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OPERATOR HILBERT SPACES WITHOUT THE OPERATOR APPROXIMATION PROPERTY

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ABSTRACT. We use a technique of Szankowski to construct operator Hilbert spaces that do not have the operator approximation property, including an example in a noncommutative L_p space for $p \neq 2$.

1. Introduction and preliminaries

A Banach space X has the approximation property, or AP, if the identity operator on X can be approximated uniformly on compact subsets of X by linear operators of finite rank. In the 50's, Grothendieck [G] investigated this property and found several equivalent statements. For example, he proved that X has the AP iff the natural map $J : X^*\hat{\otimes}X \to X^*\check{\otimes}X$ is one-to-one ($\hat{\otimes}$ is the projective tensor product of Banach spaces and $\check{\otimes}$ is the injective tensor product of Banach spaces). However, it remained unknown if every Banach space had the AP until Enflo [E] constructed the first counterexample in the early 70's. In [S], Szankowski gave a very explicit example of a subspace of ℓ_p, $1 < p < 2$, without the AP. He considered $X = (\sum_{n=1}^{\infty} \oplus \ell_2^n)_p$, which is isomorphic to ℓ_p, and defined Z to be the closed span of some vectors of length six. He then used a clever combinatorial argument to exploit the difference between the 2-norm of the blocks and the p-norm of the sum to prove that Z fails the approximation property. Szankowski's technique is fairly general. In the second section of this paper we will use it to show that the ℓ_2-sum (as defined in [P2]) of row operator spaces has a subspace without the operator space version of the approximation property, or OAP. Since this subspace is a Hilbert space at the Banach space level, it has the Banach approximation property and even a basis. Thus, this is an example of an operator space with the AP but without the OAP. This answers a question of J. Kraus. Furthermore, this is also the first example of an operator space with a basis but without a complete basis. M. Junge suggested that a similar construction using rows in the Schatten p-class S_p and Rademacher functions in $L_p[0,1]$ could lead to an example of a Hilbertian subspace of $L_p[S_p]$ failing the OAP. In the third section we verify that this is indeed possible. These are new examples of Hilbertian subspaces of noncommutative L_p spaces that are not completely complemented, even if $p > 2$.

An operator space E is a Banach space E with an isometric embedding into $B(H)$, the set of all bounded operators on a Hilbert space H. Or, equivalently, an

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operator space E is a closed subspace of $B(H)$. If $E \subset B(H_1)$ and $F \subset B(H_2)$ are operator spaces, their *minimal tensor product* $E \otimes_{\min} F$ is the closure of the algebraic tensor product $E \otimes F$ in $B(H_1 \otimes_2 H_2)$. A linear map $u : E \to F$ is *completely bounded*, or cb, if for every operator space G, the map $1_G \otimes u : G \otimes_{\min} E \to G \otimes_{\min} F$ is bounded. The completely bounded norm of u, or $\|u\|_{cb}$, is the supremum of $\|1_G \otimes u\|$, where G runs over all operator spaces G. It turns out that it is enough to verify that $1_G \otimes u$ is bounded when G is $K(\ell_2)$, the set of all compact operators on the Hilbert space ℓ_2, and that $\|u\|_{cb} = \|1_{K(\ell_2)} \otimes u\|$. The set of all cb-maps from E to F is denoted by $CB(E, F)$. Independently of each other, Blecher and Paulsen [BP] and Effros and Ruan [ER1] gave E^*, the Banach space dual of E, an operator space structure that gives $E^* \otimes_{\min} F$ the norm induced by $CB(E, F)$. This indicates that the minimal tensor product is the operator space analogue of the injective tensor product of Banach spaces. In the same papers, Blecher and Paulsen [BP] and Effros and Ruan [ER1] introduced the operator space analogue of the projective tensor product. This is denoted by $E \hat{\otimes} F$ and satisfies $(E \hat{\otimes} F)^* = CB(E, F^*)$. We refer to [ER3], [J], and [P3] for more information about operator spaces.

It is well known that the compact subsets of a Banach space X are contained in the convex hull of null sequences in X. Since there is a correspondence between null sequences in X and elements of $c_0 \check{\otimes} X$, it is easy to see that X has the AP iff for every $u \in c_0 \check{\otimes} X$ and every $\epsilon > 0$ there exists a finite rank operator on X such that $\|u - (I \otimes T)(u)\| < \epsilon$. Based on this observation, Effros and Ruan [ER2] said that an operator space V has the operator approximation property, or OAP, if for every $u \in K(H) \otimes_{min} V$ and every $\epsilon > 0$ there exists a finite rank operator T on V such that $\|u - (I \otimes T)(u)\| < \epsilon$. They proved that an operator space V has the OAP if and only if the natural map $J : V^* \hat{\otimes} V \to V^* \otimes_{min} V$ is one-to-one.

