wick operators are very similar. But, if we consider $a \in L^p(\mathbb{R}^n)$, then in general we only have $a * \sigma \in \Gamma_0^0$, that is, we are in the "limit" case $\rho = 0$ of Shubin classes (see in the next section the second proof of Theorem 1.2 and Lemma 1.1); however all the continuity properties mentioned above remain valid in view of the Calderon-Vaillancourt theorem; see [8], [4], [5]. We obtain in this case an interesting class of compact operators on $L^2(\mathbb{R}^n)$. Operators of this type, called also *localization operators*, were introduced by Daubechies as filters in signal analysis (see [6]); they were studied by Wong in [13], where the continuity of the map $a(z) \in L^p(\mathbb{R}^{2n}) \to Op_{aw}[a] \in B(L^2)$ is obtained by means of interpolation methods.

We follow here a direct approach, and in section 2 we present two different proofs of the continuity of this map, which we consider of interest for their particular simplicity; as a corollary we obtain the compactness property as in [13]. For Weyl symbols, on the contrary, we remark that the boundedness of the map $b(z) \in L^p(\mathbb{R}^{2n}) \to Op_w[b] \in B(L^2)$ fails if $p > 2$; see [11]. In section 3 we study Hilbert-Schmidt and trace class properties of these operators, pointing out the following differences between Weyl and anti-Wick symbols. An operator is Hilbert-Schmidt if and only if its Weyl symbol is in L^2, while we show that for anti-Wick symbols in L^p with $p \in [1, 2]$ one has always Hilbert-Schmidt operators; for $2 \leq p \leq \infty$ one can find operators with anti-Wick symbol in L^p which are not Hilbert-Schmidt; a sufficient condition for an operator to be trace class is that its anti-Wick symbol belongs to L^1, while the Weyl symbol needs to be in L^1 together with all its derivatives; see for instance Robert [9]. For $p > 1$, there exist operators with anti-Wick symbol in L^p which are not trace class.

We are thankful to Prof. E. Buzano and Prof. L. Rodino for their very helpful suggestions and remarks.

1. Theorems on boundedness and compactness

In this section we first give a direct proof of the continuity of the map $a \to Op_{aw}[a]$, for $a \in L^p(\mathbb{R}^{2n})$. Second, we prove the same result as a consequence of the Calderon-Vaillancourt theorem. Finally, we show the compactness of $Op_{aw}[a]$.

The following lemma, used in Theorem 1.2, clarifies the relation between L^p anti-Wick and Weyl symbols in the Γ^m_ρ classes.

Lemma 1.1. *Let $k \geq 0$ and $\langle z\rangle^k a(z) \in L^p(\mathbb{R}^{2n}), 1 \leq p < +\infty$. Then, for any multi-index α there exist positive functions $\Psi_\alpha(z)$ converging to zero as $|z| \to \infty$ such that $|\langle z\rangle^k \partial_z^\alpha (a*\sigma)(z)| \leq \|\langle z\rangle^k a(z)\|^{1/2}_{L^p} \Psi_\alpha(z)$. In particular, $a*\sigma \in \Gamma_0^{-k}$.*

Proof. Using Peetre's inequality $\langle z\rangle^k \leq 2^k \langle w\rangle^k \langle z-w\rangle^k$, we have, for suitable $M>0$,

$$\begin{aligned}
&\langle z\rangle^k |\partial_z^\alpha (a*\sigma)(z)| \\
&\leq 2^k \int \langle z-w\rangle^{-M} |a(w)| \langle w\rangle^k \langle z-w\rangle^{M+k} \partial_z^\alpha \sigma(z-w) dw \\
&\leq 2^k \|\langle z-w\rangle^{-M}\|_{L^q} \Big(\int |a(w)|^p \langle w\rangle^{pk} \langle z-w\rangle^{p(M+k)} \Big(\partial_z^\alpha \sigma(z-w)\Big)^p dw\Big)^{\frac{1}{p}} \\
&\leq 2^k \|\langle z-w\rangle^{-M}\|_{L^q} \|a(w)\langle w\rangle^k\|^{1/2}_{L^p} \\
&\quad \times \Big(\int |a(w)|^p \langle w\rangle^{pk} \langle z-w\rangle^{2p(M+k)} \Big(\partial_z^\alpha \sigma(z-w)\Big)^{2p} dw\Big)^{\frac{1}{2p}},
\end{aligned}$$

