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TOROIDAL SURGERIES ON HYPERBOLIC KNOTS

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ABSTRACT. For a hyperbolic knot K in S^3, a toroidal surgery is Dehn surgery which yields a 3-manifold containing an incompressible torus. It is known that a toroidal surgery on K is an integer or a half-integer. In this paper, we prove that all integers occur among the toroidal slopes of hyperbolic knots with bridge index at most three and tunnel number one.

1. INTRODUCTION

Let K be a knot in the 3-sphere S^3, and let $E(K) = S^3 - \mathrm{Int}N(K)$ be its exterior. A *slope* on $\partial E(K)$ is the isotopy class of an essential unoriented simple loop. As usual [11], the set of slopes on $\partial E(K)$ is parameterized by $\mathbb{Q} \cup \{\infty\}$ so that $1/0$ is the meridian slope and $0/1$ is the longitude slope. For a slope r on $\partial E(K)$, $K(r)$ denotes the closed orientable 3-manifold obtained by r-*Dehn surgery* on K. Thus $K(r) = E(K) \cup V$, where V is a solid torus glued to $E(K)$ along their boundaries in such a way that r bounds a disk in V.

Now suppose that K is a hyperbolic knot, i.e. the interior of $E(K)$ has a complete hyperbolic structure. If $K(r)$ is not hyperbolic, the surgery and the slope r are said to be *exceptional*. By the hyperbolic Dehn surgery theorem [12], K has only finitely many exceptional surgeries. A closed 3-manifold is *toroidal* if it contains an incompressible torus. If $K(r)$ is toroidal, the surgery is said to be *toroidal*. Clearly, a toroidal surgery is exceptional.

There are some results on toroidal surgeries on hyperbolic knots. Gordon and Luecke [8] showed that if $K(m/n)$ is toroidal, then $|n| \leq 2$. Hence a toroidal slope on a hyperbolic knot is either an integer or a half-integer. For hyperbolic alternating knots, toroidal slopes are integers divisible by four [1] (see also [10]). In this paper, we show that all integers can occur among the toroidal slopes of hyperbolic knots.

Theorem 1.1. *For any integer r, there exists a hyperbolic knot K in S^3 such that $K(r)$ is toroidal. Furthermore, K has bridge index at most three and tunnel number one.*

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2. Doubly Seifert-fibered knots

In this section, we will describe a construction of knots in S^3 that have toroidal surgeries done by Dean [4].

Let H be a standardly embedded handlebody of genus two in S^3. Then $H' = S^3 - \mathrm{Int}H$ is also a handlebody of genus two. Let $F = \partial H = \partial H'$. If a knot K is embedded in F, then $\partial N(K) \cap F$ defines a slope on $\partial E(K)$, which is called the *surface slope of K with respect to F*. Note that a surface slope is always integral.

Lemma 2.1. *Let K be a knot in F, and let r be the surface slope of K with respect to F. Assume that K is non-separating in F. Then $K(r) \cong W \cup W'$, where W (W') is obtained from H (H' resp.) by attaching a 2-handle along K, and they are glued along their boundaries ∂W and $\partial W'$, which are tori.*

Proof. This is a special case of [4, Lemma 2.1.1]. Let c_1 and c_2 be the curves $F \cap \partial N(K)$. Then c_i bounds a meridian disk D_i of the attached solid torus V in $K(r)$ for $i = 1, 2$. Let $\widehat{F} = (F - N(K)) \cup D_1 \cup D_2$. Since K is non-separating in F, $\widehat{F}$ is a torus. We split $K(r)$ along $\widehat{F}$ into W and W'. Then W and W' are homeomorphic to the described ones. □

For non-zero integers m and n, let $G_{m,n}$ denote the group $\langle x, y \mid x^m y^n = 1\rangle$. An element in a free group is *primitive* if it is a part of a basis. An element w in the free group $\langle x, y\rangle$ is (m, n) *Seifert-fibered* if $\langle x, y \mid w = 1\rangle \cong G_{m,n}$. If $|m| = 1$ or $|n| = 1$, then $G_{m,n} \cong \mathbb{Z}$.