The following criterion allows us to check that J is not one-to-one, when V fails the OAP.

Enflo's Criterion. *If there exists a sequence of finite rank operators $\beta_n \in V^* \otimes V$ satisfying:*

(i) *$trace(\beta_n) = 1$ for every $n \in \mathbb{N}$,*
(ii) *$\|\beta_n\|_{V^* \otimes_{min} V} \to 0$ as $n \to \infty$, and*
(iii) *$\sum_{n=1}^{\infty} \|\beta_n - \beta_{n-1}\|_{V^* \hat{\otimes} V} < \infty$,*

then V does not have the OAP.

Indeed, $\beta = \beta_1 + \sum_{n=2}^{\infty} \beta_n - \beta_{n-1} = \lim_n \beta_n$ belongs to $V^* \hat{\otimes} V$ by (iii). $J\beta = 0$ by (ii). And since $tr(\beta) = 1$, β is not zero. Hence J is not one-to-one and V fails the OAP.

2. An example using complex interpolation

For each $n \in \mathbb{N}$, let Δ_n be a partition of $\sigma_n = \{2^n, 2^n + 1, 2^n + 2, \ldots, 2^{n+1} - 1\}$. Then $\{B \in \Delta_n : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$. For each $B \in \Delta_n$, let R_B be the row Hilbert space with orthonormal basis $\{e_j : j \in B\}$. We define X to be the ℓ_2-sum of these row spaces. More precisely, X is the complex interpolation space between $(\sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus R_B)_\infty$ and $(\sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus R_B)_1$ of parameter $\theta = \frac{1}{2}$ (see [P2], page 34). That is,

$$X = \left(\sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus R_B \right)_{\ell_2} = \left(\left(\sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus R_B \right)_{\infty}, \left(\sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus R_B \right)_{1} \right)_{\frac{1}{2}}.$$

At the Banach space level, X is a Hilbert space with orthonormal basis $\{e_i : i \in \mathbb{N}\}$. But at the operator space level, X is a combination of row Hilbert spaces and OH, the self-dual operator Hilbert space introduced by Pisier in [P1]. If $A \subset \mathbb{N}$, let $X_A = \overline{span}\{e_i : i \in A\} \subset X$. It follows from the definition of X that if there exists $n \in \mathbb{N}$ such that $A \subset B$ for some $B \in \Delta_n$, then X_A is completely isometric to R_A, the row Hilbert space with orthonormal basis $\{e_i : i \in A\}$. And if for each $n \in \mathbb{N}$, A has at most one point from each element in Δ_n (i.e., card$(A \cap B) \leq 1$ for every $B \in \Delta_n$, $n \in \mathbb{N}$), then X_A is completely isometric to OH_A.

Let Z be the closed subspace of X spanned by

$$z_i = e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}, \quad i = 1, 2, \cdots .$$

Theorem 1. *With the appropriate selection of Δ_n, Z does not have the OAP.*

For each $i \in \mathbb{N}$, let $z_i^* = \frac{1}{2}(e_{2i}^* - e_{2i+1}^*)$, where the e_i^*'s are biorthogonal to the e_i's. Then let

$$\beta_n = \frac{1}{2^n} \sum_{i \in \sigma_n} z_i^* \otimes z_i \qquad \text{for } n \geq 2.$$

We need to check that the β_n's satisfy the conditions of Enflo's criterion.

Condition (i)

This is trivially verified. Since $z_i^*(z_i) = 1$ for every $i \geq 1$, we see that $trace(\beta_n) = (1/2^n) \sum_{i \in \sigma_n} z_i^*(z_i) = (1/2^n)|\sigma_n| = 1$.

Condition (ii)

Since $\|\beta_n\|_{Z^* \otimes_{min} Z} \leq \|\beta_n\|_{X^* \otimes_{min} X} = \|\beta_n\|_{cb}$, we will estimate the cb-norm of $\beta_n : X \to X$. However, it follows from the definition of β_n that we only need to estimate the cb-norm of $\beta_n : X_{\sigma_{n+1}} \to X_{\sigma_{n+1} \cup \sigma_{n+2}}$, where $X_{\sigma_k} = \text{span}\{e_i : i \in \sigma_k\}$. Let $I_1 : X_{\sigma_{n+1}} \to R_{\sigma_{n+1}}$ and $I_2 : R_{\sigma_{n+1} \cup \sigma_{n+2}} \to X_{\sigma_{n+1} \cup \sigma_{n+2}}$ be the formal identity maps, and let $\tilde{\beta}_n : R_{\sigma_{n+1}} \to R_{\sigma_{n+1} \cup \sigma_{n+2}}$ be $\tilde{\beta}_n = \frac{1}{2^n} \sum_{i \in \sigma_n} z_i^* \otimes z_i$ (that is, $\tilde{\beta}_n$ has the same matrix representation as β_n, but it is defined on row operator spaces). Then $\beta_n = I_2 \circ \tilde{\beta}_n \circ I_1$. Since the z_i's, $i \in \sigma_n$, have disjoint support, the z_i^*'s, $i \in \sigma_n$, have also disjoint support, and the row spaces are homogeneous, it is easy to see that $\|\tilde{\beta}_n\|_{cb} = \|\tilde{\beta}_n\| = \frac{1}{2^{n+1}}\sqrt{3}$. We will prove condition (ii) by checking that $\|I_1\|_{cb}\|I_2\|_{cb} \leq \sqrt{2^{n+2}}$.

