

## ON THE CHROMATIC NUMBER OF KNESER HYPERGRAPHS

JIŘÍ MATOUŠEK

(Communicated by John R. Stembridge)

ABSTRACT. We give a simple and elementary proof of Kříž's lower bound on the chromatic number of the Kneser  $r$ -hypergraph of a set system  $\mathcal{S}$ .

### 1. INTRODUCTION

Let  $\mathcal{S}$  be a system of subsets of a finite set  $X$ . The *Kneser  $r$ -hypergraph*  $\text{KG}_r(\mathcal{S})$  has  $\mathcal{S}$  as the vertex set, and an  $r$ -tuple  $(S_1, S_2, \dots, S_r)$  of sets in  $\mathcal{S}$  forms an edge if  $S_i \cap S_j = \emptyset$  for all  $i \neq j$ . In particular,  $\text{KG}(\mathcal{S}) = \text{KG}_2(\mathcal{S})$  is the *Kneser graph* of  $\mathcal{S}$ . Kneser [8] conjectured in 1955 that

$$\chi\left(\text{KG}\left(\binom{[n]}{k}\right)\right) \geq n - 2k + 2, \quad n \geq 2k,$$

where  $\binom{[n]}{k}$  denotes the system of all  $k$ -element subsets of the set  $[n] = \{1, 2, \dots, n\}$ , and  $\chi$  denotes the chromatic number. This was proved in 1978 by Lovász [12], as one of the earliest and most spectacular applications of topological methods in combinatorics. Several other proofs have been published since then, all of them topological; among them, at least those of Bárány [2], Dol'nikov [6] (also see [5] and [7]), and Sarkaria [14] can be regarded as substantially different from each other and from Lovász' original proof. Erdős' generalization of Kneser's conjecture to hypergraphs, dealing with the chromatic number of  $\text{KG}_r(\binom{[n]}{k})$ , was established by Alon, Frankl, and Lovász [1].

Kříž [10], [11] proved a remarkable lower bound for the chromatic number of  $\text{KG}_r(\mathcal{S})$  for an arbitrary set system  $\mathcal{S}$ , which easily implies the correct bound in the case when  $\mathcal{S} = \binom{[n]}{k}$  considered by Alon et al. (for  $r = 2$ , the result was obtained earlier by Dol'nikov [6]).

To state this result, we first recall that a mapping  $c: V \rightarrow [m]$  is a (*proper*) *coloring* of a hypergraph  $\mathcal{H} = (V, E)$  if none of the edges  $e \in E$  is monochromatic under  $c$ . The *chromatic number*  $\chi(\mathcal{H})$  of  $\mathcal{H}$  is the smallest  $m$  such that a proper coloring  $c: V \rightarrow [m]$  exists. We define the  *$r$ -colorability defect* of  $\mathcal{H} = (V, E)$  as the smallest number of vertices that must be removed so that the edges living completely on the remaining points form an  $r$ -colorable hypergraph, i.e.

$$\text{cd}_r(\mathcal{H}) = \min\left\{|Y| : \chi((V \setminus Y, \{e \in E : e \cap Y = \emptyset\})) \leq r\right\}.$$

---

Received by the editors January 12, 2000 and, in revised form, March 30, 2001.

2000 *Mathematics Subject Classification*. Primary 05C15, 05C65.

This research was supported by Charles University grants No. 158/99 and 159/99.

©2002 American Mathematical Society

Kříž's result can be stated as follows.

**Theorem 1.1** (Dol'nikov for  $r = 2$ ; Kříž). *For any finite set system  $(X, \mathcal{S})$  and any  $r \geq 2$ , we have*

$$\chi(\text{KG}_r(\mathcal{S})) \geq \frac{1}{r-1} \cdot \text{cd}_r((X, \mathcal{S})).$$

The proof in [10] does not work in the generality stated there (as was pointed out by Živaljević) but Theorem 1.1 remains valid [11]. We remark that  $\text{KG}_r(\mathcal{S})$  is denoted by  $[\mathcal{S}, r]$  in [10], and  $\text{cd}_r(\mathcal{S})$  is denoted by  $w(\mathcal{S}, r)$  there and called the  $r$ -width.

