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# CHARACTER DEGREE SETS THAT DO NOT BOUND THE CLASS OF A $p$-GROUP

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ABSTRACT. Suppose that we are given a set $\mathcal{S}$ of powers of a prime $p$ and that $1 \in \mathcal{S}$. A technique is presented that enables the construction of a $p$-group of specified nilpotence class $n$ such that its set of irreducible character degrees is exactly $\mathcal{S}$. If $|\mathcal{S}| \geq 2$, then this can be done for $2 \leq n \leq p$ and if $p \in \mathcal{S}$, then the only requirement is $2 \leq n$.

## 1. INTRODUCTION

If $G$ is a finite group, then as usual, we write $\mathrm{cd}(G)$ to denote the set of degrees of the irreducible characters of $G$. Now fix some prime $p$ and let $\mathcal{S}$ be a finite set of powers of $p$, subject only to the condition that $1 \in \mathcal{S}$. It is known that there necessarily exists some $p$-group $P$ such that $\mathrm{cd}(P) = \mathcal{S}$, and in fact, it is always possible to choose $P$ so that its nilpotence class $c(P)$ is at most 2. (This is the main result of [2].)

For some choices of the set $\mathcal{S}$, there exists an upper bound on the nilpotence classes of those $p$-groups $P$ for which $\mathrm{cd}(P) = \mathcal{S}$, while for other sets $\mathcal{S}$, there is no such bound. For example, it was shown in [1] that if $\mathrm{cd}(P) = \{1, p^e\}$, then the class of $P$ is at most $p$ if $e > 1$, but that the class of $P$ can be unboundedly large if $e = 1$. (The problem of determining which sets of powers of $p$ imply an upper bound on the nilpotence class of a $p$-group having that degree set was first suggested by the second author in his paper [6].)

We shall say that a set $\mathcal{S}$ of powers of $p$ (containing 1) is **class bounding** if there exists an upper bound (depending on $\mathcal{S}$, of course) for the nilpotence classes of all $p$-groups $P$ such that $\mathrm{cd}(P) = \mathcal{S}$. By the result mentioned in the previous paragraph, for example, we see that if $|\mathcal{S}| = 2$, then $|\mathcal{S}|$ is class bounding if and only if $p \notin \mathcal{S}$. It was proved in [5] that the corresponding result also holds when $|\mathcal{S}| = 3$. In this case too, the set $\mathcal{S}$ is class bounding if and only if $p \notin \mathcal{S}$. Other class-bounding sets were constructed in [5], including some having arbitrarily large cardinality, but all known class-bounding sets of powers of $p$ fail to contain the number $p$.

Of course, these results suggest the general conjecture that an arbitrary set $\mathcal{S}$ of powers of $p$ (containing 1) is class bounding if and only if $p \notin \mathcal{S}$. The main result of this paper establishes the "only if" part of this conjecture.

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**Theorem A.** *Let $p$ be any prime, let $n \geq 2$ be an integer and suppose that $\mathcal{S}$ is a finite set of powers of $p$ containing both $1$ and $p$. Then there exists a $p$-group $P$ with nilpotence class $n$ and $\mathrm{cd}(P) = \mathcal{S}$.*

Our proof of Theorem A uses a variation of the technique used in [5] to prove the corresponding result when $|\mathcal{S}| = 3$.

## 2. The construction

Our principal construction tool is the well known "pullback", as is described in the following lemma.

**(2.1) Lemma.** *Let $A \triangleleft X$ and $B \triangleleft Y$ be groups such that $X/A \cong Y/B$. Then there exists a group $P$ having normal subgroups $R$ and $S$ with the following properties:*

(a) *There exist isomorphisms $\alpha : P/R \to X$ and $\beta : P/S \to Y$.*
(b) $R \cap S = 1$.
(c) $\alpha(RS/R) = A$ *and* $\beta(RS/S) = B$.