From the definition of X, we see that $X_{\sigma_{n+1}}$ is equal to $(\sum_{B \in \Delta_{n+1}} \oplus R_B)_{\ell_2}$, the complex interpolation space $\left((\sum_{B \in \Delta_{n+1}} \oplus R_B)_\infty, (\sum_{B \in \Delta_{n+1}} \oplus R_B)_1\right)_{\frac{1}{2}}$. It is easy to check that $\|I_1 : (\sum_{B \in \Delta_{n+1}} \oplus R_B)_\infty \to R_{\sigma_{n+1}}\|_{cb} \leq \sqrt{|\Delta_{n+1}|}$ and that $\|I_1 : (\sum_{B \in \Delta_{n+1}} \oplus R_B)_1 \to R_{\sigma_{n+1}}\|_{cb} \leq 1$. Therefore $\|I_1\|_{cb} \leq (|\Delta_{n+1}|)^{\frac{1}{4}}$. Similarly, $\|I_2\|_{cb} \leq (|\Delta_{n+1}| + |\Delta_{n+2}|)^{\frac{1}{4}}$. Since $|\Delta_k| \leq 2^k$ for every $k \in \mathbb{N}$, we see that $\|I_1\|_{cb}\|I_2\|_{cb} \leq \sqrt{2^{n+2}}$.

Condition (iii)

Using the fact that $z_i^* = \frac{1}{4}(e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*)$ on Z, we get that

$$\beta_n - \beta_{n-1} = \frac{1}{2^{n+1}} \sum_{i \in \sigma_n} (e_{2i}^* - e_{2i+1}^*) \otimes (e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3})$$

$$- \frac{1}{2^{n+1}} \sum_{i \in \sigma_{n-1}} (e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*) \otimes (e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3})$$

$$= \tfrac{1}{2^{n+1}} \sum_{i\in\sigma_{n-1}} \begin{cases} e_{4i}^*\otimes(e_{4i}-e_{4i+1}+e_{8i}+e_{8i+1}+e_{8i+2}+e_{8i+3}-e_{2i}+e_{2i+1}-e_{4i}-e_{4i+1}-e_{4i+2}-e_{4i+3}) \\ e_{4i+1}^*\otimes(-e_{4i}+e_{4i+1}-e_{8i}-e_{8i+1}-e_{8i+2}-e_{8i+3}-e_{2i}+e_{2i+1}-e_{4i}-e_{4i+1}-e_{4i+2}-e_{4i+3}) \\ e_{4i+2}^*\otimes(e_{4i+2}-e_{4i+3}+e_{8i+4}+e_{8i+5}+e_{8i+6}+e_{8i+7}-e_{2i}+e_{2i+1}-e_{4i}-e_{4i+1}-e_{4i+2}-e_{4i+3}) \\ e_{4i+3}^*\otimes(-e_{4i+2}+e_{4i+3}-e_{8i+4}-e_{8i+5}-e_{8i+6}-e_{8i+7}-e_{2i}+e_{2i+1}-e_{4i}-e_{4i+1}-e_{4i+2}-e_{4i+3}). \end{cases}$$

Note that after cancellation, each of the vectors in the brace has nine terms. Two of them cancel out and two are equal. Then we can write each of them as a linear combination of nine basis vectors. Eight of them have coefficients equal to ± 1 and the other has a coefficient equal to ± 2.

Szankowski defined nine functions $f_k : \mathbb{N} \to \mathbb{N}, k \leq 9$, to index these vectors. Let $n = 4i + l$ and $l = 0, 1, 2, 3$. Then $f_1(4i + l) = 2i$ and $f_2(4i + l) = 2i + 1$. For $k = 3, 4, 5$, $f_k(4i + l) = 4i + [(l + 1) \mod 4]$. For $l = 0, 1$, $f_6(4i + l) = 8i$, $f_7(4i + l) = 8i + 1$, $f_8(4i + l) = 8i + 2$, and $f_9(4k + l) = 8i + 3$. And finally, for $l = 2, 3$, $f_6(4i + l) = 8i + 4$, $f_7(4i + l) = 8i + 5$, $f_8(4i + l) = 8i + 6$, and $f_9(4k + l) = 8i + 7$. Then we have

$$\beta_n - \beta_{n-1} = \frac{1}{2^{n+1}} \sum_{j\in\sigma_{n+1}} e_j^* \otimes y_j,$$

where $y_j = \sum_{k=1}^{9} = \lambda_{j,k} e_{f_k(j)} \in Z$. Recall that eight of the $\lambda_{j,k}$'s have absolute value equal to one, and one has absolute value equal to 2.