In this paper, we present another proof of Theorem 1.1. The basic approach is similar to that of Kříž, but our proof is somewhat simpler and, hopefully, more accessible to non-specialists in topology.

We only assume the reader's familiarity with a few basic topological notions (such as simplicial complex and its geometric realization); more special topological notions are reviewed in Section 2, in very concrete form just suitable for our purposes. We refer to Björner [3] or Živaljević [16], [17] for wider background and for nice recent overviews of topological methods in combinatorics.

After this paper has been submitted for publication, the author obtained a “de-topologized” (combinatorial) proof of Kneser's conjecture [13], by directly connecting some of the ideas of the present paper to a combinatorial lemma (Tucker's lemma) in one of the proofs of the Borsuk–Ulam theorem. This result was further extended by Ziegler [15], who proved a common generalization of Theorem 1.1 and of theorems of Alon et al. [1] and of Sarkaria [14]. The proof is based on topological ideas but uses no “continuous” structure.

## 2. PRELIMINARIES

**Simplicial complexes.** For our purposes, a (finite) simplicial complex  $K$  is a hereditary family of subsets of a finite set (i.e. if  $F \in K$  and  $F' \subseteq F$  then  $F' \in K$ ); the sets in  $K$  are called *simplices*. The *dimension* of a simplex is the number of its vertices minus 1. The vertex set of  $K$  is denoted by  $V(K)$ , and the polyhedron of a geometric realization of  $K$  is denoted by  $\|K\|$ .

Let  $(V, \leq)$  be a partially ordered set. The *order complex* of  $(V, \leq)$  is the simplicial complex with vertex set  $V$  and with all chains under  $\leq$  (i.e. subsets of  $V$  linearly ordered by  $\leq$ ) as simplices. The *first barycentric subdivision* of a simplicial complex  $K$ , denoted by  $\text{sd}(K)$ , is the order complex of the set of all nonempty simplices of  $K$  ordered by inclusion. The polyhedra of  $K$  and of  $\text{sd}(K)$  are canonically homeomorphic.

Let  $K, L$  be simplicial complexes. A *simplicial map*  $f: K \rightarrow L$  is a map  $V(K) \rightarrow V(L)$  such that the image of any simplex of  $K$  is contained in a simplex of  $L$ . A simplicial map induces a map  $\|K\| \rightarrow \|L\|$  of topological spaces.

The *join*  $K * L$  of simplicial complexes  $K$  and  $L$  with  $V(K) \cap V(L) = \emptyset$  is the simplicial complex with vertex set  $V(K) \cup V(L)$  and with simplices  $F \cup G$  for all  $F \in K$  and  $G \in L$ . If  $V(K)$  and  $V(L)$  are not disjoint, then  $K * L$  is the join of  $K$  with an isomorphic copy of  $L$  whose vertex set is disjoint from  $V(K)$ . If  $K_1, K_2, L_1, L_2$  are simplicial complexes,  $V(K_1) \cap V(L_1) = \emptyset$ , and  $f: K_1 \rightarrow K_2$  and  $g: L_1 \rightarrow L_2$  are simplicial maps, then  $f * g: K_1 * L_1 \rightarrow K_2 * L_2$  is the simplicial map given by  $(f * g)(v) = f(v)$  for  $v \in V(K_1)$  and  $(f * g)(v) = g(v)$  for  $v \in V(L_1)$ .

**Connectivity.** Let  $X, Y$  be topological spaces, and  $k \geq 0$  an integer. All mappings between topological spaces are implicitly assumed to be continuous.  $X$  is  $k$ -connected if for any  $j = 0, 1, \dots, k$ , any mapping  $f$  of the  $j$ -dimensional sphere  $S^j$  into  $X$  can be extended to a mapping of the  $(j + 1)$ -dimensional ball into  $X$ .

