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LOCALLY FINITE DIMENSIONAL SHIFT-INVARIANT SPACES IN $\mathbf{R}^d$

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ABSTRACT. We prove that a locally finite dimensional shift-invariant linear space of distributions must be a linear subspace of some shift-invariant space generated by finitely many compactly supported distributions. If the locally finite dimensional shift-invariant space is a subspace of the Hölder continuous space C^α or the fractional Sobolev space $L^{p,\gamma}$, then the superspace can be chosen to be C^α or $L^{p,\gamma}$, respectively.

1. INTRODUCTION

A function $f \in L^2$ belongs to the Paley-Wiener space $B_{1/2}$ of band-limited functions if its Fourier transform satisfies $\hat{f}(\xi) = 0$ for all $\xi \notin [-\frac{1}{2}, \frac{1}{2}]$. The Paley-Wiener space is a prototypical space for sampling theory, and for digital signal processing, and it was used to construct some of the first examples of wavelets (see for example [14]). The Whittaker representation of functions in $B_{1/2}$ ([19]) then implies that

$$B_{1/2} = \Big\{ f = \sum_{j\in\mathbf{Z}} c(j)\mathrm{sinc}(x-j) : \sum_{j\in\mathbf{Z}} |c(j)|^2 < \infty \Big\},$$

where $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. A linear space V of distributions on $\mathbf{R}^d$ is called *shift-invariant* if $f \in V$ implies $f(\cdot - \mathbf{j}) \in V$ for any $\mathbf{j} = (j_1, \dots, j_d)^T \in \mathbf{Z}^d$. Thus, the space $B_{1/2}$ is a shift-invariant space (SIS). Obviously, L^2 is also an SIS, but it does not have the same structure as $B_{1/2}$, i.e., it cannot be generated by

$$\mathrm{span}\{\phi_1(\cdot - j), \dots, \phi_M(\cdot - j) : j \in \mathbf{Z}\}$$

of some functions $\phi_1, \dots, \phi_M$ with $M < \infty$.

Since the sinc-function has infinite support and slow decay, the Paley-Wiener space may be unsuitable for some applications and some numerical implementations. Moreover, all functions in $B_{1/2}$ have infinite support since they are analytic. Hence, $B_{1/2}$ cannot be generated by the span of compactly supported functions $\{\phi_s(\cdot - j) : s = 1, \dots, r, j \in \mathbf{Z}\} \subset B_{1/2}$. In fact, the restriction of $B_{1/2}$ to the unit interval is infinite dimensional; thus, there are no compactly supported functions $\phi_1, \dots, \phi_M$ such that $B_{1/2} \subset \mathrm{span}\{\phi_1(\cdot - j), \dots, \phi_M(\cdot - j) : j \in \mathbf{Z}\}$. On the other

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hand, the shift-invariant space $V_2(\beta_o^3) = \{\sum_{j\in\mathbf{Z}} d(j)\beta_o^3(\cdot - j) : \sum_{j\in\mathbf{Z}} |d(j)|^2 < \infty\}$, generated by the infinitely supported Battle-Lemarié scaling function β_o^3, can be generated by the integer shifts $\{\beta^3(\cdot - j)\}$ of the compactly supported B-spline function of order 3. This last representation can be useful in applications. Unlike the Paley-Wiener space, the space of restrictions of functions in $V_2(\beta_o^3)$ to any bounded open set A is finite dimensional. However, there are shift-invariant spaces with finite dimensional restrictions on any bounded open set, but that do not contain any compactly supported function, as in the example below.

Example 1. Let $(\alpha(j))_{j\in\mathbf{Z}}$ be a sequence such that $\alpha(\xi) = \sum_{j\in\mathbf{Z}} \alpha(j)e^{-ij\xi}$ is a nonzero 2π-periodic C^∞ function and equals zero in a neighborhood of zero. Define $\phi = \sum_{j\in\mathbf{Z}} \alpha(j)\chi_{[j,j+1)}$, where χ_E denotes the characteristic function on a set E, and let

$$V_2(\phi) = \Big\{ \sum_{j\in\mathbf{Z}} d(j)\phi(\cdot - j) : \quad \sum_{j\in\mathbf{Z}} |d(j)|^2 < \infty \Big\}.$$

Obviously

$$V_2(\phi) \subset \Big\{ \sum_{j\in\mathbf{Z}} d_1(j)\chi_{[j,j+1)} : \sum_{j\in\mathbf{Z}} |d_1(j)|^2 < \infty \Big\}.$$

Thus, $V_2(\phi)$ has finite dimensional restrictions on any bounded open set. Now we prove, by contradiction, that there is not any compactly supported function in $V_2(\phi)$. Assume that there exists a sequence $(d(j))_{j\in\mathbf{Z}} \in \ell^2$ such that $g = \sum_{j\in\mathbf{Z}} d(j)\phi(\cdot - j)$ is a nonzero compactly supported function in $V_2(\phi)$. Then, $\hat{g}(\xi) = \hat{d}(\xi)\hat{\phi}(\xi) = \hat{d}(\xi)\hat{\alpha}(\xi)\mathrm{sinc}(\xi)$ must vanish in a neighborhood of $\xi = 0$. On the other hand, since

$$g \in \Big\{ \sum_{j\in\mathbf{Z}} d_1(j)\chi_{[j,j+1)} : \sum_{j\in\mathbf{Z}} |d_1(j)|^2 < \infty \Big\}$$

and has compact support, g can be written as $g = \sum_{j\in\mathbf{Z}} \gamma(j)\chi_{[j,j+1)}$, where γ is a nonzero finite sequence. Thus, $\hat{g}(\xi) = \hat{\gamma}(\xi)\mathrm{sinc}(\xi)$ cannot vanish in a neighborhood of $\xi = 0$, since $\hat{\gamma}(\xi)$ is a nonzero trigonometric polynomial. This contradicts our previous assertion that $\hat{g}(\xi)$ must vanish in a neighborhood of $\xi = 0$.

Let $\mathcal{D}$ denote the space of compactly supported C^∞ functions on $\mathbf{R}^d$ with the usual topology, and let $\mathcal{D}'$ be the corresponding space of distributions. A linear space V of distributions having finite dimensional restrictions on any bounded open set is said to be *locally finite dimensional*. For locally finite dimensional shift-invariant spaces, there is a long list of publications on its algebraic structure and its applications (see for instance [1, 5, 6, 8, 9, 10, 11, 13]). In the univariate case and under the additional conditions that V is a space of functions, and that it is closed under uniform convergence on compact sets, de Boor and Devore demonstrated that a locally finite dimensional SIS V can be generated by a finite set of compactly supported functions $\phi_1, \dots, \phi_M$ modulo some finite dimensional space [4].