The following lemma of Szankowski provides the key combinatorial argument (see [S] and [LT], page 108).

Lemma 2 (Szankowski)**.** *There exist partitions Δ_n and ∇_n of σ_n into disjoint sets, and a sequence $m_n \geq 2^{\frac{n}{8}-2}$, $n = 2, 3, \ldots,$ so that*

1 $\forall A \in \nabla_n, m_n \leq \mathit{card}(A) \leq 2m_n$,
2 $\forall A \in \nabla_n, \forall B \in \Delta_n, \mathit{card}(A \cap B) \leq 1$,
3 $\forall A \in \nabla_n, \forall 1 \leq k \leq 9, f_k(A)$ *is contained in an element of* Δ_{n-1}, Δ_n, *or* Δ_{n+1}.

(Notice that $f_k(\sigma_n) \subset \sigma_{n-1}$ for $k = 1, 2$, $f_k(\sigma_n) \subset \sigma_n$ for $k = 3, 4, 5$, and $f_k(\sigma_n) \subset \sigma_{n+1}$ for $k = 6, 7, 8, 9$.)

Since ∇_{n+1} is a partition of σ_{n+1}, we have that

$$\beta_n - \beta_{n-1} = \frac{1}{2^{n+1}} \sum_{A\in\nabla_{n+1}} \left[\sum_{j\in A} e_j^* \otimes y_j \right]. \tag{1}$$

Lemma 3. *For every $A \in \nabla_{n+1}$, $\| \sum_{j\in A} e_j^* \otimes y_j \|_{Z^*\hat{\otimes} Z} \leq 18\big(\mathit{card}(A)\big)^{\frac{3}{4}}$.*

The last condition of Enflo's criterion follows immediatedly from (1), Lemma 2, and Lemma 3. Indeed,

$$\begin{aligned} \|\beta_n - \beta_{n-1}\|_{Z^*\hat{\otimes} Z} &\leq \frac{1}{2^{n+1}} \mathrm{card}(\nabla_{n+1}) 18 \max_{A\in\nabla_{n+1}} \mathrm{card}(A)^{\frac{3}{4}} \\ &\leq \frac{1}{2^{n+1}} \frac{2^{n+1}}{m_{n+1}} 18 (2m_{n+1})^{\frac{3}{4}} \leq \frac{36}{m_{n+1}^{\frac{1}{4}}}, \end{aligned}$$

which is clearly summable.

We only need to prove Lemma 3. For this, we need the result of Pisier (see remark 2.11 of [P1]) that $CB(R_n, OH_n) = S_4^n$, where S_4^n is the Schatten 4-class.

Consequently, if $S : OH_n \to R_n$, then $\|S\|_{OH_n \hat{\otimes} R_n} = \|S\|_{S^n_{4/3}}$. In particular, if $I : OH_n \to R_n$ is the formal identity, $\|I\|_{OH_n \hat{\otimes} R_n} = n^{3/4}$.

Proof of Lemma 3. The element $\gamma = \sum_{j\in A} e_j^* \otimes y_j \in X^* \hat{\otimes} Z$ induces a finite rank map $\gamma : X \to Z$. The restriction of γ to Z is the map $\alpha = \gamma_{|Z} : Z \to Z$, which clearly satisfies $\alpha = \sum_{j\in A} q(e_j^*) \otimes y_j \in Z^* \hat{\otimes} Z$, where $q = (\iota_Z)^* : X^* \to Z^*$ is the adjoint of the inclusion $\iota_Z : Z \to X$. Since $(Z^* \hat{\otimes} Z)^* = CB(Z^*, Z^*)$, we have that $\|\alpha\|_{Z^* \hat{\otimes} Z} = \sup\{|\langle T, \alpha\rangle| : T : Z^* \to Z^*, \quad \|T\|_{cb} \le 1\}$, where $\langle \cdot, \cdot\rangle$ is the trace duality.

We will see that we can factor α through the formal identity map $I : OH_A \to R_A$, where R_A is the row Hilbert space with basis $\{\delta_j : j \in A\}$. Recall that the projective tensor norm of $I : OH_A \to R_A$, viewed as an element of $OH_A \hat{\otimes} R_A$, is equal to $\big(\mathrm{card}(A)\big)^{\frac{3}{4}}$.

