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ON POSSIBLE NON-HOMEOMORPHIC SUBSTRUCTURES OF THE REAL LINE

P. D. WELCH

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Abstract. We consider the problem, raised by Kunen and Tall, of whether the real continuum can have non-homeomorphic versions in different submodels of the universe of all sets. This requires large cardinals, and we obtain an exact consistency strength:

Theorem 1. *The following are equiconsistent*:

(i) $ZFC + \exists\kappa$ *a Jónsson cardinal*;

(ii) $ZFC + \exists M$ *a sufficiently elementary submodel of the universe of sets with* $\mathbb{R}_M$ *not homeomorphic to* $\mathbb{R}$.

The reverse direction is a corollary to:

Theorem 2. $\mathfrak{c}$ *is Jónsson* $\iff \exists M \prec H(\mathfrak{c}^+)\exists X_M$ *hereditarily separable, hereditarily Lindelöf,* T_3 *with* $X \neq X_M$.

We further consider the large cardinal consequences of the existence of a topological space with a proper substructure homeomorphic to Baire space.

1. Introduction

Tall in [8] and Junqueira & Tall in [3] consider the question of the behaviour of a topological space $X = \langle X, T\rangle$ when $X \in M$, where M is a submodel, not necessarily transitive, of the universe of all sets. The topology they consider for X_M is that generated by the open sets of T that happen to be in M. Thus $X_M = \langle X \cap M, T_M\rangle$, where T_M is that generated by $\{U \cap M \mid U \in T \cap M\}$. In general this will not coincide with the usual relative topology. We refer to this $\langle X_M, T_M\rangle$, somewhat incorrectly, as a "substructure" of $\langle X, T\rangle$. (In fact, as they observe, it suffices to take M a submodel of the class $H(\theta)$ of sets of hereditary cardinality θ, where $\langle X, T\rangle \in H(\theta)$.) We refer the reader to these papers for further discussion and motivation of these ideas. The purpose of this note is to be brief, and state a theorem that characterises Jónsson cardinals:

Definition 1.1. (i) A *Jónsson algebra* $\mathcal{A} = \langle A, \langle f_n\rangle_{n\in\omega}\rangle$ is an algebra on A of finitary functions $f_n : [A]^n \longrightarrow A$ that has no proper subalgebra of the same cardinality as that of A.

(ii) A cardinal κ is a *Jónsson cardinal* if there is no Jónsson algebra $\mathcal{A}$ with $\kappa \subseteq A$.

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In the first part of this note we prove the theorems of the abstract, which involve this concept. The reader is assumed familiar with, or to have access to a copy of [6] and [8]. Our definitions and terminology are standard. In the final part of the paper we look at a question of Tall's from [8] concerning the possibility of proper topological substructures being homeomorphic to the irrationals. The "large cardinal" theorem there has a proof that presupposes some basic familiarity with core model theory.

It is known that the existence of a Jónsson cardinal has mild large cardinal strength (Ramsey cardinals are Jónsson, and so their strength is less than that of a measurable cardinal). One may prove outright in ZFC that no $\aleph_n$ $(n < \omega)$ is Jónsson. It is not known whether $\aleph_\omega$ may be Jónsson, although it is known it would require very large cardinals to render this consistent. The first candidate for the continuum $\mathfrak{c}$ to be Jónsson is $\aleph_{\omega_1}$. (Simply because $cf(\mathfrak{c}) > \omega$ and no regular cardinal below $\aleph_{\omega_1}$ is Jónsson by results of Shelah [7], Woodin, and Tryba [9].)

2. Jónsson cardinals and the continuum

The following (Theorem 2 of the abstract) should be compared to Theorem 4 of [8].

Theorem 2.1. $\mathfrak{c}$ *is Jónsson* $\Longleftrightarrow \exists M \prec H(\mathfrak{c}^+)\exists X_M$ *hereditarily separable, hereditarily Lindelöf,* T_3 *with* $X \neq X_M$

Proof. ($\Rightarrow$) Let $\mathfrak{c}$ be Jónsson. Then $\exists M \prec \langle H_{\mathfrak{c}^+}, \in\rangle$ with $\mathfrak{c} \in M$, $|M \cap \mathfrak{c}| = \mathfrak{c}$, $M \cap \mathfrak{c} \neq \mathfrak{c}$. That such an M with these properties exists is a standard fact about Jónsson cardinals; see *e.g.* [4], §8. As $H_{\mathfrak{c}^+} \models \exists f : \mathbb{R} \longleftrightarrow \mathfrak{c} \in On$, it follows that $\mathbb{R} \cap M \neq \mathbb{R}$. However, as [3] shows, for the case of first countable spaces X, we have that in fact $X \cap M = X_M$. Hence $\mathbb{R}_M \neq \mathbb{R}$.