If a knot K in F represents a Seifert-fibered element of $\pi_1(H)$, then we say that K is *Seifert-fibered* with respect to H. In particular, if K represents a primitive element of $\pi_1(H)$, then K is said to be *primitive* with respect to H. Also, if K is Seifert-fibered with respect to both of H and H', then it is said to be *doubly Seifert-fibered* with respect to F. Note that the abelianization of $G_{m,n}$ is $\mathbb{Z} \oplus \mathbb{Z}_{(m,n)}$, where (m, n) is the greatest common divisor of m and n. Therefore if K is Seifert-fibered with respect to H, say, then K is non-separating on $F = \partial H$.

Lemma 2.2. *If a knot K on ∂H is (m, n) Seifert-fibered with respect to H for $|m|, |n| \geq 2$, then the manifold W obtained by adding a 2-handle to H along K is a Seifert fibered manifold with incompressible boundary.*

Proof. Note that W has Heegaard genus two. By additivity of Heegaard genus (see [3]), W is irreducible, since $\pi_1(W) = G_{m,n}$. Hence W is Haken. Then W is a Seifert fibered manifold by [13], since $G_{m,n}$ has a nontrivial center. The last part follows from the fact that the only Seifert fibered manifold with non-empty compressible boundary is a solid torus. See also [4, Lemma 2.2.1]. □

Lemma 2.3. *Let K be a doubly Seifert-fibered knot in F with surface slope r. Then $K(r)$ is toroidal.*

Proof. This immediately follows from Lemmas 2.1 and 2.2. □

3. Proof of Theorem 1.1

Let r be an integer. If r is a toroidal surgery on a knot K, then $-r$ is one on the mirror image of K. Therefore we may assume that $r \geq 0$.

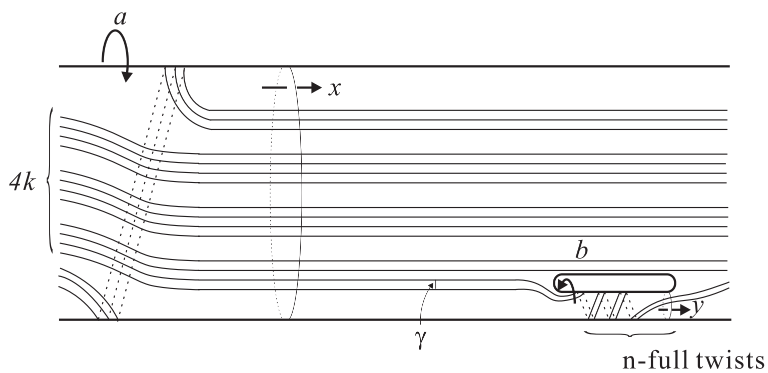

FIGURE 1. $K(4k+3,3,2,n)$

We will divide the proof into three cases.

Case 1. $r \equiv 0 \pmod 4$.

Let $K(b_1, b_2)$ be the 2-bridge knot corresponding to a continued fraction $[b_1, b_2]$. Then $K(2,-2)$ is the figure-eight knot, and 0 and 4 are toroidal surgeries [12]. If $r \geq 8$, $K(3, r/2)$ is hyperbolic, and r gives a toroidal surgery [2, Theorem 1.1(2)]. Note that any 2-bridge knot has tunnel number one.

Case 2. $r \equiv \pm 1 \pmod 4$.

Let $k \geq 1, k \not\equiv 0 \pmod 3$ and $n \leq -2$. Let $K(4k+3,3,2,n)$ be the twisted torus knot lying on H. It is obtained from the torus knot of type $(4k+3,3)$ by adding n-full twists on two strands that are parallel in the standard torus knot picture [4]. See Figure 1, where the two ends are glued to form H. We see that the surface slope with respect to ∂H is $3(4k+3)+4n$. As a knot in S^3, it is isotopic to $K(3,4k+3,2,n)$, and hence it has bridge index at most three. Also, it is clear that the arc γ shown in Figure 1 is an unknotting tunnel.

Let $\{x,y\}$ and $\{a,b\}$ be the bases of $\pi_1(H)$ and $\pi_1(H')$, respectively, as in Figure 1. The following two lemmas are checked straightforwardly.