*Proof.* Write $\overline{X} = X/A$ and $\overline{Y} = Y/B$ and let $\theta : \overline{X} \to \overline{Y}$ be the given isomorphism. Let $\Gamma = X \times Y$ be the external direct product and take $P$ to be the subgroup of $\Gamma$ consisting of all pairs $(x, y)$ such that $\theta(\overline{x}) = \overline{y}$. It is clear that for each element $x \in X$, there exists an element $y \in Y$ such that $(x, y)$ lies in $P$, and since $\theta$ is surjective, we see that the reverse is true too: For each element $y \in Y$, there exists $x \in X$ such that $(x, y) \in P$. In other words, the projection homomorphisms from $P$ to $X$ and from $P$ to $Y$ are surjective. We let $R$ and $S$ be respectively the kernels of these homomorphisms and we let $\alpha$ and $\beta$ be the corresponding isomorphisms from $P/R$ onto $X$ and from $P/S$ onto $Y$. We have now established (a).

The first component of an arbitrary element of $R$ is 1, and hence the second component can be any element $y \in Y$ such that $\theta(\overline{1}) = \overline{y}$. In other words, the second component of an element of $R$ is an arbitrary element of $B$, and thus $R = 1 \times B$. Also, since $\theta$ is injective, we see that $S = A \times 1$, and in particular, it is clear that $R \cap S = 1$, as required.

Now $\alpha(RS/R)$ is the image of $RS = A \times B$ under the projection to $X$, and it follows that $\alpha(RS/R) = A$. Similarly, $\beta(RS/S) = B$, and the proof is complete. □

In the notation of Lemma 2.1, it is easy to see that if $X$ and $Y$ are $p$-groups, then $P$ is also a $p$-group. Furthermore, it follows from the facts that $P/R \cong X$, $P/S \cong Y$ and $R \cap S = 1$ that the nilpotence class $c(P) = \max\{c(X), c(Y)\}$.

**(2.2) Definition.** Let $a \geq 0$ be an integer. We will say that a $p$-group $P$ is **suitable** with respect to $a$ if $P$ has an abelian normal subgroup $A$ such that the following hold:

(1) $P/A$ is elementary abelian of order $p^a$.
(2) Every linear character of $A$ extends to its stabilizer in $P$.

Note that if $P$ is suitable with respect to $a$, then since the subgroup $A$ is abelian and has index $p^a$, it follows that every irreducible character of $P$ has degree at most $|P : A| = p^a$. Note also that every abelian $p$-group is suitable with respect to 0 and that every $p$-group having an abelian subgroup of index $p$ is suitable with respect to 1. (To check this last assertion, it suffices to observe that an invariant irreducible character of a normal subgroup of prime index in any finite group is guaranteed to be extendible. This is Corollary 6.20 of [3].) The key to the proof of Theorem A is the following "one-new-degree" theorem.

**(2.3) Theorem.** *Let $b > a \geq 0$ be integers and suppose that $Q$ is a $p$-group that is suitable with respect to $a$. Then there exists a $p$-group $P$ that is suitable with respect to $b$ and such that $\mathrm{cd}(P) = \mathrm{cd}(Q) \cup \{p^b\}$. Furthermore, $P$ can be chosen so that its nilpotence class $c(P)$ is the maximum of 2 and the class $c(Q)$.*

*Proof.* Let $X = Q \times V$, where $V$ is elementary abelian of order $p^{b-a}$ and view $Q$ and $V$ as subgroups of $X$. Since $Q$ is suitable with respect to $a$, we know that $Q$ has an abelian normal subgroup $A$ such that $Q/A$ is elementary abelian of order $p^a$, and it follows that $A \triangleleft X$ and that $X/A$ is elementary abelian of order $p^b$.

Now let $Y$ be extraspecial of order $p^{2b+1}$ and let $B$ be a maximal abelian subgroup of $Y$. Then as is well known, $|Y : B| = p^b$ and $Y/B$ is elementary abelian, and thus $X/A \cong Y/B$. We can therefore construct the "pullback" $P$, as in Lemma 2.1, and we use the notation of that lemma. As we remarked earlier, $P$ is a $p$-group and $c(P)$ is the maximum of $c(X) = c(Q)$ and $c(Y) = 2$, as required.