where $\frac{1}{q}+\frac{1}{p}=1$ and we have used Hölder's inequality. An application of the dominated convergence theorem shows that the last integral tends to zero as $|z| \to +\infty$. □

Theorem 1.2. *Let $a \in L^p(\mathbb{R}^{2n})$, $1 \leq p \leq \infty$. Then the linear map $T : a \in L^p(\mathbb{R}^{2n}) \to Op_{aw}[a] \in B(L^2(\mathbb{R}^n))$ is continuous.*

First proof. Letting $u \in S(\mathbb{R}^n)$, we consider two different estimates of $|(u,\Phi_z)|$:

$$\begin{aligned}
|(u,\Phi_z)| &= \pi^{-n/4} \Big| \int_{\mathbb{R}^n} e^{-i\eta x} e^{-\frac{|x-y|^2}{2}} u(x)\, dx\Big| \\
&\leq \pi^{-n/4} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2}} |u(x)|\, dx = (\Phi_0 * |u|)(y),
\end{aligned} \tag{1.1}$$

with $\Phi_0(x) = \pi^{-n/4} e^{-\frac{|x|^2}{2}}$;

$$\begin{aligned}
|(u,\Phi_z)| &= \Big|\mathcal{F}_{x\to\eta}[(\tau_y\Phi_0)u](\eta)\Big| = \Big|\mathcal{F}_{x\to\eta}[(\tau_y\Phi_0)(\eta)] * \widehat{u}(\eta)\Big| \\
&= (2\pi)^{n/2} \Big|\Phi_0(\eta) * \widehat{u}(\eta)\Big| \leq (2\pi)^{n/2} (\Phi_0 * |\widehat{u}|)(\eta),
\end{aligned} \tag{1.2}$$

where $\mathcal{F}$ denotes the Fourier transform and τ_y the translation by y.

Using (1.1) and (1.2), we can estimate the L^q norm of the orthogonal projection $P_z u$ onto the vector Φ_z uniformly with respect to z:

$$\begin{aligned}
\|(u,\Phi_z)\Phi_z(x)\|_{L^q} &= \pi^{-\frac{n}{4}} \Big(\int |(u,\Phi_z)|^q e^{-q\frac{|x-y|^2}{2}}\, dz\Big)^{\frac{1}{q}} \\
&\leq (2\pi)^{n/4} \Big(\int |(\Phi_0 * |u|)(y)|^{q/2} |(\Phi_0 * |\widehat{u}|)(\eta)|^{q/2}\, dy d\eta\Big)^{\frac{1}{q}} \\
&= (2\pi)^{n/4} \|\sqrt{\Phi_0 * |u|}\,\|_{L^q_y} \|\sqrt{\Phi_0 * |\widehat{u}|}\,\|_{L^q_\eta} := K(u).
\end{aligned} \tag{1.3}$$

Now, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of functions converging to zero in $L^p(\mathbb{R}^{2n})$ as $n\to\infty$, and such that $A_n := Ta_n$ tends to an operator B in $B(L^2(\mathbb{R}^n))$; we will prove that $B=0$, so that the closed graph theorem gives the continuity of the map T.

Using (1.3) and Hölder's inequality, we get, for $\frac{1}{p}+\frac{1}{q}=1$,

$$|A_n u(x)| = \Big|\int_{\mathbb{R}^{2n}} a_n(z) P_z u \, dz\Big| \leq \int |a_n(z)| \, |(u,\Phi_z)\Phi_z(x)| \, dz$$
$$\leq \|a_n\|_p \, \|(u,\Phi_z)\Phi_z(x)\|_q \leq K(u)\|a_n\|_p.$$

Therefore the sequence of functions $A_n u(x)$ converges to the null function uniformly with respect to x.