For $1 \le p \le \infty$, denote the space of sequences with finite norm $\|\cdot\|_{\ell^p}$ by ℓ^p. For $1 \le p \le \infty$ and any functions $\phi_1, \dots, \phi_M \in L^p$ such that

$$\sum_{\mathbf{j}\in\mathbf{Z}^d} d(\mathbf{j})\phi_s(\cdot - \mathbf{j}) \in L^p \quad \text{for all} \quad 1 \le s \le M \quad \text{and} \quad (d(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d} \in \ell^p, \tag{1.1}$$

define

$$(1.2)\quad V_p(\phi_1,\dots,\phi_M) := \Big\{ \sum_{s=1}^{M}\sum_{\mathbf{j}\in\mathbf{Z}^d} d_s(\mathbf{j})\phi(\cdot-\mathbf{j}) :\ (d_s(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d}\in\ell^p,\ 1\le s\le M\Big\}.$$

It is obvious that $V_p(\phi_1,\dots,\phi_M)$ is a shift-invariant subspace of L^p.

Let ℓ_0 denote the space of sequences $(d(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d}$ such that $d(\mathbf{j})=0$ for all but finitely many $\mathbf{j}\in\mathbf{Z}^d$. Let $\phi_1,\dots,\phi_M$ be compactly supported functions in $L^p, 1\le p\le\infty$, or $\phi_1,\dots,\phi_M\in\mathcal{D}'$. Define

$$S_0(\phi_1,\cdots,\phi_M) := \Big\{ \sum_{s=1}^{M}\sum_{\mathbf{j}\in\mathbf{Z}^d} d_s(\mathbf{j})\phi_s(\cdot-\mathbf{j});\quad (d_s(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d}\in\ell_0\Big\}.$$

For $1\le p\le\infty$, denote the L^p-closure of $S_0(\phi_1,\cdots,\phi_M)$ by $S_p(\phi_1,\dots,\phi_M)$. It can be checked easily that $S_p(\phi_1,\dots,\phi_M)$ is shift-invariant and locally finite dimensional. It can also be checked easily that $S_0\subset V_p\subset S_p$.

Define the Fourier transform $\hat f$ of an integrable function f by

$$\hat f(\xi) := \int_{\mathbf{R}^d} f(x)e^{-2\pi i x\xi}dx$$

and that of a tempered distribution as usual. For $p=2$, $f\in S_2(\phi_1,\dots,\phi_M)$ if and only if $\widehat{f}(\xi)=\sum_{s=1}^M D_s(\xi)\widehat{\phi}_s(\xi)$ for some 2π-periodic functions $D_s(\xi), 1\le s\le M$, and $\sum_{s=1}^M D_s(\xi)\widehat{\phi}_s(\xi)$ is square integrable ([5]). However, it is not known whether $D_s(\xi), 1\le s\le M$, correspond to sequences. Thus unlike $V_2(\phi_1,\dots,\phi_M)$, $S_2(\phi_1,\dots,\phi_M)$ is not necessarily generated by linear combinations of generators and their shifts. Even in the case where we know that $S_2(\phi_1,\dots,\phi_M)$ is generated by such a linear combination, it is not true that the coefficients of the linear combination are in ℓ^p or in some well-defined sequence space, in general. In fact, the algebraic structure of $S_2(\phi_1,\dots,\phi_M)$ is usually very complicated. For $p\ne 2$, there are fewer treatments of the algebraic structure of the space $S_p(\phi_1,\dots,\phi_M)$. For these reasons we consider the following problem.

Problem. Given a topological linear space X of distributions, and a locally finite dimensional shift-invariant subspace V, can we find a shift-invariant subspace S of X with a simple algebraic structure such that V is a subspace of S?

For a linear topological subspace X of $\mathcal{D}'$, we say that X has *continuous translates* if $f(\cdot-\mathbf{y})\in X$ for all $f\in X$ and $\mathbf{y}\in\mathbf{R}^d$, and the translation operator $\tau_{\mathbf{y}}$,

$$\tau_{\mathbf{y}} : X\ni f\longmapsto f(\cdot-\mathbf{y})\in X,$$

is continuous for any $\mathbf{y}\in\mathbf{R}^d$; and we say that X has *continuous $\mathcal{D}$-multiplication* if the multiplication by any function $h\in\mathcal{D}$,

$$X\ni f\longmapsto hf\in X,$$

is a continuous map from X to X. Let X be a topological linear space having continuous translates and continuous $\mathcal{D}$-multiplication. In this paper, we shall prove that if V is a shift-invariant linear subspace of X and if V is locally finite dimensional, then V must be a subspace of some shift-invariant space generated by finitely many compactly supported distributions in X having linearly independent shifts (Theorem 3.1). Moreover, the intersection between the finitely generated shift-invariant space above and X has a simple algebraic structure when X is the space of Hölder continuous functions C^α, or X is the fractional Sobolev space $L^{p,\gamma}$ (Theorems 4.1 and 4.3).

2. Model case: L^p

In this section, we discuss the model case of locally finite dimensional shift-invariant subspaces of $L^p(\mathbf{R}^d)$. The ideas behind the discussion have appeared in [4, 7], and the result for the model case (Theorem 2.1) was essentially stated in [11].

