

ON POSSIBLE NON-HOMEOMORPHIC SUBSTRUCTURES OF THE REAL LINE

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(Communicated by Carl G. Jockusch, Jr.)

ABSTRACT. We consider the problem, raised by Kunen and Tall, of whether the real continuum can have non-homeomorphic versions in different submodels of the universe of all sets. This requires large cardinals, and we obtain an exact consistency strength:

Theorem 1. *The following are equiconsistent:*

- (i) $ZFC + \exists \kappa$ a Jónsson cardinal;
- (ii) $ZFC + \exists M$ a sufficiently elementary submodel of the universe of sets with \mathbb{R}_M not homeomorphic to \mathbb{R} .

The reverse direction is a corollary to:

Theorem 2. \mathfrak{c} is Jónsson $\iff \exists M \prec H(\mathfrak{c}^+) \exists X_M$ hereditarily separable, hereditarily Lindelöf, T_3 with $X \neq X_M$.

We further consider the large cardinal consequences of the existence of a topological space with a proper substructure homeomorphic to Baire space.

1. INTRODUCTION

Tall in [8] and Junqueira & Tall in [3] consider the question of the behaviour of a topological space $X = \langle X, T \rangle$ when $X \in M$, where M is a submodel, not necessarily transitive, of the universe of all sets. The topology they consider for X_M is that generated by the open sets of T that happen to be in M . Thus $X_M = \langle X \cap M, T_M \rangle$, where T_M is that generated by $\{U \cap M \mid U \in T \cap M\}$. In general this will not coincide with the usual relative topology. We refer to this $\langle X_M, T_M \rangle$, somewhat incorrectly, as a “substructure” of $\langle X, T \rangle$. (In fact, as they observe, it suffices to take M a submodel of the class $H(\theta)$ of sets of hereditary cardinality θ , where $\langle X, T \rangle \in H(\theta)$.) We refer the reader to these papers for further discussion and motivation of these ideas. The purpose of this note is to be brief, and state a theorem that characterises Jónsson cardinals:

Definition 1.1. (i) A Jónsson algebra $\mathcal{A} = \langle A, \langle f_n \rangle_{n \in \omega} \rangle$ is an algebra on A of finitary functions $f_n : [A]^n \rightarrow A$ that has no proper subalgebra of the same cardinality as that of A .

(ii) A cardinal κ is a Jónsson cardinal if there is no Jónsson algebra \mathcal{A} with $\kappa \subseteq A$.

Received by the editors January 16, 2001 and, in revised form, March 27, 2001.

2000 *Mathematics Subject Classification.* Primary 54A05, 03E35, 03E02, 54A35, 03E55; Secondary 54B05.

Key words and phrases. Real continuum, subspaces, Jónsson cardinals.

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In the first part of this note we prove the theorems of the abstract, which involve this concept. The reader is assumed familiar with, or to have access to a copy of [6] and [8]. Our definitions and terminology are standard. In the final part of the paper we look at a question of Tall's from [8] concerning the possibility of proper topological substructures being homeomorphic to the irrationals. The "large cardinal" theorem there has a proof that presupposes some basic familiarity with core model theory.

It is known that the existence of a Jónsson cardinal has mild large cardinal strength (Ramsey cardinals are Jónsson, and so their strength is less than that of a measurable cardinal). One may prove outright in *ZFC* that no \aleph_n ($n < \omega$) is Jónsson. It is not known whether \aleph_ω may be Jónsson, although it is known it would require very large cardinals to render this consistent. The first candidate for the continuum \mathfrak{c} to be Jónsson is \aleph_{ω_1} . (Simply because $cf(\mathfrak{c}) > \omega$ and no regular cardinal below \aleph_{ω_1} is Jónsson by results of Shelah [7], Woodin, and Tryba [9].)

2. JÓNSSON CARDINALS AND THE CONTINUUM

The following (Theorem 2 of the abstract) should be compared to Theorem 4 of [8].

Theorem 2.1. \mathfrak{c} is Jónsson $\iff \exists M \prec H(\mathfrak{c}^+) \exists X_M$ hereditarily separable, hereditarily Lindelöf, T_3 with $X \neq X_M$

Proof. (\Rightarrow) Let \mathfrak{c} be Jónsson. Then $\exists M \prec \langle H_{\mathfrak{c}^+}, \in \rangle$ with $\mathfrak{c} \in M$, $|M \cap \mathfrak{c}| = \mathfrak{c}$, $M \cap \mathfrak{c} \neq \mathfrak{c}$. That such an M with these properties exists is a standard fact about Jónsson cardinals; see *e.g.* [4], §8. As $H_{\mathfrak{c}^+} \models \exists f : \mathbb{R} \longleftrightarrow \mathfrak{c} \in On$, it follows that $\mathbb{R} \cap M \neq \mathbb{R}$. However, as [3] shows, for the case of first countable spaces X , we have that in fact $X \cap M = X_M$. Hence $\mathbb{R}_M \neq \mathbb{R}$.