If $u, v \in S(\mathbb{R}^n)$, we get

$$|(v, A_n u)| \leq \int_{\mathbb{R}^n} |v(x)| \, |A_n u(x)| \, dx \leq \|v\|_{L^1} \, \|A_n u\|_{L^\infty};$$

hence $(v, A_n u) \to 0$ as $n \to \infty$. For $u, v \in S(\mathbb{R}^n)$ we have

$$|(v, (A_n - B)u)| \leq \|v\|_{L^2} \, \|u\|_{L^2} \, \|A_n - B\|_{B(L^2)};$$

since $\|A_n - B\|_{B(L^2)} \to 0$ as $n \to \infty$, it follows $B = 0$. □

Second proof. In Γ_0^0 we consider the topology induced by the seminorms:

$$\sup_{\substack{z\in\mathbb{R}^{2n}\\|\alpha|\leq k}} \big|D_z^\alpha b(z)\big|, \qquad k \in \mathbb{N}, \ \alpha \in \mathbb{Z}_+^n.$$

From the Calderon-Vaillancourt theorem (see [8], [4], [5]) we get the continuity of the map $b \in \Gamma_0^0 \to Op_w[b] \in L^2(\mathbb{R}^n)$ with the norm estimate

$$\|Op_w[b]\| \leq C \sup_{\substack{z\in\mathbb{R}^{2n}\\|\alpha|\leq 2n+1}} \big|D_z^\alpha b(z)\big| \tag{1.4}$$

for suitable $C > 0$.

Using Lemma 1.1 with $k = 0$, we have the estimate $|D_z^\alpha b(z)| \leq C\|a\|_{L^p}$ (which holds also for $p = +\infty$, with a slight modification of the argument of Lemma 1.1); therefore from (1.4) it follows that $\|Op_{aw}[a]\| \leq C\|a\|_{L^p}$, for suitable constants $C > 0$. □

If $b \in L^p(\mathbb{R}^{2n})$ has compact support and for any $\phi \in S(\mathbb{R}^{2n})$ we have $\phi * b \in S(\mathbb{R}^{2n})$, then the following lemma holds:

Lemma 1.3. *Let $b \in L^p(\mathbb{R}^{2n}), 1 \leq p \leq +\infty$. If* supp b *is compact, then $B = Op_{aw}[b]$ is a regularizing operator, i.e. an operator with Weyl symbol in $S(\mathbb{R}^{2n})$.*

Corollary 1.4. *Let $a \in L^p(\mathbb{R}^{2n})$, $1 \leq p < +\infty$. Then the operator $Op_{aw}[a] : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact.*

Proof. Let $\chi_M(z)$ the characteristic function of the set $\{z \in \mathbb{R}^{2n} : |z| \leq M\}$. Then from Lemma 1.3 the operator $A_M = Op_{aw}[a\chi_M]$ is compact. By Theorem 1.2, $\|A - A_M\|_{B(L^2)} \leq C \, \|a - a\chi_M\|_{L^p} \to 0$ as $M \to +\infty$; hence A is compact. □

More generally we can consider anti-Wick symbols in the sum of L^p spaces, setting by definition $Op_{aw}[a+b] = Op_{aw}[a] + Op_{aw}[b]$ if $a \in L^{p_1}$ and $b \in L^{p_2}$, $1 \leq p_1, \ p_2 \leq +\infty$. Then the following result holds:

Corollary 1.5. *Let $p \in [1, +\infty)$ and $a(z) \in L^p(\mathbb{R}^{2n})$, with $a(z) \geq 0$. Then, for every $\epsilon > 0$, $Op_{aw}[a+\epsilon]$ is a bicontinuous bijection on $L^2(\mathbb{R}^{2n})$.*