Generally the space $V_p(\phi_1,\dots,\phi_M)$ defined by (1.2) is not a closed subspace of L^p. For $1\le p\le\infty$, let $\mathcal{L}^p$ be the space of all functions f for which

$$\|f\|_{\mathcal{L}^p} := \Big\|\sum_{\mathbf{j}\in\mathbf{Z}^d}|f(\cdot+\mathbf{j})|\Big\|_{L^p([0,1]^d)}<\infty.$$

Then $\mathcal{L}^\infty\subset\mathcal{L}^q\subset\mathcal{L}^p\subset L^p$ for any $1\le p\le q\le\infty$, $\mathcal{L}^1=L^1$. Obviously, any compactly supported L^p function belongs to $\mathcal{L}^p$. Also it is routine to check that $\mathcal{L}^p, 1\le p\le\infty$, has continuous translates and $\mathcal{D}$-multiplication. For a single generator ϕ, it can be shown that $\sum_{\mathbf{j}\in\mathbf{Z}^d} d(\mathbf{j})\phi(\cdot-\mathbf{j})\in L^p$ for any $\phi\in\mathcal{L}^p$ and $(d(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d}\in\ell^p$ (for instance, see [13]). For the case of multiple generators, $\phi_1,\dots,\phi_M\in\mathcal{L}^\infty$ and $1\le p<\infty$, the closedness of $V_p(\phi_1,\dots,\phi_M)$ in L^p was completely characterized in [2]. For any functions $\phi_1,\dots,\phi_M\in L^p$ satisfying (1.1), we say that the set of functions $\{\phi_1(\cdot-\mathbf{j}),\dots,\phi_M(\cdot-\mathbf{j}),\ \mathbf{j}\in\mathbf{Z}^d\}$ forms a strong unconditional basis (also known as a stable basis) of $V_p(\phi_1,\dots,\phi_M)$ if there exist two positive constants C_1 and C_2 such that

$$C_1\sum_{s=1}^M\|D_s\|_{\ell^p}\le\Big\|\sum_{s=1}^M\sum_{\mathbf{j}\in\mathbf{Z}^d}d_s(\mathbf{j})\phi_s(\cdot-\mathbf{j})\Big\|_{L^p}\le C_2\sum_{s=1}^M\|D_s\|_{\ell^p}\tag{2.1}$$

for all $D_1,\dots,D_M\in\ell^p$, where $D_s=(d_s(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d}\in\ell^p, 1\le s\le M$. By the completeness of ℓ^p, $V_p(\phi_1,\dots,\phi_M)$ is a Banach subspace of L^p if the set of functions $\{\phi_1(\cdot-\mathbf{j}),\dots,\phi_M(\cdot-\mathbf{j}),\ \mathbf{j}\in\mathbf{Z}^d\}$ forms a strong unconditional basis of $V_p(\phi_1,\dots,\phi_M)$.

For a locally finite dimensional shift-invariant linear subspace V of L^p, define $f_| = f\chi_{[0,1]^d}$ and $V_|=\{f_| : f\in V\}$. By the assumption on V, the space $V_|$ is finite dimensional. Let $\dim V_| = M$; then there exists an M-dimensional basis $\{\phi_1,\dots,\phi_M\}$ for $V_|$ such that $\phi_1,\dots,\phi_M$ belong to L^p with support in $[0,1]^d$. For $p=2$, the functions $\phi_1,\dots,\phi_M$ can be chosen to form an orthonormal basis of $(V_|, L^2([0,1]^d))$, and hence the set $\{\phi_1(\cdot-\mathbf{j}),\dots,\phi_M(\cdot-\mathbf{j}),\ \mathbf{j}\in\mathbf{Z}^d\}$ forms an orthonormal basis of $V_2(\phi_1,\dots,\phi_M)$. For a general $1\le p<\infty$, there exist positive constants C_1 and C_2 such that

$$C_1\sup_{1\le s\le M}|\lambda_s|\le\Big\|\sum_{s=1}^M\lambda_s\phi_s\Big\|_{L^p([0,1]^d)}\le C_2\sup_{1\le s\le M}|\lambda_s|\tag{2.2}$$

for all $(\lambda_1,\dots,\lambda_M)\in\mathbf{R}^M$. By the definition of $\phi_1,\dots,\phi_M$, for any $f\in V$ there exist unique sequences $(d_s(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d}, 1\le s\le M$, such that

$$f=\sum_{s=1}^M\sum_{\mathbf{j}\in\mathbf{Z}^d}d_s(\mathbf{j})\phi_s(\cdot-\mathbf{j}).\tag{2.3}$$

Recall that the $\phi_s, 1 \le s \le M$, are supported in $[0,1]^d$. Then, by (2.3),

$$\|f\|_p = \Big\| \Big(\Big\| \sum_{s=1}^{M} d_s(\mathbf{j})\phi_s \Big\|_{L^p([0,1]^d)} \Big)_{\mathbf{j}\in\mathbf{Z}^d} \Big\|_{\ell^p}. \tag{2.4}$$

This together with (2.2) yields:

Theorem 2.1. *Let $1 \le p < \infty$, and let V be a locally finite dimensional shift-invariant linear subspace of L^p. Then there exist compactly supported functions $\phi_1, \dots, \phi_M$ in L^p such that $\{\phi_1(\cdot - \mathbf{j}), \dots, \phi_M(\cdot - \mathbf{j}),\ \mathbf{j} \in \mathbf{Z}^d\}$ forms a stable basis for $V_p(\phi_1, \dots, \phi_M)$, and such that V is a subspace of $V_p(\phi_1, \dots, \phi_M)$.*

3. Shift-invariant spaces of distributions

For any compactly supported distributions $\phi_1, \dots, \phi_M$, define the corresponding semi-convolution map S from $\ell^{(M)}$ to $\mathcal{D}'$ by

$$S:\quad \Big((d_1(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d}, \dots, (d_M(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d} \Big) \longmapsto \sum_{s=1}^{M} \sum_{\mathbf{j}\in\mathbf{Z}^d} d_s(\mathbf{j})\phi_s(\cdot - \mathbf{j}), \tag{3.1}$$

where $\ell^{(M)}$ is the linear space consisting of ordered M-tuples of sequences. We say that the compactly supported distributions $\phi_1, \dots, \phi_M$ have *linearly independent shifts* if the corresponding map S in (3.1) is one-to-one. The image of the map S in (3.1), which we denote by $S(\phi_1, \dots, \phi_M)$, is said to be the *shift-invariant space generated by* $\phi_1, \dots, \phi_M$. It is obvious that $S(\phi_1, \dots, \phi_M)$ is a locally finite dimensional SIS. Moreover, $S(\phi_1, \dots, \phi_M)$ has a simple algebraic structure. However, as proved by Example 1, locally finite dimensional SIS of distributions need not be generated by finitely many compactly supported distributions.

In the proof of Theorem 2.1, we use the restrictions of functions in V to unit cubes $[k, k+1]^d, k \in \mathbf{Z}^d$. Generally some properties such as Hölder continuity may be lost when restricting functions in V to cubes. Moreover, the restriction procedure cannot be used when the elements in V are distributions that are not generated by functions. Hence the procedure in the proof of Theorem 2.1 cannot be generalized in a straightforward way to the shift-invariant subspaces of C^α and $L^{p,\gamma}$, two important classes of function spaces. Thus, for a shift-invariant space with its elements in some linear topological space, we need to develop new methods to construct an "appropriate" shift-invariant space generated by finitely many distributions.