**$Z_p$ -spaces.** A  $Z_p$ -space is a pair  $(X, \nu)$ , where  $\nu : X \rightarrow X$  is a homeomorphism  $X \rightarrow X$  with  $\nu^p = \text{id}_X$ ;  $\nu$  is called a  $Z_p$ -action on  $X$  (here  $Z_p$  denotes the group  $Z/pZ$ , i.e. integers modulo  $p$  with addition). The  $Z_p$ -action  $\nu$  is called *free* if for each  $x \in X$  the points  $x, \nu(x), \nu^2(x), \dots, \nu^{p-1}(x)$  are pairwise distinct. For prime  $p$ , it suffices to require  $\nu(x) \neq x$  for all  $x$ . A simplicial  $Z_p$ -complex is a simplicial complex  $K$  with a  $Z_p$ -action on  $\|K\|$  given by a simplicial map  $K \rightarrow K$ .

For  $Z_p$ -spaces  $(X, \nu), (Y, \omega)$ , a  $Z_p$ -mapping  $f : (X, \nu) \rightarrow (Y, \omega)$  is a mapping of  $X$  into  $Y$  that commutes with the  $Z_p$ -actions, i.e.  $f \circ \nu = \omega \circ f$ .

**The  $Z_p$ -index.** For integers  $k$  and  $p$ , we define the simplicial complex  $E_{k,p}$  whose maximal simplices are the edges of the complete  $(k+1)$ -uniform  $(k+1)$ -partite hypergraph with classes of size  $p$ . More formally, the vertex set is  $[k+1] \times [p]$  and the simplices have the form  $\{(j_1, i_1), (j_2, i_2), \dots, (j_q, i_q)\}$ ,  $1 \leq j_1 < j_2 < \dots < j_q \leq k+1$  and  $i_t \in [p]$ ,  $t = 1, 2, \dots, q$ . The mapping  $\omega : V(E_{k,p}) \rightarrow V(E_{k,p})$  given by  $(j, i) \mapsto (j, i+1)$ , where  $p+1$  means 1, is a free simplicial  $Z_p$ -action on  $E_{k,p}$ . In particular,  $E_{k,2}$  is the  $k$ -dimensional sphere represented as the unit sphere of the  $L_1$ -norm, and the  $Z_2$ -action is the antipodality  $x \mapsto -x$ . The important property of  $E_{k,p}$  is that its polyhedron is a  $k$ -dimensional,  $(k-1)$ -connected free  $Z_p$ -space;<sup>1</sup> any  $k$ -dimensional  $(k-1)$ -connected free simplicial  $Z_p$ -complex (or  $Z_p$ -CW-complex) would do equivalently in the definition below.

For a free  $Z_p$ -space  $(X, \nu)$ , the  $Z_p$ -index is defined by

$$\text{ind}_{Z_p}(X) = \min\{k : \text{there is a } Z_p\text{-map } (X, \nu) \rightarrow (\|E_{k,p}\|, \omega)\}$$

(the action  $\nu$  is not shown in the notation  $\text{ind}_{Z_p}$  but is understood from context). This kind of index, under the name *genus*, was introduced by Krasnosel'skiĭ [9] (for  $Z_2$ -spaces); our presentation follows [16]. Let us remark that, while various definitions of indices and deep theories related to them have been developed in algebraic topology, the index just introduced is mainly a convenient notational shorthand.

The key fact about the  $Z_p$ -index is  $\text{ind}_{Z_p}(\|E_{k,p}\|) = k$ , i.e. there is no  $Z_p$ -map  $\|E_{k,p}\| \rightarrow \|E_{k-1,p}\|$ . For  $p = 2$ , this is one of the versions of the well-known Borsuk–Ulam theorem, and for larger  $p$ , it is a particular case of a theorem of Dold [4]; see e.g. [17] for a sketch of a proof using only basic homology theory.

Clearly, if there is a  $Z_p$ -map  $(X, \nu) \rightarrow (Y, \omega)$ , then  $\text{ind}_{Z_p}(X) \leq \text{ind}_{Z_p}(Y)$ . For a free simplicial  $Z_p$ -complex, we have  $\text{ind}_{Z_p}(\|K\|) \leq \dim(K)$  (this can be shown easily using the  $(k-1)$ -connectedness of  $E_{k,p}$ ; see, for example, [17]). For free simplicial  $Z_p$ -complexes  $K$  and  $L$ , we have

$$(1) \quad \text{ind}_{Z_p}(K * L) \leq \text{ind}_{Z_p}(K) + \text{ind}_{Z_p}(L) + 1,$$

where the  $Z_p$ -action on  $K * L$  is the join of the  $Z_p$ -actions on  $K$  and on  $L$ . This is easily derived from the isomorphism of  $E_{k,p} * E_{\ell,p}$  with  $E_{k+\ell+1,p}$ .