(\Leftarrow) Suppose \mathfrak{c} is not Jónsson. It suffices by the proof of [8], Theorem 4, to show that $[0, 1] \subseteq M$. For this it is enough to show $\mathfrak{c} \subseteq M$. But $H_{\mathfrak{c}^+} \models \exists \langle \mathfrak{c}, f_n \rangle_{n < \omega}$ a Jónsson algebra on \mathfrak{c} . Hence M is a model of the above sentence. Let $\langle \mathfrak{c}, g_n \rangle_{n < \omega} \in M$ be a witness to this. (It is unproblematic to assume that the field of the algebra is \mathfrak{c} itself.) Then each $g_n \in M$, and, if $g_n : [\mathfrak{c}]^n \rightarrow \mathfrak{c}$, $\langle M \cap \mathfrak{c}, g_n \upharpoonright [M \cap \mathfrak{c}]^n \rangle_{n < \omega}$ is a subalgebra of $\langle \mathfrak{c}, g_n \rangle_{n < \omega}$. By definition of $\langle g_n \rangle$ we have $M \cap \mathfrak{c} = \mathfrak{c}$. \square

Corollary 2.2. If \mathfrak{c} is not Jónsson, and if X_M is an uncountable complete separable metric space, then $X_M = X$.

Corollary 2.3. If \mathfrak{c} is not Jónsson, and X_M is homeomorphic to an uncountable Borel subspace of \mathbb{R} , then $X_M = X$.

In [8] already some anti-large cardinal assumption was used. In the former corollary the weaker hypothesis " \mathfrak{c} is not Jónsson" replaces " $CH + \neg 0^\#$ " of [8], Theorem 21 a), and just " $\neg 0^\#$ " of Cor. 25., *op.cit.* The proofs are the same.

In [6] Kunen and Tall asked:

Question: What is a lower bound on the consistency strength of:

(*) $\exists 2^\mathfrak{c}$ -many non-homeomorphic \mathbb{R}_M of size \mathfrak{c} ?

They show ([6], Theorems 16 and 8, respectively):

(i) $Con(ZFC + \exists \text{ a Ramsey cardinal}) \Rightarrow Con(ZFC + (*))$.

(ii) $ZFC + \exists \mathbb{R}_M (|\mathbb{R}_M| = \mathfrak{c} \wedge \mathbb{R}_M \text{ is not homeomorphic to } \mathbb{R}) \vdash 0^\# \text{ exists.}$

We raise this latter lower bound to that of a Jónsson cardinal.

Theorem 2.4. $ZFC + \exists \mathbb{R}_M (|\mathbb{R}_M| = \mathfrak{c} \wedge \mathbb{R}_M \text{ is not homeomorphic to } \mathbb{R}) \vdash \mathfrak{c} \text{ is a Jónsson cardinal}$

Proof. This is just the argument of (\Leftarrow) of Theorem 2.1. □

In their model of (i) in which (*) holds, \mathfrak{c} has been made equal to the Ramsey cardinal κ of the ground model by c.c.c. forcing. In such a model κ is Jónsson (see [1]). As Jónsson cardinals are equiconsistent with Ramseys, this shows that the authors of [6] were using exactly the right hypothesis. In their model \mathfrak{c} is weakly inaccessible.

Their proof that there were at least $2^{\mathfrak{c}}$ non-homeomorphic \mathbb{R}_M amongst their $2^{\mathfrak{c}}$ many different submodels M used Lavrentieff’s theorem on extending putative homeomorphisms to G_δ ’s and a counting argument. We next observe that in their construction *all* the \mathbb{R}_M are non-homeomorphic—again using Lavrentieff, but using a further property of indiscernibles arising from Ramsey cardinals:

Definition 2.1. Let $\mathcal{A} = \langle L_\kappa[A], A, \vec{R}, \vec{f}, \dots \rangle$ be a first order structure with $\kappa \subseteq |A|$. $I \subseteq \kappa$ is a *good set of indiscernibles* for \mathcal{A} if for all $\gamma \in I$

- (i) $\langle L_\gamma[A \cap \gamma], \in, A \cap \gamma, \vec{R} \upharpoonright \gamma, \vec{f} \upharpoonright \gamma, \dots \rangle \prec \mathcal{A}$;
- (ii) $I \setminus \gamma$ is a set of indiscernibles for the structure $\langle L_\delta[A], \in, A, \vec{R}, \vec{f}, \dots \langle \xi \rangle_{\xi \leq \gamma} \rangle$.