Proof. Since $a(z)+\epsilon>0$, we have $Op_{aw}[a+\epsilon]>0$; so it is injective on $L^2(\mathbb{R}^{2n})$. As $Op_{aw}[a]$ is compact, $Op_{aw}[a+\epsilon]=Op_{aw}[a]+\epsilon I$ is a Fredholm operator of index zero, so it is also surjective. The continuity of $Op_{aw}[a+\epsilon]^{-1}$ follows from the inverse mapping theorem. □

We notice that, extending to Γ_0^0 the argument in [10], Chapter 4, the compactness of Weyl operators also follows from Lemma 1.1.

2. Hilbert-Schmidt and trace class operators

Let $S_2(L^2(\mathbb{R}^n))$ be the space of Hilbert-Schmidt operators and $S_1(L^2(\mathbb{R}^n))$ the space of trace class operators on $L^2(\mathbb{R}^n)$. We give sufficient conditions for operators with anti-Wick symbols in $L^p(\mathbb{R}^{2n})$ to belong to these classes. Our results are the following.

Theorem 2.1. *Let $a\in L^p(\mathbb{R}^{2n})$, $p\in[1,2]$. Then $A=Op_{aw}[a]\in S_2(L^2(\mathbb{R}^n))$ and $\|A\|_2=\|b\|_{L^2(\mathbb{R}^{2n})}$, where $b=(2\pi)^{-n}a*\sigma\in L^2(\mathbb{R}^{2n})$ is the Weyl symbol of A.*

Proof. Using the same arguments as in [10], we have also for simbols b in the limit class Γ_0^0 that $b\in L^2(\mathbb{R}^{2n})$ if and only if $Op_w[b]\in S_2(L^2(\mathbb{R}^n))$, with $\|A\|_2=\|b\|_{L^2(\mathbb{R}^{2n})}$. Using Young's inequality $\|b\|_2\le(2\pi)^{-n}\|a\|_p\|\sigma\|_q$, $p\ge1$, $q\ge1$, $\frac{1}{p}=\frac{3}{2}-\frac{1}{q}$, we then have $b\in L^2(\mathbb{R}^{2n})$ under the condition $p\in[1,2]$. □

The previous result cannot be improved, as shown by the following:

Theorem 2.2. *For all $p>2$, there exist $a\in L^p(\mathbb{R}^{2n})$ such that $Op_{aw}[a]\notin S_2(L^2(\mathbb{R}^n))$.*

Proof. Assume, to the contrary, that there exists $p>2$ such that $Op_{aw}[a]\in S_2(L^2(\mathbb{R}^n))$ for all $a\in L^p$. Then, for all $a\in L^p$, the convolution $b=(2\pi)^{-n}a*\sigma$ should belong to $L^2(\mathbb{R}^{2n})$, since every symbol $b\in\Gamma_0^0$ satisfies $b\in L^2\Leftrightarrow Op_w[b]\in S_2(L^2(\mathbb{R}^n))$. Then the map $a\in L^p\to a*\sigma\in L^2,(p>2)$ is linear and continuous; moreover it commutes with translations, so by a property of the convolution transform it would be the null operator (see [12]), a contradiction. □

Now we give an explicit example:

Example 2.3. For all $p>2$ we determine anti-Wick symbols $a\in L^p(\mathbb{R}^{2n})$ such that $Op_w(a*\sigma)\notin S_2(L^2(\mathbb{R}^n))$. Using again the property $Op_w(a*\sigma)\in S_2(L^2(\mathbb{R}^n))$ $\Longleftrightarrow a*\sigma\in L^2(\mathbb{R}^{2n})$, we search for an a such that $|a*\sigma|^2\notin L^1(\mathbb{R}^{2n})$.