($\Leftarrow$) Suppose $\mathfrak{c}$ is not Jónsson. It suffices by the proof of [8], Theorem 4, to show that $[0, 1] \subseteq M$. For this it is enough to show $\mathfrak{c} \subseteq M$. But $H_{c^+} \models$ "$\exists\langle \mathfrak{c}, f_n\rangle_{n<\omega}$ a Jónsson algebra on $\mathfrak{c}$". Hence M is a model of the above sentence. Let $\langle \mathfrak{c}, g_n\rangle_{n<\omega} \in M$ be a witness to this. (It is unproblematic to assume that the field of the algebra is $\mathfrak{c}$ itself.) Then each $g_n \in M$, and, if $g_n : [\mathfrak{c}]^n \longrightarrow \mathfrak{c}$, $\langle M \cap \mathfrak{c}, g_n \restriction [M \cap \mathfrak{c}]^n\rangle_{n<\omega}$ is a subalgebra of $\langle \mathfrak{c}, g_n\rangle_{n<\omega}$. By definition of $\langle g_n\rangle$ we have $M \cap \mathfrak{c} = \mathfrak{c}$. □

Corollary 2.2. *If* $\mathfrak{c}$ *is not Jónsson, and if* X_M *is an uncountable complete separable metric space, then* $X_M = X$.

Corollary 2.3. *If* $\mathfrak{c}$ *is not Jónsson, and* X_M *is homeomorphic to an uncountable Borel subspace of* $\mathbb{R}$*, then* $X_M = X$.

In [8] already some anti-large cardinal assumption was used. In the former corollary the weaker hypothesis "$\mathfrak{c}$ is not Jónsson" replaces "$CH + \neg 0^{\#}$" of [8], Theorem 21 a), and just "$\neg 0^{\#}$" of Cor. 25., *op.cit.* The proofs are the same.

In [6] Kunen and Tall asked:

Question: What is a lower bound on the consistency strength of:

$$(*) \qquad \exists 2^{\mathfrak{c}}\text{-many non-homeomorphic } \mathbb{R}_M \text{ of size } \mathfrak{c}?$$

They show ([6], Theorems 16 and 8, respectively):

(i) $Con(ZFC + \exists$ a Ramsey cardinal $) \Rightarrow Con(ZFC + (*))$.

(ii) $ZFC + \exists\mathbb{R}_M(|\mathbb{R}_M| = \mathfrak{c} \wedge \mathbb{R}_M$ is not homeomorphic to $\mathbb{R}) \vdash 0^{\#}$ exists.

We raise this latter lower bound to that of a Jónsson cardinal.

Theorem 2.4. *$ZFC + \exists\mathbb{R}_M(|\mathbb{R}_M| = \mathfrak{c} \wedge \mathbb{R}_M$ is not homeomorphic to $\mathbb{R}) \vdash \mathfrak{c}$ is a Jónsson cardinal*

Proof. This is just the argument of ($\Leftarrow$) of Theorem 2.1. $\square$

In their model of (i) in which $(*)$ holds, $\mathfrak{c}$ has been made equal to the Ramsey cardinal κ of the ground model by c.c.c. forcing. In such a model κ is Jónsson (see [1]). As Jónsson cardinals are equiconsistent with Ramseys, this shows that the authors of [6] were using exactly the right hypothesis. In their model $\mathfrak{c}$ is weakly inaccessible.

Their proof that there were at least $2^{\mathfrak{c}}$ non-homeomorphic $\mathbb{R}_M$ amongst their $2^{\mathfrak{c}}$ many different submodels M used Lavrentieff's theorem on extending putative homeomorphisms to G_δ's and a counting argument. We next observe that in their construction *all* the $\mathbb{R}_M$ are non-homeomorphic—again using Lavrentieff, but using a further property of indiscernibles arising from Ramsey cardinals:

Definition 2.1. Let $\mathcal{A} = \langle L_\kappa[A], A, \vec{R}, \vec{f}, \dots\rangle$ be a first order structure with $\kappa \subseteq |\mathcal{A}|$. $I \subseteq \kappa$ is a *good set of indiscernibles for* $\mathcal{A}$ if for all $\gamma \in I$

(i) $\langle L_\gamma[A\cap\gamma], \in, A\cap\gamma, \vec{R}\cap\gamma, \vec{f}\restriction\gamma, \dots\rangle \prec \mathcal{A}$;

(ii) $I\backslash\gamma$ is a set of indiscernibles for the structure $\langle L_\delta[A], \in, A, \vec{R}, \vec{f}, \dots \langle\xi\rangle_{\xi\leq\gamma}\rangle$.