Let $\Psi : R_A \to Z$ be the map defined by $\Psi(\delta_j) = y_j$. We claim that $\|\Psi\|_{cb} \le 18$. Indeed, if $a_j \in B(\mathcal{H})$ for $j \in A$, then

$$\sum_{j\in A} a_i \otimes y_j = \sum_{j\in A} a_j \otimes \sum_{k=1}^{9} \lambda_{j,k} e_{f_k(j)} = \sum_{k=1}^{9} \left[\sum_{j\in A} \lambda_{j,k} a_j \otimes e_{f_k(j)} \right].$$

It follows from (3) of Lemma 2 that $\{f_k(j) : j \in A\} \subset B$ for some B in Δ_n, Δ_{n+1}, or Δ_{n+2}. Then the definition of X implies that the span of the $e_{f_k(j)}$'s for $j \in A$ is a row operator space. Hence,

$$\Big\| \sum_{j\in A} a_i \otimes y_j \Big\| \le (9)(2) \Big\| \sum_{j\in A} a_j a_j^* \Big\|^{\frac{1}{2}} = 18 \Big\| \sum_{j\in A} a_j \otimes \delta_j \Big\|.$$

Let $X_A = \mathrm{span}\{e_j : j \in A\}$. By (2) of Lemma 2, all the elements of A belong to different elements of the partition Δ_{n+1}. This implies that X_A is completely isometric to OH_A. Let $P_A : X \to X_A$ be the completely contractive projection onto X_A, and let $I : X_A \to R_A$ be the formal identity. Then we have that

$$\alpha = \Psi \circ I \circ P_A \circ \iota_Z.$$

If $T : Z^* \to Z^*$ is completely bounded, then

$$\begin{aligned} |\langle T, \alpha\rangle| &= |tr(T^* \circ \alpha)| = |tr(T^* \circ \Psi \circ I \circ P_A \circ \iota_Z)| = |tr(P_A \circ \iota_Z \circ T^* \circ \Psi \circ I)| \\ &\le \|P_A \circ \iota_Z \circ T^* \circ \Psi\|_{cb} \|I\|_{OH_A \hat{\otimes} R_A} \le 18 \|T\|_{cb} \big(\mathrm{card}(A)\big)^{\frac{3}{4}}. \end{aligned}$$

This finishes the proof of Lemma 3. □

Remarks. We can replace the ℓ_2 sum of R_n's by the ℓ_2 sum of C_n's or by the ℓ_2 sum of any other family of homogeneous Hilbert spaces which is "far" from OH (e.g., $R_p^n = (R_n, C_n)_{\frac{1}{p}}$, $p \ne 2$). The same proof gives that all of these spaces have subspaces failing the operator approximation property. We can also replace the ℓ_2 sum of R_n's by the ℓ_p sum of R_n's. However, when $p \ne 2$, then X is no longer a Hilbert space. The subspace Z of X will fail the operator approximation property, but Szankowski's theorem actually gives that Z already fails the Banach approximation property.

3. An example in noncommutative L_p

We recall standard facts about noncommutative L_p spaces. S_p is the Schatten p-class on ℓ_2 with the operator space structure induced by the complex interpolation $S_p = (S_\infty, S_1)_{\frac{1}{p}}$ (see [P2]). $R_p = \overline{\text{span}}\{e_{1n} : n \in \mathbb{N}\}$ is the row of S_p, and $C_p = \overline{\text{span}}\{e_{n1} : n \in \mathbb{N}\}$ is the column of S_p. Both of them are taken with the operator space structure they inherit from S_p. It follows from Lemma 1.7 of [P2] that the operator space structure of any operator space E is determined by $S_p[E]$, the noncommutative E-valued S_p-space. In particular, the operator space structures of R_p and S_p are determined by $S_p[R_p]$ and $S_p[C_p]$. That is, if $a_n \in S_p$ is a finite family, then

$$\left\|\sum_n a_n \otimes e_{1n}\right\|_{S_p[R_p]} = \left\|\left(\sum_n a_n a_n^*\right)^{\frac{1}{2}}\right\|_{S_p}$$

and

$$\left\|\sum_n a_n \otimes e_{n1}\right\|_{S_p[C_p]} = \left\|\left(\sum_n a_n^* a_n\right)^{\frac{1}{2}}\right\|_{S_p}.$$

We will also consider R_p^n and C_p^n, the row and column of S_p^n; and more generally, if B is a subset of $\mathbb{N}$, R_p^B and C_p^B are the row and column of S_p^B, the Schatten p-class on the Hilbert space with orthonormal basis indexed by B.