Now $RS \cong R \times S \cong A \times B$, which is abelian, and $P/RS \cong X/A$, which is elementary abelian of order $p^b$. To prove that $P$ is suitable with respect to $b$, therefore, it suffices to show that every linear character of the subgroup $RS$ extends to its stabilizer in $P$. Every such linear character is uniquely of the form $\lambda\mu$, where $R \subseteq \ker(\lambda)$ and $S \subseteq \ker(\mu)$, and because of the uniqueness, we see that the stabilizer of $\lambda\mu$ in $P$ is the intersection of the stabilizers in $P$ of $\lambda$ and $\mu$.

There are just two possibilities for the stabilizer of $\mu$ in $P$. To see why this is so, observe that $\mu$ can be identified with a linear character $\mu_0$ of $B \cong RS/S$. (Recall that $B$ is a maximum abelian subgroup of the extraspecial group $Y \cong P/S$.) If $Y' \subseteq \ker(\mu_0)$, then $\mu_0$ is extendible to $Y$ and hence is invariant in $Y$. Otherwise, $\mu_0$ induces irreducibly to $Y$, and hence $B$ is its stabilizer in $Y$. We deduce that either $\mu$ is invariant in (and extends to) $P$ or else the stabilizer of $\mu$ in $P$ is the subgroup $RS$. In the latter case, we see that the stabilizer of $\lambda\mu$ in $P$ is $RS$, and in this situation it is trivially true that $\lambda\mu$ extends to its stabilizer, as required. We can thus suppose that $\mu$ is invariant in $P$ and we let $T$ be the stabilizer of $\lambda$ in $P$. Then $T$ is also the stabilizer of $\lambda\mu$ in $P$ and we must show that $\lambda\mu$ extends to $T$. But since $\mu$ extends to $P$ in this situation, it suffices to show that $\lambda$ extends to $T$.

Now $\lambda$ corresponds to a linear character $\lambda_0$ of $A \cong RS/R$. Also, the stabilizer $T_0$ of $\lambda_0$ in $X = Q \times V$ is the subgroup that corresponds to $T/R$ under the isomorphism $X \cong P/R$. Since $Q$ is suitable, we know that $\lambda_0$ extends to its stabilizer in $Q$, and thus it also extends to its stabilizer $T_0$ in $X = Q \times V$. It follows from this that $\lambda$ extends to its stabilizer $T$ in $P$, and this completes the proof that $P$ is suitable with respect to $b$, as desired.

Finally, we must show that $\mathrm{cd}(P) = \mathrm{cd}(Q) \cup \{p^b\}$. Since $P/R \cong X = Q \times V$, we see that $Q$ is a homomorphic image of $P$, and thus $\mathrm{cd}(Q) \subseteq \mathrm{cd}(P)$. Similarly, the extraspecial group $Y \cong P/S$ is a homomorphic image of $P$ and we know that $\mathrm{cd}(Y) = \{1, p^b\}$, and thus $p^b \in \mathrm{cd}(P)$, as desired. To complete the proof, we must consider an arbitrary character $\chi \in \mathrm{Irr}(P)$ and show that either $\chi(1) = p^b$ or $\chi(1) \in \mathrm{cd}(Q)$.

Let $\lambda\mu$ be a linear constituent of $\chi_{RS}$, where, as before, $R \subseteq \ker(\lambda)$ and $S \subseteq \ker(\mu)$. We saw previously that the stabilizer of $\mu$ is either $RS$ or $P$. If the stabilizer of $\mu$ is $RS$, we know that the stabilizer of $\lambda\mu$ is also $RS$, and thus $(\lambda\mu)^P$ is irreducible. In this case, $\chi = (\lambda\mu)^P$ has degree $|P : RS| = p^b$, and there is nothing further to prove. In the remaining case, we know that $\lambda\mu$ extends to its stabilizer $T$ in $P$, and thus since $T/RS$ is abelian, it follows that every irreducible character of $T$ that lies over $\lambda\mu$ is linear. We conclude from this that $\chi(1) = |P : T|$. But we know in this case that $T$ corresponds to the stabilizer $T_0$ of some linear character $\lambda_0$ of $A$ in the group $X = Q \times V$, and we also know that $\lambda_0$ extends to $T_0$. It follows that the number $|P : T| = |X : T_0|$ lies in the set $\mathrm{cd}(X) = \mathrm{cd}(Q)$. This completes the proof. □

Our principal application of Theorem 2.3 is the following, from which Theorem A is immediate.