Fact 2.1. *Suppose κ is a Ramsey cardinal. Then for evey first order structure $\mathcal{A}$ as above there is a good sequence of indiscernibles of order type κ. (See,* e.g.*,* [2]*, Ch. 16.)*

Using this and the notation of [6], Theorem 16, if I is decomposed into a sequence $\langle I_\alpha \mid \alpha < \kappa\rangle$ of mutually disjoint subsets each of size κ, then each $M_\alpha =_{df} \{\tau_G \mid \tau$ is a $\mathbb{P}$-name $\tau \in \mathcal{H}(I_\alpha)\}$ has a version of $\mathbb{R} : \mathbb{R}_M$. (We are taking here $\mathcal{H}(Y)$ to be defined as the Skolem hull of Y in $H(\lambda^+)$.) We have then:

Lemma 2.5. *The collection of $\mathbb{R}_\alpha =_{df} \mathbb{R}_{M_\alpha}$ are all pairwise non-homeomorphic.*

Proof. As Kunen and Tall argue, by Lavrentieff's theorem, if $\mathbb{R}_\alpha$ were homeomorphic with $\mathbb{R}_\beta$ *via* a homoeomorphism g say, g could be extended to a homeomorphism between two G_δ subsets of $\mathbb{R}$; $\tilde{g} : G_\alpha \longrightarrow G_\beta$. But such a homeomorphism, and G_α, G_β, are essentially coded by a real, r say. But any real of $V[G]$ (where G is the generic for the c.c.c. forcing $\mathbb{P}$ adding κ many reals) is added at some initial stage of the forcing. We may thus factor $\mathbb{P}$ as $\mathbb{P}_0 * \mathbb{P}_1$ with r being added by $G_0 =_{df} G \cap \mathbb{P}_0$. Let $\dot{r}$ be a $\mathbb{P}$-name (and a $\mathbb{P}_0$-name) for r. But for all sufficiently large γ, the sets I_γ are good indiscernibles for $\langle L_\kappa[A], \in, \mathbb{P}, \Vdash_{\mathbb{P}}, \mathbb{P}_0, \Vdash_{\mathbb{P}_0}, \dot{r}, \langle\xi\rangle_{\xi<\gamma}\rangle$, where $\mathbb{P}_0, \dot{r} \in V_\gamma$ and $A \subseteq \kappa$ is assumed to code up V_κ so that for any strong inaccessible $\eta < \kappa$ we have $V_\eta = L_\eta[A\cap\eta]$. They are thus easily seen to be good indiscernibles in $V[G_0]$. Now the argument finishes as in [6]: let $f : \kappa \leftrightarrow \mathbb{R}$ be a bijection in $V[G]$ named by a term in $\mathcal{H}(\emptyset)$; if $i > \gamma$ is any indiscernible in $I_\beta\backslash I_\alpha$, we have $f(i) \in \mathbb{R}_\beta\backslash\mathbb{R}_\alpha$. But then $\tilde{g}^{-1}(f(i)) \in \mathbb{R}_\alpha$, and thus $i \in M_\alpha$ although i cannot be named by any term defined from the indiscernibles of I_α. $\square$

To make $\mathfrak{c}$ singular and Jónsson requires the consistency of larger cardinals.

Theorem 2.6.

$$\begin{aligned} &Con(ZFC + \mathfrak{c} > cf(\mathfrak{c}) = \lambda + \exists\mathbb{R}_M(|\mathbb{R}_M| = \mathfrak{c} \wedge \mathbb{R}_M \neq \mathbb{R}) \\ &\quad\Longleftrightarrow Con(ZFC + \exists\langle\kappa_i \,|\, i < \lambda\rangle \text{ an increasing sequence} \\ &\qquad\qquad \text{of measurable cardinals, with } \kappa_0 > \lambda).\end{aligned}$$

Proof (Sketch) ($\Rightarrow$). The assumption implies again that $\mathfrak{c}$ is Jónsson, by the argument of Theorem 2.1, and that together with the cofinality hypothesis gives an inner model with the sequence of λ measurable cardinals by Koepke [5].

($\Leftarrow$) Assuming we have measures $\mathcal{U}_i$ on κ_i, find a sequence $\langle X_i \mid i < \lambda \rangle$ of measure one sets which are mutually coherently indiscernible, and use the argument of [6]: make $\mathfrak{c} = \sup_i\{\kappa_i\}$ by c.c.c. forcing adding reals, and get $2^{\mathfrak{c}}$ many models with different versions of $\mathbb{R}_M$. $\square$

3. On the possibility of a topological space with a proper substructure homeomorphic to Baire space

In [8] an interesting question is raised:

Question: Can there be $\langle X, T\rangle \in M \prec H(\lambda^+)$ with X_M homeomorphic to the irrationals, but $X_M \neq X$?

