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PARABOLIC SUBGROUPS OF VERSHIK-KEROV'S GROUP

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ABSTRACT. In this note we show that all parabolic subgroups of Vershik-Kerov's group $GLB(R)$ (i.e. subgroups containing $T(\infty, R)$—the group of infinite dimensional upper triangular matrices) are net subgroups for a wide class of semilocal rings R.

1. INTRODUCTION

The classical result for finite dimensional general linear group over a field states that all "parabolic" subgroups, that is, containing the group of all upper triangular matrices, are "staircase groups" (see [1], p. 53, or [3] for more general result in the context of groups with $BN-$pair).

In [2] Borevich introduced a concept of a net of ideals σ and a net subgroup $G(\sigma)$ (see definitions in the next section). Theorem 1 of [2] gives the following generalization:

Theorem 1.1. *Let R be a semilocal ring, in which 1 is a sum of two invertible elements. If H is a parabolic subgroup of $GL_n(R)$, then there exists a unique $T-$net $\sigma = (\sigma_{ij})$ of two-sided ideals of R, such that $H = G(\sigma)$.*

In our paper we extend this result to one infinite dimensional linear group. For R an associative ring with 1 by $GL(\infty, R)$ we denote the group of all column-finite invertible infinite matrices over R (indexed by positive integers $\mathbb{N}$). Let $T(\infty, R)$ denote a group of all infinite upper triangular matrices over R (with invertible elements on the main diagonal). We define $GLB(R)$ as a subgroup of $GL(\infty, R)$ of all matrices which have a finite number of nonzero entries below the main diagonal (clearly, $T(\infty, R) < GLB(R) < GL(\infty, R)$). This group was considered in the case of finite field k by Vershik and Kerov [6] and has applications in representation theory. $GLB(k)$ is infinite dimensional, locally compact, totally disconnected, and amenable in topological sense and unimodular group. The stable general linear group $GL_\infty(k)$, i.e. direct limit of $GL_n(k)$ under natural embeddings $g \to \operatorname{diag}(g, 1)$, is its dense subgroup and the quotient group of $GLB(k)$ over the center is topologically simple.

In this paper we give a purely algebraic description of parabolic subgroups of $GLB(R)$ (i.e. containing $T(\infty, R)$). Our main result is the following theorem.

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Theorem 1.2. *Let R be a semilocal ring, in which 1 is a sum of two invertible elements. If H is a parabolic subgroup of $GLB(R)$, then there exists a unique $T-$net $\sigma = (\sigma_{ij})$ of two-sided ideals of R, such that $H = G(\sigma)$.*

Using this theorem we can prove the "standard properties" (see [3], §2) of parabolic subgroups in $GLB(R)$.

Theorem 1.3. *If R is a semilocal ring, in which 1 is a sum of two invertible elements, then:*

(i) *If P_1, P_2 are two parabolic subgroups of $GLB(R)$ and $gP_1g^{-1} \subset P_2$ for some $g \in GLB(R)$, then $g \in P_2$ and $P_1 \subset P_2$.*
(ii) *Two different parabolic subgroups of $GLB(R)$ are not conjugate.*
(iii) *Every parabolic subgroup of $GLB(R)$ is self-normalized.*

Let $S_{\text{fin}}(\mathbb{N})$ denote the regular matrix representation of all permutations of positive integers $\mathbb{N}$ with finite support. We have

Theorem 1.4 (Bruhat Decomposition Theorem)**.** *For any field K,*

$$GLB(K) = T(\infty, K) \cdot S_{\text{fin}}(\mathbb{N}) \cdot T(\infty, K).$$

If $K = \mathbb{C}$ (complex numbers), then Theorem 1.4 follows from [6]. In [6] the Hecke algebra of double cosets of $GLB(\mathbb{C})$ over the Borel subgroup $B = T(\infty, \mathbb{C})$ was introduced, and because this Hecke algebra is isomorphic to the group algebra $\mathbb{C}(S_{\text{fin}}(\mathbb{N}))$, we obtain the Bruhat decomposition.