Lemma 3.1. *In* $\pi_1(H)$,

$$K(4k+3,3,2,n) \textit{ represents} \begin{cases} x^{\frac{8k+7}{3}} y x^{\frac{4k+2}{3}} y & \textit{if} \quad k \equiv 1 \pmod 3, \\ x^{\frac{8k+5}{3}} y x^{\frac{4k+4}{3}} y & \textit{if} \quad k \equiv 2 \pmod 3. \end{cases}$$

Lemma 3.2. *In* $\pi_1(H')$, $K(4k+3,3,2,n)$ *represents* $a^2 b^{-n} a b^{-n}$.

Lemma 3.3. *With respect to* H,

$$K(4k+3,3,2,n) \textit{ is} \begin{cases} (\frac{4k+5}{3}, 2) \textit{ Seifert-fibered} & \textit{if} \quad k \equiv 1 \pmod 3, \\ (\frac{4k+1}{3}, 2) \textit{ Seifert-fibered} & \textit{if} \quad k \equiv 2 \pmod 3. \end{cases}$$

Proof. We prove the case where $k \equiv 1 \pmod 3$. The other case is similar.

$$\begin{aligned} \langle x, y \mid x^{\frac{8k+7}{3}} y x^{\frac{4k+2}{3}} y = 1 \rangle &= \langle x, y \mid x^{\frac{4k+5}{3}} (x^{\frac{4k+2}{3}} y)^2 = 1 \rangle \\ &= \langle x, y, z \mid x^{\frac{4k+5}{3}} z^2 = 1, z = x^{\frac{4k+2}{3}} y \rangle \\ &= \langle x, z \mid x^{\frac{4k+5}{3}} z^2 = 1 \rangle. \end{aligned}$$

□

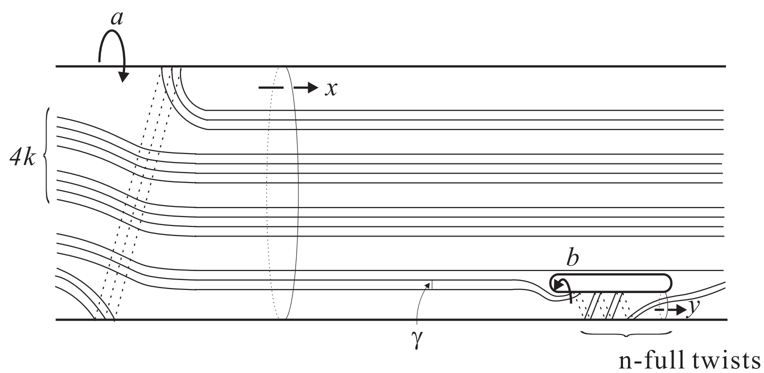

FIGURE 2. $K(4k+6,3,2,n)$

Lemma 3.4. *$K(4k+3,3,2,n)$ is $(3,n)$ Seifert-fibered with respect to H'.*

Proof.

$$\begin{aligned}\langle a,b \mid a^2b^{-n}ab^{-n}=1\rangle &= \langle a,b \mid a(ab^{-n})^2=1\rangle \\ &= \langle a,b,c \mid ac^2=1, c=ab^{-n}\rangle \\ &= \langle b,c \mid cb^nc^2=1\rangle \\ &= \langle b,c \mid c^3b^n=1\rangle.\end{aligned}$$

□

Proposition 3.5. *For $K(4k+3,3,2,n)$, the surface slope $3(4k+3)+4n$ with respect to F is toroidal. Therefore the knot is hyperbolic.*

Proof. The knot $K(4k+3,3,2,n)$ is doubly Seifert-fibered by Lemmas 3.3 and 3.4. Hence the surface slope is toroidal by Lemma 2.3. Since it has bridge index at most three, it is either a torus knot or a hyperbolic knot. But a torus knot has no non-zero toroidal surgery. (In fact, 0-surgery on the trefoil is the only toroidal surgery for torus knots. See [9].) Thus the knot is hyperbolic. □

For a given integer $r>0$ such that $r\equiv 1 \pmod 4$, we choose k so that $r\le 12k+1, k\not\equiv 0 \pmod 3$. Then the knot $K(4k+3,3,2,n)$, $n=-2-\frac{(12k+1)-r}{4}$, has the surface slope r exactly. If $r\equiv -1 \pmod 4$, then consider the knot $K(7,3,2,n)$ with $n=-6+\frac{3-r}{4}$. Then it has the surface slope $-r$. Hence its mirror image is a desired one.