The operator space structure of $L_p[0,1]$ is determined by $S_p[L_p]$. We note that by Proposition 2.1 of [P2], $S_p[L_p]$ is completely isometric to the more familiar $L_p[S_p]$. We will consider $\mathcal{R}_p \subset L_p[0,1]$, the subspace generated by the Rademacher functions $(\epsilon_n)_{n\in\mathbb{N}}$, with the operator space structure inherited from L_p. It is known that the operator space structure of $\mathcal{R}_p$ is determined by the noncommutative Khintchine's inequalities of [L-P] and [L-PP]. If $p \geq 2$, $\mathcal{R}_p$ is completely isomorphic to $R_p \cap C_p$ and $\mathcal{R}_p^n$ is completely isomorphic to $R_p^n \cap C_p^n$, with a constant depending only on p. If $1 \leq p \leq 2$, $\mathcal{R}_p$ is completely isomorphic to $R_p + C_p$ and $\mathcal{R}_p^n$ is completely isomorphic to $R_p^n + C_p^n$, with a constant depending only on p (see [P2], Section 8.4, for details).

Recall that the classical Khintchine's inequality states that there exist constants A_p and B_p such that for any square summable sequence (α_n)

$$A_p\left\|\sum_n \alpha_n\epsilon_n\right\|_2 \leq \left\|\sum_n \alpha_n\epsilon_n\right\|_p \leq B_p\left\|\sum_n \alpha_n\epsilon_n\right\|_2. \tag{2}$$

We will now construct a Hilbertian subspace of $L_p[S_p]$ as a combination of Rademacher functions in L_p and the row of S_p. The idea is to put a finite set of Rademacher functions in the (1,1) row of S_p, then to put another finite set of Rademacher functions in the (1,2) row of S_p, a third finite set of Rademacher functions in the (1,3) row of S_p, and so on. At the end we get a space X which plays the same role as the space X of the previous section. To make the construction more precise, recall that for each $n \in \mathbb{N}$, Δ_n is a partition of $\sigma_n = \{2^n, 2^n+1, \ldots, 2^{n+1}\}$. Then $\mathcal{P} = \{B \in \Delta_n : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$. For convenience, we index the row of S_p by the countable set $\mathcal{P}$. That is, $R_p = \overline{\text{span}}\{e_{1B} : B \in \mathcal{P}\}$. Define

$$X = \overline{\text{span}}\{e_{1B} \otimes \epsilon_k : B \in \mathcal{P}, k \in B\} \subset S_p[L_p] \equiv L_p[S_p].$$

Alternatively, define $e_k := e_{1B} \otimes \epsilon_k \in S_p[L_p]$, where $B \in \mathcal{P}$ is the element in the partition that contains $k \in \mathbb{N}$, and let X be the closed span of the e_k's.

Proposition 4. *At the Banach space level, X is isomorphic to a Hilbert space.*

Proof. An element $f \in X$ has the form $f = \sum_{B\in\mathcal{P}} e_{1B} \otimes f_B$, where f_B belongs to $\mathrm{span}\{e_k : k \in B\} = \mathcal{R}_p^B$. Since $S_p[L_p]$ is completely isometric to $L_p[S_p]$, we view $f : [0,1] \to S_p$ as an S_p-valued function with norm

$$\|f\|_{L_p[S_p]} = \left(\int_0^1 \|f(t)\|_{S_p}^p dt\right)^{\frac{1}{p}} = \left[\int_0^1 \left(\sum_{B\in\mathcal{P}} |f_B(t)|^2\right)^{\frac{p}{2}} dt\right]^{\frac{1}{p}}.$$

If $p \geq 2$, then $\|f\|_{L_p[S_p]} \geq \left(\int_0^1 (\sum_{B\in\mathcal{B}} |f_B(t)|^2)^{\frac{2}{2}} dt\right)^{\frac{1}{2}} = \sqrt{\sum_{B\in\mathcal{P}} \|f_B\|_2^2}$. To check the opposite direction, we use that $L_p[S_p]$ has type 2, and that $\sum_{B\in\mathcal{P}} e_{1B} \otimes f_B$ is an unconditional sum. Then, $\|f\|_{S_p[L_p]} \leq C\sqrt{\sum_{B\in\mathcal{P}} \|f_B\|_p^2}$. Since $f_B \in \mathcal{R}_p^B$, it follows from (2) that $\|f_B\|_p \leq B_p\|f_B\|_2$. Therefore, $\|f\|_{S_p[L_p]}$ is equivalent to $\sqrt{\sum_{B\in\mathcal{P}} \|f_B\|_2^2}$, which is the norm of a Hilbert space.