It is easy to see, by the reasoning above, that if the question had a positive answer with $\lambda = \mathfrak{c}$ then $\mathfrak{c}$ would have to be Jónsson (and hence at least $\aleph_{\omega_1}$). However the possibility exists that X may be of size $\lambda > \mathfrak{c}$, with $|M| = \mathfrak{c}$ and X_M homeomorphic to Baire space. This would, *prima facie*, only imply that $\langle \lambda, \mathfrak{c}\rangle \longrightarrow \langle \mathfrak{c}, {<}\mathfrak{c}\rangle$, a Chang-conjecture-like property weaker than that of a Jónsson cardinal. I cannot answer the above question, but I can show that large cardinals are still involved.

Theorem 3.1. *Suppose the answer to the question is positive, and let X, T, λ, M witness this. Then there is an inner model with λ a $\mathfrak{c}$-Erdős cardinal.*

Proof. We suppose there is no inner model with a measurable cardinal $\kappa \leq \lambda$, for otherwise we have nothing to do: in the core model K below a measurable cardinal every V-cardinal $\overline{\lambda} \geq \kappa$ is a Ramsey cardinal, and hence certainly a $\mathfrak{c}$-Erdős cardinal! We show then that in this core model K λ satisfies the partition relation $\lambda \longrightarrow (\mathfrak{c})^{<\omega}_2$. This suffices. Recall that $\lambda \longrightarrow (\mathfrak{c})^{<\omega}_2$ is defined to mean that for every function $F : [\lambda]^{<\omega} \longrightarrow 2$ there is a set $H \subseteq \lambda$ with $|H| = \mathfrak{c}$ and for each $n < \omega$ we have $|F``[H]^n| = 1$, that is, H is *homogeneous* for F. We assume only a slight familiarity with the construction of core models. Such a core model is of the form $K = L[E]$, where E is a sequence of extenders. The point is that, unlike for Gödel's constructible universe L, we must have a failure of condensation. Suppose $F \in K$ is a function as above, which is a counterexample to the partition property in K. We may assume that F has been chosen least in the natural ordering of $H(\lambda^+)^K$ and hence is a definable point in the substructure M, as is also the predicate $E \restriction \lambda'$ coding the whole model's construction up to λ', the latter a sufficiently large ordinal so that $F \in L_{\lambda'}[E \restriction \lambda']$. As X_M is assumed homeomorphic to Baire space, M has size $\mathfrak{c}$. $|M \cap \mathfrak{c}| < \mathfrak{c}$, as otherwise M would contain all of $\mathbb{R}$. Let $\sigma : N \longrightarrow M$ be the inverse of the transitive collapse of M, with N transitive. Set $\sigma(\overline{\lambda}, \overline{E}, \overline{F}) = \lambda, E, F$. We then note that $On \cap N \geq \mathfrak{c}$ and $N \models$ "$\overline{\lambda}$ is the largest cardinal". Hence $\overline{\lambda} \geq \mathfrak{c}$. The point now is to note that $L_{\mathfrak{c}}[\overline{E}] \neq K_{\mathfrak{c}} =_{df} L_{\mathfrak{c}}[E]$ (the core model constructed up to $\mathfrak{c}$). For if it were, as the least ordinal moved by $\sigma, crit(\sigma) \leq \delta \leq \mathfrak{c}$ for some regular δ. We should then be able to define an iterable K-ultrafilter on an ordinal $\beta < \mathfrak{c}$, and hence an elementary $j : K \longrightarrow K$. This implies there is an inner model with a measurable cardinal, contrary to our global hypothesis. Hence, if $\overline{K} =_{df} K^N$, then $\overline{K}_{\mathfrak{c}} \neq K_{\mathfrak{c}}$. If we now perform the comparison iteration of K with $\overline{K}$, we shall see that there is a "mouse" structure M of cardinality $< \mathfrak{c}$ that iterates past $\overline{K}$. If

$$C = \{\kappa_i | \kappa_i \text{ is the critical point used at stage } i \text{ in the comparison}\},$$

then standard arguments show that an end segment $C_0 \subseteq C$ is cub in $\mathfrak{c}$ and moreover forms an unbounded set of good indiscernibles for the structure $\langle \overline{K}_{\overline{\lambda}}, \in \overline{F} \rangle$. Let $D = \sigma``C_0$. Then D is a good set of indiscernibles of order type $\mathfrak{c}$ for $\langle K_\lambda, \in F \rangle$. We now appeal to the Jensen Indiscernibles Lemma (see [2], 16.10), and use the uncountable cofinality of $\mathfrak{c}$, to claim that there is $E \supseteq D$ with $E \in K$ and elements of E good indiscernibles for the same structure. But this entails E being a homogeneous set for the function F. This contradicts our choice of F. $\square$

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Department of Mathematics, University of Bristol, Bristol BS8 1TW, England – and – Department Institut für Formale Logik, Währingerstr 25, A-1090 Wien, Austria

E-mail address: welch@logic.univie.ac.at

Current address: Mathematisches Institut, Beringstrasse 6, Bonn, D-53115, Germany