Theorem 3.1. *Let X be a linear topological subspace of $\mathcal{D}'$ having continuous translates and $\mathcal{D}$-multiplication, and let V be a locally finite dimensional shift-invariant linear topological subspace of X. Then there exist compactly supported distributions $\phi_1, \dots, \phi_M \in X$ such that $\phi_1, \dots, \phi_M$ have linearly independent shifts, and such that V is a subspace of $S(\phi_1, \dots, \phi_M)$.*

To prove Theorem 3.1, we need the following decomposition of any finite set of compactly supported distributions, whose proof is given at the end of this section. This decomposition is interesting by itself, and has been used in the study of convergence of cascade algorithms and smoothness of refinable distributions ([17]).

Theorem 3.2. *Let $\{g_1, \dots, g_N\}$ be any finite set of compactly supported distributions. Then there exist compactly supported distributions $\phi_1, \dots, \phi_M$ such that*

(i) *$\phi_1, \dots, \phi_M$ have linearly independent shifts;*

(ii) *$\phi_s, 1 \le s \le M$, are finite linear combinations of $g_i(\cdot - \mathbf{j})$ and $h_k g_i(\cdot - \mathbf{j})$, where $1 \le i \le N$, $\mathbf{j} \in \mathbf{Z}^d$ and $h_k \in \mathcal{D}$, i.e., there exist finitely many $h_k \in \mathcal{D}$ and $\mathbf{j} \in \mathbf{Z}^d$ such that*

$$\phi_s = \sum_{i=1}^N \sum_{k,\mathbf{j}} c_{i,s,k,\mathbf{j}} h_k g_i(\cdot - \mathbf{j}) + \sum_{i=1}^N \sum_{\mathbf{j}} \tilde{c}_{i,s,\mathbf{j}} g_i(\cdot - \mathbf{j})$$

for some coefficients $c_{i,s,k,\mathbf{j}}$ and $\tilde{c}_{i,s,\mathbf{j}}$;

(iii) *$g_i, 1 \le i \le N$, are finite linear combinations of $\phi_s(\cdot - \mathbf{j}), 1 \le s \le M, \mathbf{j} \in \mathbf{Z}^d$, i.e., $g_i = \sum_{s=1}^M \sum_{\mathbf{j} \in \mathbf{Z}^d} d_{is}(\mathbf{j}) \phi_s(\cdot - \mathbf{j})$ for some sequences $(d_{is}(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d} \in \ell_0$.*

For the moment, we assume that Theorem 3.2 is true and prove Theorem 3.1.

Proof of Theorem 3.1. Let $A_0 = (-1,1)^d$ and let $h_0 \in \mathcal{D}$ be so chosen that supp $h_0 \subset A_0$ and

$$\sum_{\mathbf{k} \in \mathbf{Z}^d} h_0(\mathbf{x} - \mathbf{k}) = 1 \quad \text{for all} \quad \mathbf{x} \in \mathbf{R}^d. \tag{3.2}$$

Denote the linear space of restrictions on A_0 of all distributions in V by V_{A_0}. By the assumption on V, V_{A_0} is a finite dimensional linear space. Let $N = \dim V_{A_0}$; then there exist finitely many distributions $f_1, \ldots, f_N$ in V such that their restrictions on A_0 are bases of V_{A_0}. Therefore by the shift-invariance of V, we conclude that for any $f \in V$ there exist unique sequences $(c_i(\mathbf{k}))_{\mathbf{k} \in \mathbf{Z}^d}, i = 1, \ldots, N$, such that

$$f(\cdot + \mathbf{k}) - \sum_{i=1}^N c_i(\mathbf{k}) f_i = 0 \quad \text{on} \quad A_0 \quad \text{for all} \quad \mathbf{k} \in \mathbf{Z}^d.$$

This together with supp $h_0 \subset A_0$ leads to

$$h_0(\cdot - \mathbf{k}) f = \sum_{i=1}^N c_i(\mathbf{k}) h_0(\cdot - \mathbf{k}) f_i(\cdot - \mathbf{k}) \quad \text{for all} \quad \mathbf{k} \in \mathbf{Z}^d. \tag{3.3}$$

Set $g_i = h_0 f_i, 1 \le i \le N$, and let $\phi_1, \ldots, \phi_M$ and $(d_{is}(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d},\ i = 1, \ldots, N,\ s = 1, \ldots, M$, be the compactly supported distributions and the finitely supported sequences in Theorem 3.2 respectively such that $\phi_1, \ldots, \phi_M$ have linearly independent shifts and

$$h_0 f_i = \sum_{s=1}^M \sum_{\mathbf{j} \in \mathbf{Z}^d} d_{is}(\mathbf{j}) \phi_s(\cdot - \mathbf{j}) \quad \text{for all } 1 \le i \le N. \tag{3.4}$$

Then by (ii) of Theorem 3.2 and the continuous translates and $\mathcal{D}$-multiplication properties of X, it follows that $\phi_1, \ldots, \phi_M$ belong to X. Combining (3.2), (3.3) and (3.4), we get

$$\begin{aligned} f &= \sum_{\mathbf{k} \in \mathbf{Z}^d} h_0(\cdot - \mathbf{k}) f = \sum_{i=1}^N \sum_{\mathbf{k} \in \mathbf{Z}^d} c_i(\mathbf{k}) h_0(\cdot - \mathbf{k}) f_i(\cdot - \mathbf{k}) \\ &= \sum_{i=1}^N \sum_{s=1}^M \sum_{\mathbf{j} \in \mathbf{Z}^d} \sum_{\mathbf{k} \in \mathbf{Z}^d} c_i(\mathbf{k}) d_{is}(\mathbf{j}) \phi_s(\cdot - \mathbf{j} - \mathbf{k}) \in S(\phi_1, \ldots, \phi_M). \end{aligned}$$

□

For any bounded open set A of $\mathbf{R}^d$ and compactly supported distributions $g_1, \ldots, g_N$, let $S_0(g_1, \ldots, g_N)|_A$ be the space of all restrictions of distributions in $S_0(g_1, \ldots, g_N)$ on A, i.e.,

$$S_0(g_1, \ldots, g_N)|_A := \{g|_A : \ g \in S_0(g_1, \ldots, g_N)\},$$

where $g|_A$ is the restriction of g on A.