The case $1 \leq p \leq 2$ is similar. We first check that $\|f\|_{L_p[S_p]} \leq \sqrt{\sum_{B\in\mathcal{P}} \|f_B\|_2^2}$. Then we use the cotype 2 of $L_p[S_p]$ and the Rademacher estimate of (2) to conclude that $\|f\|_{L_p[S_p]} \geq C\sqrt{\sum_{B\in\mathcal{P}} \|f_B\|_2^2}$. $\square$

The operator space structure of X can be described easily. The blocks of X are indexed by $B \in \mathcal{P}$, and they are completely isometric to $\mathrm{span}\{\epsilon_k : k \in B\} = \mathcal{R}_p^B$. When $p \geq 2$, $\mathcal{R}_p^B$ is completely isomorphic (with constant depending only on p) to $R_p^B \cap C_p^B$, and when $1 \leq p \leq 2$, $\mathcal{R}_p^B$ is completely isomorphic to $R_p^B + C_p^B$. Since the sum is taken in an R_p sense, we obtain the following complete isomorphisms:

$$X \approx \left(\sum_{n=1}^{\infty} \sum_{B\in\Delta_n} \oplus\big[R_p^B \cap C_p^B\big]\right)_{R_p} \qquad \text{for } p \geq 2,$$

$$X \approx \left(\sum_{n=1}^{\infty} \sum_{B\in\Delta_n} \oplus\big[R_p^B + C_p^B\big]\right)_{R_p} \qquad \text{for } 1 \leq p \leq 2.$$

To apply Enflo's criterion, we need the following estimates.

Lemma 5. *Let $1 \leq p \leq \infty$. Then the normed spaces $CB(C_p^n, R_p^n)$, $CB(R_p^n, C_p^n)$, $CB(C_p^n, R_p^n \cap C_p^n)$, and $CB(R_p^n, R_p^n \cap C_p^n)$ are isomorphic to S_r^n, the Schatten r-class, where r satisfies $\frac{1}{r} = |\frac{1}{p} - \frac{1}{2}|$.*

Proof. It is enough to prove that $CB(C_p^n, R_p^n)$ is isomorphic to S_r^n. The proof for the second space is similar, and the proof for the last two spaces follows from the first two. By duality, it is enough to assume that $1 \leq p \leq 2$. Let $T \in CB(C_p^n, R_p^n)$. Write $T = UDV$, where $U : R_p^n \to R_p^n$ and $V : C_p^n \to C_p^n$ are unitary, and $D : C_p^n \to R_p^n$ is the diagonal operator $De_{i1} = \lambda_i e_{1i}$ with λ_i equal to the ith singular number of T. Since R_p^n and C_p^n are homogeneous Hilbert spaces, we conclude that $\|T\|_{cb} = \|D\|_{cb}$. By Lemma 1.7 of [P2],

$$\|D\|_{cb} = \sup\left\{\frac{\left\|\sum_{i\leq n} \lambda_i e_{1i} \otimes a_i\right\|_{R_p^n[S_p]}}{\left\|\sum_{i\leq n} e_{i1} \otimes a_i\right\|_{C_p^n[S_p]}} : a_i \in S_p, \quad i \leq n\right\}.$$

It is well known that, for $1 \le p \le 2$, $\|\sum_{i\le n}\lambda_i e_{1i}\otimes a_i\| \le \left(\sum_{i\le n}|\lambda_i|^p\|a_i\|_{S_p}^p\right)^{\frac{1}{p}}$ and $\|\sum_{i\le n} e_{i1}\otimes a_i\| \ge \left(\sum_{i\le n}\|a_i\|_{S_p}^2\right)^{\frac{1}{2}}$. Moreover, if $a_i = b_i e_{i1} \in S_p^n$ for some scalars $b_i \in \mathbb{C}$, the inequalities are attained. Since $\|a_i\|_{S_p} = |b_i|$, we have

$$\|D\|_{cb} = \sup\left\{\left(\sum_{i\le n}|\lambda_i b_i|^p\right)^{\frac{1}{p}} : \sqrt{\sum_{i\le n}|b_i|^2} \le 1\right\} = \left(\sum_{i\le n}|\lambda_i|^r\right)^{\frac{1}{r}} = \|T\|_{S_r^n}.$$

This finishes the proof. □

Since $CB(C_p^n, R_p^n \cap C_p^n)$ is the dual of $(R_p^n)^*\hat{\otimes}(R_p^n + C_p^n)$, it follows from Lemma 5 that for every $T \in (R_p^n)^*\hat{\otimes}(R_p^n + C_p^n)$ we have $\|T\|_{(R_p^n)^*\hat{\otimes}(R_p^n+C_p^n)} = \|T\|_{S_{r'}^n}$, where $\frac{1}{r}+\frac{1}{r'}=1$. In particular, if $T = I_n$ is the "formal" identity map, then

$$\|I_n\|_{(R_p^n)^*\hat{\otimes}(R_p^n+C_p^n)} = n^{\frac{1}{r'}}. \tag{3}$$

Similarly,

$$\|I_n\|_{(R_p^n\cap C_p^n)^*\hat{\otimes}R_p^n} = n^{\frac{1}{r'}}. \tag{4}$$

With these estimates, we can follow Szankowski's construction to obtain the following theorem. We sketch the proof.