**(2.4) Corollary.** *Suppose that $Q$ is a $p$-group that is suitable with respect to $e$, where $p^e$ is the largest member of $\mathrm{cd}(Q)$, and let $\mathcal{B}$ be a set of powers of $p$ all of which exceed $p^e$. Then there exists a $p$-group $P$ such that $\mathrm{cd}(P) = \mathrm{cd}(Q) \cup \mathcal{B}$. Furthermore, we can choose $P$ so that $c(P) = c(Q)$ except when $c(Q) = 1$ and $\mathcal{B}$ is nonempty, in which case $c(P) = 2$.*

*Proof.* We can certainly assume that $\mathcal{B}$ is nonempty and we write $\mathcal{B} = \{p^{e_1}, p^{e_2}, \ldots, p^{e_r}\}$, where $e_1 < e_2 < \cdots < e_r$. We set $P_0 = Q$ and we use Theorem 2.3 repeatedly to construct a sequence of $p$-groups $P_i$, where $1 \leq i \leq r$. We can do this so that $\mathrm{cd}(P_i) = \mathrm{cd}(P_{i-1}) \cup \{p^{e_i}\}$ and each of the groups $P_i$ is suitable with respect to $e_i$. Furthermore, the nilpotence classes of the groups $P_i$ will all be equal to $c(Q)$ except when $c(Q) = 1$, in which case $c(P_i) = 2$ for $i > 0$. The group $P = P_r$ has the desired properties. □

*Proof of Theorem* A. By [1], we know that there exists a $p$-group $Q$ such that $c(Q) = n$ and $\mathrm{cd}(Q) = \{1, p\}$. Furthermore, $Q$ has an abelian subgroup of index $p$, and thus $Q$ is suitable with respect to 1. We can now apply Corollary 2.4 with this group $Q$ and with $\mathcal{B} = \mathcal{S} - \{1, p\}$ to obtain a group $P$ such that $c(P) = n$ and $\mathrm{cd}(P) = \mathrm{cd}(Q) \cup \mathcal{B} = \mathcal{S}$, as desired. □

## 3. Further applications

We can also use Corollary 2.4 to give a new proof of the main theorem of [2].

**(3.1) Theorem.** *Let $\mathcal{S}$ be a set of powers of $p$ and assume that $1 \in \mathcal{S}$. Then there exists a $p$-group $P$ with nilpotence class 2 and $\mathrm{cd}(P) = \mathcal{S}$.*

*Proof.* Let $Q$ be any abelian $p$-group. Then $Q$ is suitable with respect to 0 and $\mathrm{cd}(Q) = \{1\}$. Now apply Corollary 2.4 with $\mathcal{B} = \mathcal{S} - \{1\}$ to obtain the desired group $P$. □

If $\mathcal{S}$ is a class-bounding set of powers of $p$, we ask how small the corresponding bound can be. We can use Corollary 2.4 to show that this bound can never be smaller than $p$, and in fact, we have a bit more.

**(3.2) Theorem.** *Let $\mathcal{S}$ be a set of powers of $p$ and assume that $1 \in \mathcal{S}$ and $|\mathcal{S}| > 1$. If $n$ is an integer and $2 \leq n \leq p$, then there exists a $p$-group $P$ with nilpotence class $n$ and $\mathrm{cd}(P) = \mathcal{S}$.*

*Proof.* Let 1 and $p^e$ be the two smallest members of $\mathcal{S}$. We claim that there exists a $p$-group $R$ such that $\mathrm{cd}(R) = \{1, p^e\}$ and class $c(R) = p$, and that in addition, $R$ can be chosen so that it has an abelian normal subgroup $B$ such that $R/B$ is elementary abelian of order $p^e$. If $e = 1$, it is easy to construct such a group: Let $R$ be the wreath product of a cyclic group of order $p$ with itself. If $e > 1$, then Example 3.9 of [1] shows that $R$ exists.