As stated before, if a hyperbolic 2-bridge knot has a toroidal slope, then the slope is an integer divisible by 4 [1]. Therefore our knots are 3-bridge.

Case 3. $r\equiv 2 \pmod 4$.

Let $k\ge 1, k\not\equiv 0 \pmod 3$ and $n\le -2$. Let $K(4k+6,3,2,n)$ be the twisted torus knot lying on H. See Figure 2. We see that the surface slope with respect to ∂H is $3(4k+6)+4n$.

As in Case 2, the knot has bridge index at most three, and tunnel number one. We use the same bases of $\pi_1(H)$ and $\pi_1(H')$ as in Case 1.

The following two lemmas are checked straightforwardly.

Lemma 3.6. *In* $\pi_1(H)$,

$$K(4k+6,3,2,n) \text{ represents} \begin{cases} x^{\frac{8k+13}{3}}yx^{\frac{4k+5}{3}}y & \text{if} \quad k \equiv 1 \pmod 3, \\ x^{\frac{8k+11}{3}}yx^{\frac{4k+7}{3}}y & \text{if} \quad k \equiv 2 \pmod 3. \end{cases}$$

Lemma 3.7. *In* $\pi_1(H')$, $K(4k+6,3,2,n)$ *represents* $a^2b^{-n}ab^{-n}$.

The next lemmas are proved by the same way as in the proofs of Lemmas 3.3 and 3.4.

Lemma 3.8. *With respect to* H,

$$K(4k+6,3,2,n) \text{ is} \begin{cases} (\frac{4k+8}{3},2) \text{ Seifert-fibered} & \text{if} \quad k \equiv 1 \pmod 3, \\ (\frac{4k+4}{3},2) \text{ Seifert-fibered} & \text{if} \quad k \equiv 2 \pmod 3. \end{cases}$$

Lemma 3.9. $K(4k+6,3,2,n)$ *is* $(3,n)$ *Seifert-fibered with respect to* H'.

Proposition 3.10. *For* $K(4k+6,3,2,n)$, *the surface slope* $3(4k+6)+4n$ *with respect to* F *is toroidal. Therefore the knot is hyperbolic.*

Proof. The arguments in the proof of Proposition 3.5 work well. □

For a given integer $r > 0$ such that $r \equiv 2 \pmod 4$, we choose k so that $r \leq 12k+10, k \not\equiv 0 \pmod 3$. Then the knot $K(4k+6,3,2,n)$, $n = -2 - \frac{(12k+10)-r}{4}$, has the surface slope r. As in Case 2, the knot is 3-bridge.

Thus we have proved Theorem 1.1.

Remark 3.11. Eudave-Muñoz [5] gave infinitely many hyperbolic knots with half-integer toroidal surgeries. For example, the $(-2,3,7)$-pretzel knot has a toroidal slope 37/2. Among his knots, $k(l,m,n,0)$ (in his notation) has a non-integral toroidal slope

$$-\frac{1}{2} - l + l^2m + 2lm - 2l^2m^2 + (2lm-1)^2n.$$

Indeed, $k(3,1,1,0)$ is the $(-2,3,7)$-pretzel knot. Also, $k(l,m,0,p)$ has a non-integral toroidal slope

$$-\frac{1}{2} - l + l^2m + 2lm - 2l^2m^2 + (2lm-1-l)^2p.$$

See also [6, Propositions 5.3, 5.4]. Since his knots are expected to give all hyperbolic knots with non-integral toroidal surgeries [7], it seems to be reasonable to conjecture that not all $n/2$ can be realized as toroidal slopes of hyperbolic knots. In fact, we may conjecture that $|n/2| \geq 37/2$.

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