<sup>1</sup>The  $(k-1)$ -connectedness can be derived in several ways, for example by representing  $E_{k,p}$  as the  $(k+1)$ -fold join  $[p]^{*(k+1)}$ , where  $[p]$  is the  $p$ -point discrete space, and using the fact that the join of a  $j$ -connected simplicial complex and of an  $\ell$ -connected simplicial complex is  $(j + \ell + 2)$ -connected (see e.g. [3]).

## 3. PROOF OF THEOREM 1.1

First, let  $r = p$  be a prime number. Let  $X = [n]$ , and let  $\mathcal{S}$  be a set system on  $X$  with  $\text{cd}_p((X, \mathcal{S})) > \ell$ .

We define a partial ordering  $\leq$  on the set of all ordered  $p$ -tuples  $(A_1, A_2, \dots, A_p)$  of subsets of  $X$  by letting  $(A_1, \dots, A_p) \leq (A'_1, \dots, A'_p)$  iff  $A_i \subseteq A'_i$  for all  $i = 1, 2, \dots, p$ .

Consider the set of all ordered  $p$ -tuples  $(A_1, A_2, \dots, A_p)$  such that the  $A_i$  are pairwise disjoint subsets of  $X$  whose union covers all but at most  $\ell$  points of  $X$ , and let  $\mathbf{K} = \mathbf{K}(X, p, \ell)$  be the order complex of this set with the ordering  $\leq$  defined above. A simplicial free  $Z_p$ -action  $\nu$  is defined on  $\mathbf{K}$  by the cyclic shift:

$$\nu: (A_1, \dots, A_p) \mapsto (A_2, A_3, \dots, A_p, A_1).$$

Suppose that  $c: \mathcal{S} \rightarrow [m]$  is a proper  $m$ -coloring of the Kneser  $p$ -hypergraph  $\text{KG}_p(\mathcal{S})$ . This time we consider the set of all ordered  $p$ -tuples  $(C_1, \dots, C_p)$  of subsets of  $[m]$  with  $\bigcup_{i=1}^p C_i \neq \emptyset$  and  $\bigcap_{i=1}^p C_i = \emptyset$ . Let  $\mathbf{L}$  be the order complex of this set with the componentwise inclusion ordering  $\leq$  as above. The simplicial  $Z_p$ -action on  $\mathbf{L}$ , again given by the cyclic shift of coordinates (i.e.  $(C_1, \dots, C_p) \mapsto (C_2, \dots, C_p, C_1)$ ), is free—here we use that  $p$  is a prime.

Using the  $m$ -coloring  $c$ , we are going to define a simplicial  $Z_p$ -map  $f: \mathbf{K} \rightarrow \mathbf{L}$ . For a subset  $A \subseteq X$ , let

$$g(A) = \{c(S) : S \subseteq A, S \in \mathcal{S}\},$$

and for a vertex  $(A_1, A_2, \dots, A_p)$  of  $\mathbf{K}$ , put

$$f((A_1, A_2, \dots, A_p)) = (g(A_1), g(A_2), \dots, g(A_p)).$$

If  $c$  is a proper coloring, then no  $p$  pairwise disjoint sets of  $\mathcal{S}$  can have the same color, and it follows that  $\bigcap_{i=1}^p g(A_i) = \emptyset$ . Since we assume  $\text{cd}_p((X, \mathcal{S})) > \ell$ , for any ordered  $p$ -tuple  $(A_1, \dots, A_p) \in V(\mathbf{K})$ , there are  $i \in [p]$  and  $S \in \mathcal{S}$  with  $S \subseteq A_i$ . Therefore,  $\bigcup_{i=1}^p g(A_i) \neq \emptyset$ , so  $f((A_1, \dots, A_p)) \in V(\mathbf{L})$ , and it is easy to see that  $f$  is a simplicial  $Z_p$ -map  $\mathbf{K} \rightarrow \mathbf{L}$ .