Denote the N copies of ℓ_0 by $\ell_0^{(N)}$. To prove Theorem 3.2, we need the decomposition (3.5) of a vector-valued distribution on some shift-invariant open set.

Lemma 3.3. *Let $g_1, \ldots, g_N$ be compactly supported distributions, and let A be a bounded open set of $\mathbf{R}^d$ such that its closure has disjoint integer shifts, i.e., $\bar{A} \cap (\bar{A} + \mathbf{j}) = \emptyset$ for any $\mathbf{j} \in \mathbf{Z}^d \backslash \{0\}$. Then there exist compactly supported distributions $\psi_s \in S_0(g_1, \ldots, g_N)$ and vector-valued sequences $\mathbf{E}_s = (\mathbf{e}_s(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d} \in \ell_0^{(N)}, 1 \leq s \leq \dim S_0(g_1, \ldots, g_N)|_A$, such that $\psi_s, 1 \leq s \leq \dim S_0(g_1, \ldots, g_N)|_A$, are linear independent on A, and such that*

$$(3.5) \quad (g_1, \ldots, g_N)^T = \sum_{s=1}^{\dim S_0(g_1, \ldots, g_N)|_A} \sum_{\mathbf{j} \in \mathbf{Z}^d} \mathbf{e}_s(\mathbf{j})(h\psi_s)(\cdot - \mathbf{j}) \quad \text{on} \quad \bigcup_{\mathbf{j} \in \mathbf{Z}^d} (A + \mathbf{j}),$$

where $h \in \mathcal{D}$ is chosen so that $h = 1$ on A and $h = 0$ on $\bigcup_{\mathbf{j} \in \mathbf{Z}^d \backslash \{0\}} (A + \mathbf{j})$.

Proof. Clearly $S_0(g_1, \ldots, g_N)|_A$ is a finite dimensional linear space. Let $\psi_s, 1 \leq s \leq \dim S_0(g_1, \ldots, g_N)|_A$, be compactly supported distributions in $S_0(g_1, \ldots, g_N)$ chosen so that $\{\psi_s|_A : 1 \leq s \leq \dim S_0(g_1, \ldots, g_N)|_A\}$ is a basis of $S_0(g_1, \ldots, g_N)|_A$. Then $\psi_s, 1 \leq s \leq \dim S_0(g_1, \ldots, g_N)|_A$, are linearly independent on A.

By the shift-invariance of $S_0(g_1, \ldots, g_N)$, the restriction of $g_i(\cdot + \mathbf{j}), 1 \leq i \leq N$, on A belongs to $S_0(g_1, \ldots, g_N)|_A$ for any $\mathbf{j} \in \mathbf{Z}^d$. Thus for any $\mathbf{j} \in \mathbf{Z}^d$, the restriction of $g_1(\cdot + \mathbf{j}), \ldots, g_N(\cdot + \mathbf{j})$ on A is a finite linear combination of the ψ_s, i.e.,

$$(3.6) \qquad (g_1(\cdot + \mathbf{j}), \ldots, g_N(\cdot + \mathbf{j}))^T = \sum_{s=1}^{\dim S_0(g_1, \ldots, g_N)|_A} \mathbf{e}_s(\mathbf{j})\psi_s \quad \text{on} \quad A$$

for some $N \times 1$ vectors $\mathbf{e}_s(\mathbf{j}), 1 \leq s \leq \dim S_0(g_1, \ldots, g_N)|_A$. Recall that $\{\psi_s, 1 \leq s \leq \dim S_0(g_1, \ldots, g_N)|_A\}$ is a basis of $S_0(g_1, \ldots, g_N)|_A$, and that $g_1, \ldots, g_N$ have compact support. Hence $\mathbf{e}_s(\mathbf{j}) = 0$ for all $\mathbf{j} \in \mathbf{Z}^d$ with sufficiently large $|\mathbf{j}|$. This proves that $(\mathbf{e}_i(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d} \in \ell_0^{(N)}$.

Finally we prove (3.5) for the distributions ψ_s and sequences $(\mathbf{e}_s(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d}$ chosen above. Let $h \in \mathcal{D}$ be chosen so that $h = 1$ on A and the support of h is disjoint from $\bigcup_{\mathbf{j} \in \mathbf{Z}^d \backslash \{0\}} (A + \mathbf{j})$. The existence of such a function h follows from the assumption on the open set A. Therefore it follows from (3.6) that for any $\mathbf{j} \in \mathbf{Z}^d$,

$$\begin{aligned}
(g_1, \ldots, g_N)^T &= \sum_{s=1}^{\dim S_0(g_1, \ldots, g_N)|_A} \mathbf{e}_s(\mathbf{j})\psi_s(\cdot - \mathbf{j}) \\
&= \sum_{s=1}^{\dim S_0(g_1, \ldots, g_N)|_A} \mathbf{e}_s(\mathbf{j})(h\psi_s)(\cdot - \mathbf{j}) \\
&= \sum_{s=1}^{\dim S_0(g_1, \ldots, g_N)|_A} \sum_{\mathbf{j}' \in \mathbf{Z}^d} \mathbf{e}_s(\mathbf{j}')(h\psi_s)(\cdot - \mathbf{j}') \quad \text{on } A + \mathbf{j},
\end{aligned}$$

where we have used the fact that $h=1$ on A and that $h\psi_s(\cdot-\mathbf{j}')=0$ on $A+\mathbf{j}$ for any $\mathbf{j}\neq\mathbf{j}'$. Then (3.5) follows. $\square$

We remark that the representation (3.6) of a distribution is true for any open set A. Such a representation has been used in studying the shift-invariant space $S(g_1,\ldots,g_N)$ generated by $g_1,\ldots,g_N$ and in studying linear independence of shifts of $g_1,\ldots,g_N$ (see for instance [11, 12, 13]).

Proof of Theorem 3.2. Let $A_k, 1\le k\le K$, be bounded open sets such that

$$\bigcup_{k=1}^{K}\bigcup_{\mathbf{j}\in\mathbf{Z}^d}(A_k+\mathbf{j})=\mathbf{R}^d$$

and the closure of A_k has disjoint integer shifts for any $1\le k\le K$, i.e.,

$$\bar{A}_k\cap(\bar{A}_k+\mathbf{j})=\emptyset\quad\text{for any } \mathbf{j}\in\mathbf{Z}^d\backslash\{0\}.$$

For instance, $A_k=(1/5,4/5)^d+\mathbf{x}_k$ and $\{\mathbf{x}_k, 1\le k\le 2^d\}=-\{0,1/2\}^d\subset\mathbf{R}^d$ is such a family of bounded open sets. Let $h_k\in\mathcal{D}, 1\le k\le K$, be chosen so that $h_k=1$ on A_k and $h_k=0$ on $\bigcup_{\mathbf{j}\in\mathbf{Z}^d\backslash\{0\}}(A_k+\mathbf{j})$.