Let $\langle z\rangle^{-s}=(1+|z|^2)^{-s/2}$; since $\langle z\rangle^{-s}>0$, it follows that

$$|\langle z\rangle^{-s}*\sigma(z)|^2=2^n\left|\int_{\mathbb{R}^{2n}}e^{-|t|^2}\langle z-t\rangle^{-s}dt\right|^2=2^n\left(\int_{\mathbb{R}^{2n}}e^{-|t|^2}\langle z-t\rangle^{-s}dt\right)^2.$$

We want to determine s such that

$$\left(\int_{\mathbb{R}^{2n}}e^{-|t|^2}\langle z-t\rangle^{-s}dt\right)^2\ge\frac{C}{(1+|z|^2)^n}\notin L^1(\mathbb{R}^{2n}),$$

i.e. $\int_{\mathbb{R}^{2n}} e^{-|t|^2}\langle z-t\rangle^{-s}\langle z\rangle^n dt \geq C$. Using Peetre's inequality, we get

$$\begin{aligned}&\int(1+|z|^2)^{(n-s)/2}(1+|z|^2)^{s/2}e^{-|t|^2}(1+|z-t|^2)^{-s/2}\,dt\\ &\geq 2^{-s}\int(1+|z|^2)^{(n-s)/2}(1+|t|^2)^{-s/2}e^{-|t|^2}\,dt\\ &= C'_s(1+|z|^2)^{(n-s)/2}\; > C\end{aligned}$$

if $s \leq n$. On the other hand, since $\langle z\rangle^{-s} \in L^p(\mathbb{R}^{2n})$ we get $s > \frac{2n}{p}$. Therefore, $a(z) = \langle z\rangle^{-s}$ is the example we have searched for, under the condition $\frac{2n}{p} < s \leq n$, which we can satisfy for some s if $p > 2$.

Now we state our result about trace class operators.

Theorem 2.4. *Let $a \in L^1(\mathbb{R}^{2n})$. Then $Op_{aw}[a] \in S_1(L^2(\mathbb{R}^n))$ and*

$$Tr(Op_{aw}[a]) = (2\pi)^{-2n}\int a(y,\eta)\,\sigma(x-y,\xi-\eta)\,dxdyd\xi d\eta. \tag{2.1}$$

Proof. Let $b = (2\pi)^{-n}a * \sigma$. By Young's inequality $\|b\|_1 \leq (2\pi)^{-n}\,\|a\|_1\,\|\sigma\|_1$, and, for all $\alpha \in \mathbb{Z}^n_+$, $\|\partial^\alpha b\|_1 \leq (2\pi)^{-n}\,\|a\|_1\,\|\partial^\alpha\sigma\|_1$. Then $b \in L^1$ and $\partial^\alpha b \in L^1$, $\forall\alpha \in \mathbb{Z}^n_+$, and we have that $Op_w[b] \in S_1(L^2(\mathbb{R}^n))$ and trace's formula (2.1) holds (see Theorem II-53 of [9]). □

We notice that a similar result under different hypothesis is in Berezin [1]. We also remark that, with a minor modification of the same argument, Theorem 2.4 also holds for complex Borel measures as well as for distributions with compact support.

Remark 2.5. Let $a \in L^p(\mathbb{R}^{2n})$, $p > 1$. Then $Op_{aw}[a]$ is not, in general, a trace class operator. An easy example can be constructed from the harmonic oscillator (for simplicity in dimension 1) $H = -\frac{d^2}{dx^2} + |x|^2$. Denote by G^m_ρ the space of pseudo-differential operators with Weyl symbols in Γ^m_ρ, and consider the operator $A = H + I \in G^2_1$; its Weyl symbol is $h_0(x,\xi) = \xi^2 + x^2 + 1 = |z|^2 + 1$, with $z = (x,\xi)$. The operator $A^{-1} = (H+I)^{-1} \in G^{-2}_1$ has Weyl symbol $\sigma_w(A^{-1}) = (|z|^2+1)^{-1} + r(z)$, with $r(z) \in \Gamma^{-3}_1$.