It remains to bound the indices  $\text{ind}_{Z_p}(\mathbf{K})$  and  $\text{ind}_{Z_p}(\mathbf{L})$ . As for the latter, we have  $\text{ind}_{Z_p}(\mathbf{L}) \leq \dim(\mathbf{L}) = m(p-1)$ . Indeed, supposing that  $(C_1, \dots, C_p)$  is the largest element in a chain of vertices of  $\mathbf{L}$ , each  $j \in [m]$  is in at most  $p-1$  of the  $C_i$ , and each time we pass to a smaller element of the chain, some  $j \in [m]$  is omitted from at least one of the sets; thus, the chain has at most  $m(p-1)+1$  elements.

The  $Z_p$ -index of  $\mathbf{K}$  can be bounded from below in several ways (homology computation, inductive argument showing an appropriate connectivity, shelling argument); we use a simple approach inspired by Sarkaria's papers.

First we consider the larger complex  $\mathbf{K}_0 = \mathbf{K}(X, p, n-1)$ , with all  $p$ -tuples of pairwise disjoint subsets of  $X$ , not all of them empty, as vertices. It is well-known that  $\text{ind}_{Z_p}(\mathbf{K}_0) = n-1$  (for those familiar with deleted joins, we remark that  $\mathbf{K}_0$  is the first barycentric subdivision of the  $p$ -fold 2-wise deleted join of the  $(n-1)$ -simplex—see e.g. [14]). In fact,  $\mathbf{K}_0$  is  $Z_p$ -isomorphic to  $\text{sd}(\mathbf{E}_{n-1,p})$ : the isomorphism  $\varphi: V(\text{sd}(\mathbf{E}_{n-1,p})) \rightarrow V(\mathbf{K}_0)$  is given by  $\{(j_1, i_1), (j_2, i_2), \dots, (j_q, i_q)\} \mapsto (A_1, A_2, \dots, A_p)$ , where  $A_i = \{j_t : i_t = i, t = 1, 2, \dots, q\}$ .

Let  $\mathbf{K}_1$  be the subcomplex of  $\mathbf{K}_0$  with  $V(\mathbf{K}_1) = V(\mathbf{K}_0) \setminus V(\mathbf{K})$  and with simplices inherited from  $\mathbf{K}_0$ , i.e. the simplices are the  $F \in \mathbf{K}_0$  with  $F \subseteq V(\mathbf{K}_1)$ . The

vertices of  $K_1$  are  $p$ -tuples  $(A_1, \dots, A_p)$  of disjoint sets with  $|\bigcup_{i=1}^p A_i| \leq n - \ell - 1$ , and  $\text{ind}_{Z_p}(K_1) \leq \dim(K_1) = n - \ell - 2$ . We have  $K_0 \subseteq K * K_1$ , and so by (1)

$$\text{ind}_{Z_p}(K) \geq \text{ind}_{Z_p}(K_0) - \text{ind}_{Z_p}(K_1) - 1 = n - 1 - (n - \ell - 2) - 1 = \ell.$$

Since we have constructed the  $Z_p$ -map  $f: K \rightarrow L$ , we have  $\ell \leq \text{ind}_{Z_p}(K) \leq \text{ind}_{Z_p}(L) \leq m(p-1)$ . This proves Theorem 1.1 for all prime  $r$ .

The non-prime case is handled by a short combinatorial argument, which is given in [11] and which we omit.  $\square$

*Remark.* As we have seen, the simplicial complex  $K_0$  is the subdivision of  $E_{n-1,p}$ ; in particular, for  $p = 2$ , it is an  $(n-1)$ -sphere. The subcomplex  $K_1$  is the subdivision of the  $(n-\ell-2)$ -skeleton of  $E_{n-1,p}$ . For  $p = 2$ , the simplicial complex  $K$  also has a nice interpretation (noted by G. Ziegler): it can be regarded as the subdivision of the  $\ell$ -skeleton of the cube  $[0, 1]^n$  (interpreted as a cell complex, with faces being the usual faces of the cube, i.e. cubes of various dimensions). Indeed, a vertex  $(A, B)$  of  $K$  can be encoded by a sequence  $v \in \{0, 1, *\}^X$ , where  $v_x = 0$  if  $x \in A$ ,  $v_x = 1$  if  $x \in B$ , and  $v_x = *$  otherwise. Each such  $v$  specifies a face of the  $n$ -cube.