2. Proofs of main results

By e (e_n) we denote the unit matrix in $GL(\infty, R)$ $(GL_n(R))$ and by e_{ij} a matrix with the only nontrivial element 1 in the $i-$th row and $j-$th column. We denote $t_{ij}(\zeta) = e + \zeta e_{ij}$, $\zeta \in R$, $i, j \in \mathbb{N}$, $d_i(\theta) = e + (\theta - 1)e_{ii}$, $\theta-$invertible, and $[x, y] = xyx^{-1}y^{-1}$.

Definition 2.1. A system $\sigma = (\sigma_{ij})$ $(i, j \in \mathbb{N})$ of two sided ideals σ_{ij} of R is called a *net* if

$$(\star) \qquad \sigma_{ir} \cdot \sigma_{rj} \subseteq \sigma_{ij} \quad \text{for all} \quad i, j, r \in \mathbb{N}.$$

We call σ *a $T-$net* if $\sigma_{ij} = R$ for $i \le j$. If the set of indexes is $I = \{1, 2, \dots, n\}$ we have the finite nets of ideals in $GL_n(R)$.

Let the set $M(\sigma)$ consist of all matrices a, such that $a_{ij} \in \sigma_{ij}$. If σ satisfies $(\star)$, then $e + M(\sigma) = \{e + a : a \in M(\sigma)\}$ is closed under multiplication of matrices and by $G(\sigma)$ we denote its maximal subgroup. Let $G(m, \infty)$ denote the subgroup of $GL(\infty, R)$ of all matrices a for which $a_{ij} = 0$ for $i > max\{j, m\}$. It is clear that $GLB(R)$ is a direct limit of $G(m, \infty)$ under natural embeddings.

Proof of Theorem 1.2. For H, $T(\infty, R) < H < GLB(R)$, we define $T-$nets σ and $\sigma(m)$ as follows:

$$\sigma_{ij} = \begin{cases} \{\zeta \in R : t_{ij}(\zeta) \in H\} & \text{for} \ \ i > j, \\ R \ \ \text{for} \ \ i \le j, \end{cases}$$

and

$$\sigma(m)_{ij} = \begin{cases} 0 \ \ \text{if} \ \ i > \max\{m, j\}, \\ \sigma_{ij} \ \ \text{otherwise}. \end{cases}$$

We put $H(m) = H \cap G(m,\infty)$. It is clear that H is a direct limit of $H(m)$ and $G(\sigma)$ is a direct limit of $G(\sigma(m))$. We now show that $H(m) = G(\sigma(m))$. If $g \in H(m)$, then $g = \left(\begin{array}{c|c} g_1 & g_3 \\ \hline 0 & g_2 \end{array}\right)$ where $g_1 \in GL_m(R)$, $g_2 \in T(\infty,R)$. Since $H(m) \geq T(\infty,R)$, multiplying $g \in H(m)$ by matrices $\left(\begin{array}{c|c} e_m & 0 \\ \hline 0 & g_2^{-1} \end{array}\right)$ and $\left(\begin{array}{c|c} e_m & g_3 \\ \hline 0 & e \end{array}\right)$, we see that $g' = \left(\begin{array}{c|c} g_1 & 0 \\ \hline 0 & e \end{array}\right) \in H$. We denote by $\hat{H}(m)$ the subgroup of $GL_m(R)$ generated by all such g_1. From Theorem 1.1 it follows that there exists a unique finite $T-$net $\hat{\sigma}(m)$ of ideals of R such that $\hat{H}(m) = G(\hat{\sigma}(m))$. From the construction of $\hat{\sigma}(m)$ and $\sigma(m)$ we deduce the equality $H(m) = G(\sigma(m))$ which implies $H = G(\sigma)$.