Theorem 6. *For every $1 \le p < \infty$, $p \ne 2$, there exists a Hilbertian subspace of $L_p[S_p]$ without the operator approximation property.*

Recall that X is the closed span of $e_k := e_{1B}\otimes\epsilon_k$, $k \in \mathbb{N}$. For each $i = 1, 2, \dots$, let $z_i = e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}$ and $z_i^* = \frac{1}{2}(e_{2i}^* - e_{2i+1}^*)$, where the e_i^*'s are biorthogonal to the e_i's. For each $n \ge 2$, let $\beta_n = \frac{1}{2^n}\sum_{i\in\sigma_n} z_i^*\otimes z_i$. Finally, let Z be the closed span of the z_i's. The claim is that there exist partitions $(\Delta_n)_n$ such that Z fails the OAP. We prove this by checking the three conditions of Enflo's criterion. The first one is trivial. The second one follows easily. We estimate the cb-norm of β_n through the cb-norm of $\tilde{\beta}_n$, which has the same matrix representation as β_n but is defined on R_p, the row of S_p.

We use Szankowski's partitions of Lemma 2 to check the third condition of Enflo's criterion for $1 \le p < 2$. Recall that $\beta_n - \beta_{n-1} = \frac{1}{2^{n+1}}\sum_{j\in\sigma_{n+1}} e_j^*\otimes y_j = \frac{1}{2^{n+1}}\sum_{A\in\nabla_{n+1}}\left[\sum_{j\in A} e_j^*\otimes y_j\right]$, where $y_j = \sum_{k=1}^9 \lambda_{j,k}e_{f_k(j)} \in Z$ for some $|\lambda_{j,k}| = 1$ or 2. Let $A \in \nabla_{n+1}$. It follows from Lemma 2 (2) that $X_A = \operatorname{span}\{e_i : i \in A\}$ is completely isomorphic to R_p^A. And if $\Psi : R_p^A + C_p^A \to Z$ is defined by $\Psi(\delta_j) = y_j$, it follows from Lemma 2 (3) that $\|\Psi\|_{cb} \le 18$. Note that

$$\sum_{j\in A} e_j^*\otimes y_j = \sum_{j\in A} e_j^*\otimes\Psi(\delta_j) = (I\otimes\Psi)\sum_{j\in A} e_j^*\otimes\delta_j,$$

and that $\sum_{j\in A} e_j^*\otimes\delta_j$ is the "formal" identity map on $(R_p^A)^*\hat{\otimes}(R_p^A + C_p^A)$. Then it follows from (3) that

$$\left\|\sum_{j\in A} e_j^*\otimes y_j\right\|_{Z^*\hat{\otimes}Z} \le \|\Psi\|_{cb}\left\|\sum_{j\in A} e_j^*\otimes\delta_j\right\|_{(R_p^A)^*\hat{\otimes}(R_p^A+C_p^A)} \le 18\big(\operatorname{card}(A)\big)^{\frac{1}{r'}},$$

where $\frac{1}{r}+\frac{1}{r'}=1$ and $\frac{1}{r} = |\frac{1}{p}-\frac{1}{2}|$. Now, we easily check $\sum_n\|\beta_n - \beta_{n-1}\|_{Z^*\hat{\otimes}Z} < \infty$.

To prove the third condition of Enflo's criterion for $p > 2$, we consider the variation of Lemma 2 described in [LT], page 111. Namely, we find partitions Δ_n, ∇_n such that every $A \in \nabla_n$ is contained in some element of Δ_n while, for every

$A \in \nabla_n$, $k = 1, \ldots, 9$, and B in Δ_{n-1}, Δ_n, or Δ_{n+1}, $\mathrm{card}(B \cap f_k(A)) \leq 1$. Then, if $A \in \nabla_{n+1}$, X_A is completely isomorphic to $R_p^A \cap C_p^A$, and if $\Psi : R_p^A \to Z$ is defined by $\Psi(\delta_j) = y_j$, then $\|\Psi\|_{cb} \leq 18$. Since $\sum_{j \in A} e_j^* \otimes \delta_j$ is the "formal" identity map on $(R_p^A \cap C_p^A)^* \hat{\otimes} R_p^A$, it follows from (4) that $\| \sum_{j \in A} e_j^* \otimes y_j \|_{Z^* \hat{\otimes} Z} \leq 18 \big(\mathrm{card}(A)\big)^{\frac{1}{r'}}$, and the third condition of Enflo's criterion follows.

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