Since $n \leq p$, there is some homomorphic image $Q$ of $R$ with $c(Q) = n$, and we know that $\mathrm{cd}(Q) \subseteq \mathrm{cd}(R) = \{1, p^e\}$. Also, because $n \geq 2$, we see that $Q$ is nonabelian, and thus $\mathrm{cd}(Q) = \{1, p^e\}$. Furthermore, the image $A \subseteq Q$ of $B$ is abelian and $|Q : A| \leq |R : B| = p^e$. It follows that $|Q : A| = p^e$ and $Q/A \cong R/B$ is elementary abelian.

Now if $\lambda$ is any linear character of $A$, we show that $\lambda$ extends to its stabilizer in $Q$. If some linear character of $Q$ lies over $\lambda$, this is obvious, and otherwise there exists $\chi \in \mathrm{Irr}(Q)$ lying over $\lambda$ with $\chi(1) = p^e$. Since $|P : A| = p^e$, it follows that $\lambda^G = \chi$, and thus the full stabilizer of $\lambda$ in $P$ is the subgroup $A$. In this case too, we see that $\lambda$ extends to its stabilizer, and thus $Q$ is suitable with respect to $e$.

We can now apply Corollary 2.4 with $\mathcal{B} = \mathcal{S} - \{1, p^e\}$ to obtain a group $P$ with the desired properties. $\square$

## 4. Some questions

The big remaining question, of course, is the following.

**Question 1.** If $\mathcal{S}$ is a set of powers of $p$ that contains 1 but not $p$, must $\mathcal{S}$ be class bounding?

It seems hard to find sets that one can prove to be class bounding, and so the answer to Question 1 may well be "no". In fact, the assumption that $\mathcal{S}$ is class bounding may turn out to be extremely restrictive. It may be, for example, that if $\mathcal{S}$ is class bounding and $\mathrm{cd}(P) = \mathcal{S}$, then the structure of $P$ is very tightly constrained. For example, we do not know the answer to the following.

**Question 2.** Does there exist a $p$-group $P$ such that $\mathrm{cd}(P)$ is class bounding and $P$ is not metabelian?

If the answer to Question 2 is "no", then the answer to Question 1 must also be "no". This is because there exist $p$-groups $P$ of arbitrarily large derived length for which $p \notin \mathrm{cd}(P)$. For example, we can take $P$ to be a Sylow $p$-subgroup of $GL(n, p^e)$ where $e > 1$ and $n$ is large. In this situation, it is known (see [4]) that $\mathrm{cd}(P)$ consists of powers of $p^e$, and so does not contain $p$, but the nilpotence class and derived length of $P$ are unboundedly large as $n$ grows.

For every class-bounding set $\mathcal{S}$, there is some integer $b(\mathcal{S})$, which is the largest possible nilpotence class for a $p$-group $P$ such that $\mathrm{cd}(P) = \mathcal{S}$. By Theorem 3.2, we know that $b(\mathcal{S}) \geq p$ for every class-bounding set other than $\{1\}$, but is there an upper bound for $b(\mathcal{S})$?

**Question 3.** Can $b(\mathcal{S})$ be arbitrarily large for a class-bounding set $\mathcal{S}$?

If the answer to Question 3 is "no", then again, by considering a Sylow $p$-subgroup of $GL(n, p^e)$ with $e > 1$ and $n$ large, we see that the answer to Question 1 must be "no".

We know of just one class-bounding set $\mathcal{S}$ for which we can show that $b(\mathcal{S})$ exceeds $p$: the set $\mathcal{S} = \{1, 4, 16\}$. (This set is class bounding because it has cardinality 3 and does not contain $p = 2$.) If $P$ is a Sylow 2-subgroup of $GL(4, 4)$, then it is well known that $P$ has class 3. Also, all members of $\mathrm{cd}(P)$ are powers of 4 and it is easy to see that $P$ has an abelian subgroup of index 16. It follows that $\mathrm{cd}(P) \subseteq \{1, 4, 16\}$, and it is not too hard to show that all three of these degrees occur, and thus $\mathrm{cd}(P) = \{1, 4, 16\}$. This suggests what is probably the least ambitious of our questions.

**Question 4.** If $p > 2$, does there exist a $p$-group $P$ of class $p+1$ such that $\mathrm{cd}(P) = \{1, p^2, p^4\}$?

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