Then $\sigma_w(A^{-1}) \in L^p(\mathbb{R}^2)$ for $p > 1$, but $\sigma_w(A^{-1}) \notin L^1(\mathbb{R}^2)$, and, therefore, $A^{-1} \in S_2(L^2(\mathbb{R}))$, but $A^{-1} \notin S_1(L^2(\mathbb{R}))$ because $\|A^{-1}\|_{tr} = \sum_{j\in\mathbb{N}}\frac{1}{2j+2} = +\infty$.

Every $K \in G^m_\rho$ has an anti-Wick symbol, if we consider it modulo regularizing operators (see, for instance, [2], [10]). Hence there exists $\tilde{A}^{-1} \in G^2_1$, such that $\tilde{A}^{-1} - A^{-1} \in G^{-\infty}$, and its anti-Wick simbol is $\tilde{a}(z) = (1+|z|^2)^{-1} + r(z) + \tilde{r}(z)$, with $\tilde{r}(z) \in \Gamma^{-4}_1$.

Then, $\tilde{a}(z) = (1+|z|^2)^{-1} + r(z)$, with $r(z) \in \Gamma^{-3}_1$. Therefore $\tilde{A}^{-1} \notin S_1(L^2(\mathbb{R}))$, and its anti-Wick simbol satisfies $\tilde{a}(z) \notin L^1(\mathbb{R}^2)$ but $\tilde{a}(z) \in L^p(\mathbb{R}^2)$, $\forall p > 1$.

References

1. F.A. Berezin, *Wick and Anti-Wick operator symbols*, Math. Sb. **86**(128), (1971), pp. 578–610; Math. USSR Sb. **15** (1971), pp. 577–606. MR **45:**929
2. B. Boggiatto, E. Buzano, L. Rodino, *Global Hypoellipticity and Spectral Theory*, Akademie Verlag, Berlin, Vol. **92**, (1996). MR **97m:**35040
3. P. Boggiatto, L. Rodino, *Partial Differential Equations of Multi-Quasi-Elliptic Type*, Ann. Univ. Ferrara - Sez. VII - Sc. Mat. Suppl. Vol. XLV, pp. 275–291 (1999). CMP 2001:07

4. A. Calderon, R. Vaillancourt, *On the Boundedness of Pseudo-differential Operators*, J. Math Soc. Japan, **23**, (1971), pp.374–378. MR **44:**2096
5. A. Calderon, R. Vaillancourt, *A Class of Bounded Pseudo-differential Operators*, Proc. Nat. Acad. Sci. USA, **69** (1972) pp.1185–1187. MR **45:**7532
6. I. Daubechies, *Ten Lectures on Wavelets*, Capital City Press, CBMS-NSF Regional Conference Series in Appl. Math. (1992). MR **93e:**42045
7. J. Du, M.W. Wong, *A Product Formula for Localization Operators*, Bull. Korean Math. Soc. **37** (2000), No. 1, pp. 77–84. MR **2000m:**47066
8. G. B. Folland, *Harmonic Analysis in Phase Space*. Annals of Math. Studies, **122** (1989). Princeton University Press, Princeton, NJ. MR **92k:**22017
9. D. Robert, *Autour de l'Approximation Semi-Classique*, Birkhäuser-Verlag, Progress in Mathematics, Vol. **68** (1987). MR **89g:**81016
10. M. A. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag, Berlin (1987). MR **88c:**47105
11. B. Simon, *The Weyl Transform and L^p Functions on Phase Space*, Proc.Amer.Math.Soc.**4** (1992). MR **93b:**81131
12. E. M. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, **32** Princeton University Press, (1971). MR **46:**4102
13. M.W. Wong, *Weyl Transforms*, Springer-Verlag, New York, Universitext, 1998. MR **2000c:**47098

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
E-mail address: boggiatto@dm.unito.it

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
E-mail address: cordero@dm.unito.it