Proof of Theorem 1.3. We now prove (i). Then (ii) and (iii) follow easily from (i). If $g \cdot G(\sigma) \cdot g^{-1} \subset G(\sigma')$, then for some m we have $g \in G(m,\infty)$. So

$$\left(\begin{array}{c|c} g_1 & 0 \\ \hline 0 & e \end{array}\right) \cdot G(\sigma) \cdot \left(\begin{array}{c|c} g_1^{-1} & 0 \\ \hline 0 & e \end{array}\right) \subset G(\sigma')$$

or equivalently $g_1 \cdot G(\hat{\sigma}(m)) \cdot g_1^{-1} \subset G(\hat{\sigma}'(m))$ in the group $GL_m(R)$. We show that $g_1 \in G(\hat{\sigma}'(m))$ which implies $g \in G(\sigma'(m))$. From decomposition $g_1 = uvdw$ where u, w are upper unitriangular, d is diagonal and v is lower unitriangular ([2], Thm. 1) it suffices to show that $v \in G(\sigma'(m))$. We have $v = v_2 \cdot \ldots \cdot v_m$, where $v_i = \prod_{j=1}^{i-1} t_{ij}(v_{ij})$. We proceed by induction. Assume that for some r, $2 \leq r \leq m$, we proved that $v_k \in G(\sigma'(m))$, $2 \leq k < r$. Thus $b \cdot G(\sigma(m)) \cdot b^{-1} \subset G(\sigma'(m))$, where $b = v_r \cdot \ldots \cdot v_m$. We have $c = [d_s(\theta)^{-1}, b] \in G(\sigma')$ and hence $c_{rs} = v_{rs}(\theta - 1) \in \sigma'_{rs}$. This implies $v_{rs} \in \sigma'_{rs}$ and $v_r \in G(\sigma')$.

Proof of Theorem 1.4. From [1], p. 45, for any field K we have $GL_m(K) = T_m(K) \cdot S_m \cdot T_m(K)$, where S_m is a regular matrix representation of symmetric group on m elements. It means that $G(m,\infty) = T(\infty,K) \cdot S_m \cdot T(\infty,K)$ and since $S_{\text{fin}}(\mathbb{N})$ is direct limit of S_m under natural embeddings Theorem 1.4 follows.

3. Remarks

a) In a semilocal ring R the unit element 1 is a sum of two invertible elements if and only if every summand in the decomposition of a factor ring of R over a Jacobson radical is different from two elements field ([2], Thm. 3).

b) As was claimed in [5] it is possible to extend Theorem 1.1 to rings R such that R is additively generated by all invertible elements and 1 is a sum of two invertible elements. It means that our results are also valid in this case.

c) Under the same assumption on R as in the remark above, in [4] there is description of subgroups of $T(\infty,R)$ containing $D_{\text{fin}}(\infty,R)$ — the subgroup of finitary diagonal matrices. This result together with Theorem 1.2 give a description of two important intervals in the lattice of subgroups of $GLB(R)$.

References

1. J. L. Alperin, R. B. Bell, *Groups and representations*, Springer-Verlag New York Berlin Heidelberg, 1995. MR **96m:**20001
2. Z.I.Borevich, *On parabolic subgroups in linear groups over a semilocal ring*, (Russian) Vestnik Leningr. Univ., Mat. Meh. Astr. **13** (1976) 16-24 (English translation: Vestn. Leningr. Univ., Math. **9** (1981) 187-196). MR **57:**6217

3. N. Bourbaki, *Groupes et algebres de Lie, Chap.IV Groupes de Coxeter and systemes de Tits*, Hermann Paris, 1968. MR **39:**1590
4. W.Hołubowski, *Subgroups of infinite triangular matrices containing diagonal matrices*, to appear in Publ. Math. Debrecen. **59** (2001). CMP 2002:01
5. N.A.Vavilov, *Bruhat decomposition for subgroups containing the group of diagonal matrices.* (Russian) Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst. Steklova **114** (1982) 50-61 (English translation: J. Sov. Math. **27** (1984) 2865-2874). MR **84e:**20050
6. A. M. Vershik, S. V. Kerov, *On an infinite-dimensional group over a finite field.*(Russian) Funktsional. Anal. i Prilozhen. **32** (1998), no. 3, 3–10, 95 (English translation: Funct. Anal. Appl. **32** (1998), no. 3, 147–152 (1999)). MR **2000b:**22006

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