Fact 2.1. *Suppose κ is a Ramsey cardinal. Then for every first order structure \mathcal{A} as above there is a good sequence of indiscernibles of order type κ . (See, e.g., [2], Ch. 16.)*

Using this and the notation of [6], Theorem 16, if I is decomposed into a sequence $\langle I_\alpha \mid \alpha < \kappa \rangle$ of mutually disjoint subsets each of size κ , then each $M_\alpha =_{df} \{ \tau_G \mid \tau \text{ is a } \mathbb{P}\text{-name } \tau \in \mathcal{H}(I_\alpha) \}$ has a version of $\mathbb{R} : \mathbb{R}_M$. (We are taking here $\mathcal{H}(Y)$ to be defined as the Skolem hull of Y in $H(\lambda^+)$.) We have then:

Lemma 2.5. *The collection of $\mathbb{R}_\alpha =_{df} \mathbb{R}_{M_\alpha}$ are all pairwise non-homeomorphic.*

Proof. As Kunen and Tall argue, by Lavrentieff’s theorem, if \mathbb{R}_α were homeomorphic with \mathbb{R}_β via a homeomorphism g say, g could be extended to a homeomorphism between two G_δ subsets of \mathbb{R} ; $\tilde{g} : G_\alpha \rightarrow G_\beta$. But such a homeomorphism, and G_α, G_β , are essentially coded by a real, r say. But any real of $V[G]$ (where G is the generic for the c.c.c. forcing \mathbb{P} adding κ many reals) is added at some initial stage of the forcing. We may thus factor \mathbb{P} as $\mathbb{P}_0 * \mathbb{P}_1$ with r being added by $G_0 =_{df} G \cap \mathbb{P}_0$. Let \dot{r} be a \mathbb{P} -name (and a \mathbb{P}_0 -name) for r . But for all sufficiently large γ , the sets I_γ are good indiscernibles for $\langle L_\kappa[A], \in, \mathbb{P}, \Vdash_{\mathbb{P}}, \mathbb{P}_0, \Vdash_{\mathbb{P}_0}, \dot{r}, \langle \xi \rangle_{\xi < \gamma} \rangle$, where $\mathbb{P}_0, \dot{r} \in V_\gamma$ and $A \subseteq \kappa$ is assumed to code up V_κ so that for any strong inaccessible $\eta < \kappa$ we have $V_\eta = L_\eta[A \cap \eta]$. They are thus easily seen to be good indiscernibles in $V[G_0]$. Now the argument finishes as in [6]: let $f : \kappa \leftrightarrow \mathbb{R}$ be a bijection in $V[G]$ named by a term in $\mathcal{H}(\emptyset)$; if $i > \gamma$ is any indiscernible in $I_\beta \setminus I_\alpha$, we have $f(i) \in \mathbb{R}_\beta \setminus \mathbb{R}_\alpha$. But then $\tilde{g}^{-1}(f(i)) \in \mathbb{R}_\alpha$, and thus $i \in M_\alpha$ although i cannot be named by any term defined from the indiscernibles of I_α . □

To make \mathfrak{c} singular and Jónsson requires the consistency of larger cardinals.

Theorem 2.6.

$$\begin{aligned} \text{Con}(ZFC + \mathfrak{c} > cf(\mathfrak{c}) = \lambda + \exists \mathbb{R}_M (|\mathbb{R}_M| = \mathfrak{c} \wedge \mathbb{R}_M \neq \mathbb{R}) \\ \iff \text{Con}(ZFC + \exists \langle \kappa_i \mid i < \lambda \rangle \text{ an increasing sequence} \\ \text{of measurable cardinals, with } \kappa_0 > \lambda). \end{aligned}$$

Proof (Sketch) (\Rightarrow). The assumption implies again that \mathfrak{c} is Jónsson, by the argument of Theorem 2.1, and that together with the cofinality hypothesis gives an inner model with the sequence of λ measurable cardinals by Koepke [5].

(\Leftarrow) Assuming we have measures \mathcal{U}_i on κ_i , find a sequence $\langle X_i \mid i < \lambda \rangle$ of measure one sets which are mutually coherently indiscernible, and use the argument of [6]: make $\mathfrak{c} = \sup_i \{\kappa_i\}$ by c.c.c. forcing adding reals, and get $2^{\mathfrak{c}}$ many models with different versions of \mathbb{R}_M . \square

3. ON THE POSSIBILITY OF A TOPOLOGICAL SPACE WITH A PROPER SUBSTRUCTURE HOMEOMORPHIC TO BAIRE SPACE

In [8] an interesting question is raised:

Question: Can there be $\langle X, T \rangle \in M \prec H(\lambda^+)$ with X_M homeomorphic to the irrationals, but $X_M \neq X$?