Set $g_{0,i}=g_i, 1\le i\le N$. Inductively for $1\le k\le K$, we let $\psi_{k,s}$ and $\{\mathbf{e}_{k,s}(\mathbf{j})\}_{\mathbf{j}\in\mathbf{Z}^d}, 1\le s\le N_k$, be the compactly supported distributions ψ_s and finitely supported sequences $(\mathbf{e}_s(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d}$ in Lemma 3.3 with g_i replaced by $g_{k-1,i}, 1\le i\le N$, and A by A_k, and we define

$$(g_{k,1},\ldots,g_{k,N})^T=(g_{k-1,1},\ldots,g_{k-1,N})^T-\sum_{s=1}^{N_k}\sum_{\mathbf{j}\in\mathbf{Z}^d}\mathbf{e}_{k,s}(\mathbf{j})(h_k\psi_{k,s})(\cdot-\mathbf{j}),\tag{3.7}$$

where we denote $N_k=\dim S_0(g_{k-1,1},\ldots,g_{k-1,N})|_{A_k}$.

By Lemma 3.3, for any $1\le k\le K$, we have

$$\psi_{k,s}\in S_0(g_{k-1,1},\ldots,g_{k-1,N})\quad\text{for all } 1\le s\le N_k,\tag{3.8}$$

$$\psi_{k,s}, 1\le s\le N_k, \text{ are linearly independent on }\quad A_k,\tag{3.9}$$

and

$$(g_{k,1},\ldots,g_{k,N})^T=0\quad\text{on}\quad\bigcup_{\mathbf{j}\in\mathbf{Z}^d}(A_k+\mathbf{j}).\tag{3.10}$$

Set

$$\phi_{k,s}=h_k\psi_{k,s},\quad 1\le s\le N_k, 1\le k\le K.\tag{3.11}$$

It remains to show that $\phi_{k,s}, 1\le s\le N_k, 1\le k\le K$, are the distributions having linearly independent shifts that we are seeking in Theorem 3.2. By (3.7) and (3.8), for any $1\le k\le K$, $g_{k,1}-g_{k-1,1},\ldots,g_{k,N}-g_{k-1,N}$ are finite linear combinations of $h_k(\cdot-\mathbf{j}')g_{k-1,i}(\cdot-\mathbf{j})$, where $1\le i\le N$ and $\mathbf{j},\mathbf{j}'\in\mathbf{Z}^d$. This implies that for any $0\le k\le K$, $g_{k,1},\ldots,g_{k,N}$ are finite linear combinations of $g_i(\cdot-\mathbf{j})$ and $h_{k,l}g_i(\cdot-\mathbf{j})$, where $h_{k,l}\in\mathcal{D}, 1\le i\le N$, and $\mathbf{j}\in\mathbf{Z}^d$. The same is true for $\phi_{k,s}$ since $\psi_{k,s}\in S_0(g_{k-1,1},\ldots,g_{k-1,N})$ and $h_k\in\mathcal{D}$. This proves assertion (ii) of Theorem 3.2.

From (3.7), (3.8), and (3.10), we have

$$(g_{k,1},\ldots,g_{k,N})^T=0\quad\text{on}\quad\bigcup_{k'=1}^{k}\bigcup_{\mathbf{j}\in\mathbf{Z}^d}(A_{k'}+\mathbf{j}).\tag{3.12}$$

Taking $k = K$ in (3.12) and using the assumption $\bigcup_{k=1}^{K} \bigcup_{\mathbf{j}\in\mathbf{Z}^d}(A_k + \mathbf{j}) = \mathbf{R}^d$, we get

$$(g_{K,1}, \ldots, g_{K,N})^T \equiv 0. \tag{3.13}$$

Assertion (iii) follows from (3.7), (3.11), (3.13) and $g_{0,i} = g_i, 1 \le i \le N$.

Finally, we prove assertion (i), i.e., $\phi_{k,s}, 1 \le s \le N_k, 1 \le k \le K$, have linearly independent shifts. Let $C_{k,s} = (c_{k,s}(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}^d}, 1 \le s \le N_k, 1 \le k \le K$, be sequences such that

$$\sum_{k=1}^{K} \sum_{s=1}^{N_k} \sum_{\mathbf{j}\in\mathbf{Z}^d} c_{k,s}(\mathbf{j})\phi_{k,s}(\cdot - \mathbf{j}) \equiv 0 \quad \text{on} \quad \mathbf{R}^d. \tag{3.14}$$

Then it suffices to prove

$$C_{k,s} = 0 \quad \forall\ 1 \le s \le N_k \quad \text{and} \quad 1 \le k \le K. \tag{3.15}$$

We prove (3.15) by contradiction. Assume that,

$$C_{k_0,s_0} \ne 0 \quad \text{for some} \quad 1 \le s_0 \le N_{k_0} \quad \text{and} \quad 1 \le k_0 \le K, \tag{3.16}$$

and

$$C_{k,s} = 0 \quad \text{for all} \quad 1 \le s \le N_k \quad \text{and} \quad 1 \le k < k_0. \tag{3.17}$$

Set $F_k = \sum_{s=1}^{N_k} \sum_{\mathbf{j}\in\mathbf{Z}^d} c_{k,s}(\mathbf{j})\phi_{k,s}(\cdot - \mathbf{j})$ for any $1 \le k \le K$. Using (3.17), we get that

$$F_k \equiv 0 \quad \forall\ 1 \le k < k_0. \tag{3.18}$$

Moreover, using (3.8), (3.11), and (3.12), we conclude that

$$F_k = 0 \quad \text{on} \quad \bigcup_{\mathbf{j}\in\mathbf{Z}^d}(A_{k_0} + \mathbf{j}) \quad \text{for all } k_0 < k \le K. \tag{3.19}$$

Combining (3.14), (3.18), and (3.19), we obtain

$$F_{k_0} = 0 \quad \text{on} \quad \bigcup_{\mathbf{j}\in\mathbf{Z}^d}(A_{k_0} + \mathbf{j}). \tag{3.20}$$