#### REFERENCES

- [1] N. Alon, P. Frankl, and L. Lovász. The chromatic number of Kneser hypergraphs. *Trans. Amer. Math. Soc.*, 298:359–370, 1986. MR **88g**:05098
- [2] I. Bárány. A short proof of Kneser’s conjecture. *J. Combinatorial Theory Ser. A*, 25:325–326, 1978. MR **81g**:05056
- [3] A. Björner. Topological methods. In *Handbook of Combinatorics (eds. R. Graham, M. Grötschel, L. Lovász)*, pages 1819–1872. North-Holland, Amsterdam, 1995. MR **96m**:52012
- [4] A. Dold. Simple proofs of some Borsuk-Ulam results. In *Contemp. Math. 19*, pages 65–69. Amer. Math. Soc., 1983. MR **85e**:55003
- [5] V. L. Dol’nikov. Transversals of families of sets. In *Studies in the theory of functions of several real variables (Russian)*, pages 30–36, 109. Yaroslav. Gos. Univ., Yaroslavl’, 1981. CMP 20:03
- [6] V. L. Dol’nikov. A combinatorial inequality. *Siberian Math. J.*, 29, 1988. Translated from *Sibirsk. Mat. Zh.*, 29:53–58, 1988. MR **89k**:05079
- [7] V. L. Dol’nikov. Transversals of families of sets in  $R^n$  and a connection between the Helly and Borsuk theorems. *Russian Acad. Sci. Sb. Math.*, 79:93–107, 1994. Translated from *Ross. Akad. Nauk Matem. Sbornik*, Tom 184, No. 5, 1993. MR **94g**:52005
- [8] M. Kneser. Aufgabe 360. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 2. Abteilung, 58:27, 1955.
- [9] M. A. Krasnosel’skiĭ. On the estimation of the number of critical points of functionals (in Russian). *Uspehi Mat. Nauk*, 7:157–164, 1952. MR **14**:55f
- [10] I. Kříž. Equivariant cohomology and lower bounds for chromatic numbers. *Trans. Amer. Math. Soc.*, 333:567–577, 1992. MR **92m**:05085
- [11] I. Kříž. A correction to “Equivariant cohomology and lower bounds for chromatic numbers” (Trans. Amer. Math. Soc. 333 (1992) 567–577). *Trans. Amer. Math. Soc.*, 352(4):1951–1952, 2000. MR **2000i**:05078
- [12] L. Lovász. Kneser’s conjecture, chromatic number and homotopy. *J. Combinatorial Theory Ser. A*, 25:319–324, 1978. MR **81g**:05059
- [13] J. Matoušek. A combinatorial proof of Kneser’s conjecture. *Combinatorica*, 2001, in press.
- [14] K. S. Sarkaria. A generalized Kneser conjecture. *J. Combinatorial Theory Ser. B*, 49:236–240, 1990. MR **91g**:05003
- [15] G. M. Ziegler. Generalized Kneser coloring theorems with combinatorial proofs. *Invent. Math.*, in press.

- [16] R. T. Živaljević. User's guide to equivariant methods in combinatorics. *Publ. Inst. Math. Beograd*, 59(73):114–130, 1996. MR **98c**:55001
- [17] R. T. Živaljević. Topological methods. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 11, pages 209–224. CRC Press LLC, Boca Raton, FL, 1997. MR **2000j**:52001

DEPARTMENT OF APPLIED MATHEMATICS AND INSTITUTE OF THEORETICAL COMPUTER SCIENCE  
(ITI), CHARLES UNIVERSITY, MALOSTRANSKÉ NÁM. 25, 118 00 PRAHA 1, CZECH REPUBLIC  
*E-mail address:* `matousek@kam.mff.cuni.cz`