It is easy to see, by the reasoning above, that if the question had a positive answer with $\lambda = \mathfrak{c}$ then \mathfrak{c} would have to be Jónsson (and hence at least \aleph_{ω_1}). However the possibility exists that X may be of size $\lambda > \mathfrak{c}$, with $|M| = \mathfrak{c}$ and X_M homeomorphic to Baire space. This would, *prima facie*, only imply that $\langle \lambda, \mathfrak{c} \rangle \longrightarrow \langle \mathfrak{c}, < \mathfrak{c} \rangle$, a Chang-conjecture-like property weaker than that of a Jónsson cardinal. I cannot answer the above question, but I can show that large cardinals are still involved.

Theorem 3.1. *Suppose the answer to the question is positive, and let X, T, λ, M witness this. Then there is an inner model with λ a \mathfrak{c} -Erdős cardinal.*

Proof. We suppose there is no inner model with a measurable cardinal $\kappa \leq \lambda$, for otherwise we have nothing to do: in the core model K below a measurable cardinal every V -cardinal $\bar{\lambda} \geq \kappa$ is a Ramsey cardinal, and hence certainly a \mathfrak{c} -Erdős cardinal! We show then that in this core model K λ satisfies the partition relation $\lambda \longrightarrow (\mathfrak{c})_2^{<\omega}$. This suffices. Recall that $\lambda \longrightarrow (\mathfrak{c})_2^{<\omega}$ is defined to mean that for every function $F : [\lambda]^{<\omega} \longrightarrow 2$ there is a set $H \subseteq \lambda$ with $|H| = \mathfrak{c}$ and for each $n < \omega$ we have $|F \upharpoonright [H]^n| = 1$, that is, H is *homogeneous* for F . We assume only a slight familiarity with the construction of core models. Such a core model is of the form $K = L[E]$, where E is a sequence of extenders. The point is that, unlike for Gödel's constructible universe L , we must have a failure of condensation. Suppose $F \in K$ is a function as above, which is a counterexample to the partition property in K . We may assume that F has been chosen least in the natural ordering of $H(\lambda^+)^K$ and hence is a definable point in the substructure M , as is also the predicate $E \upharpoonright \lambda'$ coding the whole model's construction up to λ' , the latter a sufficiently large ordinal so that $F \in L_{\lambda'}[E \upharpoonright \lambda']$. As X_M is assumed homeomorphic to Baire space, M has size \mathfrak{c} . $|M \cap \mathfrak{c}| < \mathfrak{c}$, as otherwise M would contain all of \mathbb{R} . Let $\sigma : N \longrightarrow M$ be the inverse of the transitive collapse of M , with N transitive. Set $\sigma(\bar{\lambda}, \bar{E}, \bar{F}) = \lambda, E, F$. We then note that $On \cap N \geq \mathfrak{c}$ and $N \models \text{“}\bar{\lambda} \text{ is the largest cardinal”}$. Hence $\bar{\lambda} \geq \mathfrak{c}$. The point now is to note that $L_{\mathfrak{c}}[\bar{E}] \neq K_{\mathfrak{c}} =_{df} L_{\mathfrak{c}}[E]$ (the core model constructed up to \mathfrak{c}). For if it were, as the least ordinal moved by σ , $\text{crit}(\sigma) \leq \delta \leq \mathfrak{c}$ for some regular δ . We should then be able to define an iterable K -ultrafilter on an ordinal $\beta < \mathfrak{c}$, and hence an elementary $j : K \longrightarrow K$. This implies there is an inner model with a measurable cardinal, contrary to our global hypothesis. Hence, if $\bar{K} =_{df} K^N$, then $\bar{K}_{\mathfrak{c}} \neq K_{\mathfrak{c}}$. If we now perform the comparison iteration of K with \bar{K} , we shall see that there is a “mouse” structure M of cardinality $< \mathfrak{c}$ that iterates past \bar{K} . If

$$C = \{\kappa_i \mid \kappa_i \text{ is the critical point used at stage } i \text{ in the comparison}\},$$

then standard arguments show that an end segment $C_0 \subseteq C$ is cub in \mathfrak{c} and moreover forms an unbounded set of good indiscernibles for the structure $\langle \overline{K_\lambda}, \in \overline{F} \rangle$. Let $D = \sigma "C_0$. Then D is a good set of indiscernibles of order type \mathfrak{c} for $\langle K_\lambda, \in F \rangle$. We now appeal to the Jensen Indiscernibles Lemma (see [2], 16.10), and use the uncountable cofinality of \mathfrak{c} , to claim that there is $E \supseteq D$ with $E \in K$ and elements of E good indiscernibles for the same structure. But this entails E being a homogeneous set for the function F . This contradicts our choice of F . \square

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