Recall that $\phi_{k_0,s} = 0$ on $A_{k_0} + \mathbf{j}$ for any $\mathbf{j} \in \mathbf{Z}^d\backslash\{0\}$, and $\phi_{k_0,s} = \psi_{k_0,s}$ on A_{k_0} by (3.11). Using (3.20), it follows that

$$F_{k_0} = \sum_{i=1}^{N_{k_0}} c_{k_0,s}(\mathbf{j})\psi_{k_0,s}(\cdot - \mathbf{j}) \quad \text{on} \quad A_{k_0} + \mathbf{j}$$

for any $\mathbf{j} \in \mathbf{Z}^d$, which together with (3.9) leads to $c_{k_0,s}(\mathbf{j}) = 0$ for all $1 \le i \le N_{k_0}$ and $\mathbf{j} \in \mathbf{Z}^d$. Thus C_{k_0,s_0} is a zero sequence, which contradicts (3.16). This proves (3.15), and hence assertion (i) of the theorem. □

4. Hölder continuous and Sobolev shift-invariant spaces

For any locally finite dimensional shift-invariant subspace V of a topological linear space X, by Theorem 3.1, we can find a shift-invariant space $S(\phi_1, \ldots, \phi_M)$ such that $V \subset S(\phi_1, \ldots, \phi_M) \cap X$, where $\phi_1, \ldots, \phi_M \in X$ have compact support and linearly independent shifts. The algebraic structure of the space $S(\phi_1, \ldots, \phi_M) \cap X$

is not clear even though $S(\phi_1, \dots, \phi_M)$ has a very simple algebraic structure. For the case $X = L^p$ and $\phi_1, \dots, \phi_M$ in Theorem 2.1, we have

$$S(\phi_1, \dots, \phi_M) \cap L^p = V_p(\phi_1, \dots, \phi_M) \tag{4.1}$$

([11, Theorem 7.2]). In particular, the equality

$$S(\phi_1, \dots, \phi_M) \cap L^p = S_p(\phi_1, \dots, \phi_M)$$

was proved in [11]. As a consequence, assertion (4.1) holds since $V_p(\phi_1, \dots, \phi_M) = S_p(\phi_1, \dots, \phi_M)$ for all $1 < p < \infty$ under the assumption that $\phi_1, \dots, \phi_M$ have stable shifts. This inspires us to discuss the algebraic structure of $S(\phi_1, \dots, \phi_M) \cap X$ for the Hölder continuous functions $X = C^\alpha$, and the fractional Sobolev space $X = L^{p,\gamma}$. For a linear topological space X of distributions and $\phi_1, \dots, \phi_M \in X$ such that

$$\sum_{s=1}^{M} \sum_{\mathbf{j} \in \mathbf{Z}^d} d_s(\mathbf{j}) \phi_s(\cdot - \mathbf{j}) \in X \quad \text{for all } (d_s(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d} \in \ell^p \text{ and } 1 \le s \le M,$$

let

$$V_p(\phi_1, \dots, \phi_M) = \Big\{ \sum_{s=1}^{M} \sum_{\mathbf{j} \in \mathbf{Z}^d} d_s(\mathbf{j}) \phi_s(\cdot - \mathbf{j}) : \quad (d_s(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d} \in \ell^p, 1 \le s \le M \Big\}.$$

We shall prove the following results: Let $\phi_1, \dots, \phi_M$ be compactly supported distributions in X and have linearly independent shifts. If $X = C^\alpha$ for some $\alpha \ge 0$, then

$$S(\phi_1, \dots, \phi_M) \cap X = V_\infty(\phi_1, \cdots, \phi_M).$$

If instead $X = L^{p,\gamma}$ for some $1 < p < \infty$ and $-\infty < \gamma < \infty$, then

$$S(\phi_1, \dots, \phi_M) \cap X = V_p(\phi_1, \cdots, \phi_M).$$

4.1. **Hölder continuous space.** Take a nonnegative real number α, and let α_0 be the greatest integer smaller than or equal to α and $\delta = \alpha - \alpha_0$. Let C^α be the space of all continuous functions f on $\mathbf{R}^d$ such that f has continuous $\mathbf{k}$-th derivative $D^{\mathbf{k}} f$ for any $\mathbf{k}$ satisfying $\sum_{i=1}^d \mathbf{k}_i \le \alpha_0$, and such that $\|f\|_{C^\alpha} < \infty$, where

$$\|f\|_{C^\alpha} := \sum_{\sum_{i=1}^d \mathbf{k}_i \le \alpha_0} \|D^{\mathbf{k}} f\|_\infty + \sup_{\mathbf{x}, \mathbf{y} \in \mathbf{R}^d, \sum_{i=1}^d \mathbf{k}_i = \alpha_0} |D^{\mathbf{k}} f(\mathbf{x}) - D^{\mathbf{k}} f(\mathbf{y})| / |\mathbf{x} - \mathbf{y}|^\delta,$$

and $\mathbf{k} = (k_1, \dots, k_d) \in \mathbf{Z}^d$ satisfies $k_i \ge 0$ for $i = 1, \dots, d$. We say that compactly supported distributions $\phi_1, \dots, \phi_M$ have *stable shifts* if the restriction of the semi-convolution map S in (3.1) to the space of all vector-valued bounded sequences is one-to-one.

Theorem 4.1. *Let $\alpha \ge 0$, and let $\phi_1, \dots, \phi_M$ be compactly supported distributions in C^α and have stable shifts. Then*

$$S(\phi_1, \dots, \phi_M) \cap C^\alpha = V_\infty(\phi_1, \dots, \phi_M).$$

Proof of Theorem 4.1. It is easy to show that any function in $V_\infty(\phi_1, \dots, \phi_M)$ belongs to C^α. Hence

$$V_\infty(\phi_1, \dots, \phi_M) \subset S(\phi_1, \dots, \phi_M) \cap C^\alpha. \tag{4.2}$$

From $C^\alpha \subset L^\infty$ and Theorem 7.2 in [11], it follows that

$$S(\phi_1, \dots, \phi_M) \cap C^\alpha \subset S(\phi_1, \dots, \phi_M) \cap L^\infty = V_\infty(\phi_1, \dots, \phi_M). \tag{4.3}$$

Combining (4.2) and (4.3) leads to the desired assertion. $\square$

By the definition of Hölder continuous space, C^α has continuous translates and $\mathcal{D}$-multiplication. Hence, by Theorems 3.1 and 4.1, we have the following result for a locally finite dimensional shift-invariant subspace of C^α.

Corollary 4.2. *Let $\alpha \geq 0$, and let V be a locally finite dimensional shift-invariant subspace of C^α. Then there exist compactly supported functions $\phi_1, \dots, \phi_M \in C^\alpha$ such that $\phi_1, \dots, \phi_M$ have linearly independent shifts, and such that V is a subspace of $V_\infty(\phi_1, \dots, \phi_M)$.*

4.2. **Fractional Sobolev space.** For any real number γ, define the Bessel potential operator $\mathcal{J}_\gamma$ on the space of tempered distributions by

$$\mathcal{J}_\gamma f := \left(\widehat{f}(1 + |\cdot|^2)^{\gamma/2}\right)^\vee.$$

Here $f^\vee$ is the inverse Fourier transform of a tempered distribution f. For any $1 \leq p < \infty$ and real number γ, let $L^{p,\gamma}$ denote the fractional Sobolev space that consists of all distributions f with $\|f\|_{p,\gamma} < \infty$, where $\|f\|_{p,\gamma} =: \|\mathcal{J}_\gamma f\|_p$. Obviously, $L^{p,0} = L^p$ and for $d = 1$, the δ distribution belongs to $L^{2,\gamma}$ for any $\gamma < -1/2$. It is known that $L^{p,\gamma_2} \subset L^{p,\gamma_1}$ if $\gamma_1 < \gamma_2$. If γ is a nonnegative integer, and $1 < p < \infty$, then

$$L^{p,\gamma} = \{f : D^{\mathbf{k}} f \in L^p \quad \text{for all } |\mathbf{k}| \leq \gamma\}. \tag{4.4}$$

Let $\triangle = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ denote the Laplacian. Results in [16, 18] show that for any $k \geq -\gamma/2$, there exists a positive constant C such that

$$\|(1 - \triangle)^{-k} g\|_p \leq C\|g\|_{p,\gamma}, \quad \forall\, g \in L^{p,\gamma}. \tag{4.5}$$

Theorem 4.3. *Let $1 < p < \infty$, $-\infty < \gamma < \infty$, and let $\phi_1, \dots, \phi_M$ be compactly supported distributions in $L^{p,\gamma}$ and have linearly independent shifts. Then*

$$S(\phi_1, \dots, \phi_M) \cap L^{p,\gamma} = V_p(\phi_1, \dots, \phi_M).$$

By direct computation, $\|f(\cdot - \mathbf{y})\|_{p,\gamma} = \|f\|_{p,\gamma}$ for any $\mathbf{y} \in \mathbf{R}^d$. By the classical multiplier theorem ([16], p. 96) and the fact that $\widehat{hf} = \int_{\mathbf{R}^d} \hat{h}(\eta)\hat{f}(\cdot - \eta)d\eta$, we obtain

$$\begin{aligned} \|hf\|_{p,\gamma} &\leq \int_{\mathbf{R}^d} |\hat{h}(\eta)| \times \left\| \left(m_\eta \hat{f}(1 + |\cdot|^2)^{\gamma/2}\right)^\vee \right\|_p d\eta \\ &\leq C_1 \|f\|_{p,\gamma} \int_{\mathbf{R}^d} |\hat{h}(\eta)|(1 + |\eta|^2)^{(|\gamma|+d)/2} d\eta \leq C_2 \|f\|_{p,\gamma}, \end{aligned}$$

for all $h \in \mathcal{D}$, where $m_\eta(\xi) = (1+|\xi|^2)^{-\gamma/2}(1+|\xi+\eta|^2)^{\gamma/2}$. Therefore the fractional Sobolev space $L^{p,\gamma}$ has continuous translates and $\mathcal{D}$-multiplication for any $1 < p < \infty$ and $-\infty < \gamma < \infty$. This together with Theorems 3.1 and 4.3 leads to the following result:

Corollary 4.4. *Let $1 < p < \infty, -\infty < \gamma < \infty$, and let V be a locally finite dimensional shift-invariant subspace of $L^{p,\gamma}$. Then there exist compactly supported distributions $\phi_1, \dots, \phi_M \in L^{p,\gamma}$ such that $\phi_1, \dots, \phi_M$ have linearly independent shifts, and V is a subspace of $V_p(\phi_1, \dots, \phi_M)$.*

Let $1 < p < \infty$ and $-\infty < \gamma < \infty$. Since the space $L^{p,\gamma}$ has continuous translates and $\mathcal{D}$-multiplication, by Theorem 3.2, any compactly supported distribution in $L^{p,\gamma}$ can be decomposed as a finite linear combination of the shifts of some compactly supported distributions in $L^{p,\gamma}$ having linearly independent shifts. Therefore, using the above decomposition for $L^{p,\gamma}$ distributions instead of the decomposition for L^p functions in [11] (or instead of (2.3)), the assertion in Theorem 4.3 instead of an estimate in [11] similar to (2.4), and the ideas about compatibility of linear system in the proof of Theorem 7.2 in [11], we have the following slight generalization of Theorem 4.3.

Theorem 4.5. *Let $1 < p < \infty$, $-\infty < \gamma < \infty$, and let $\phi_1, \dots, \phi_M$ be compactly supported distributions in $L^{p,\gamma}$ and have stable shifts. Then*

$$S(\phi_1, \dots, \phi_M) \cap L^{p,\gamma} = V_p(\phi_1, \dots, \phi_M).$$

For the special case $\gamma = 0$, the assertion of Theorem 4.5 was proved in [11, Theorem 7.2]. To prove Theorem 4.3, we need the inclusion

$$S(\phi_1, \dots, \phi_M) \cap L^{p,\gamma} \subset V_p(\phi_1, \dots, \phi_M) \tag{4.6}$$

and the estimate

$$\sum_{\mathbf{j}\in\mathbf{Z}^d} d(\mathbf{j})\phi(\cdot - \mathbf{j}) \in L^{p,\gamma} \tag{4.7}$$

for any sequence $D = (d(\mathbf{j}))_{\mathbf{j}\in\mathbf{Z}} \in \ell^p$ and any compactly supported distribution $\phi \in L^{p,\gamma}$. The proof of (4.6) is non-trivial when γ is a negative number (when $\gamma \ge 0$, (4.6) follows easily from $L^{p,\gamma} \subset L^p$ and a consequence of [11, Theorem 7.2]). The estimate (4.7) is also non-trivial when γ is a real number other than a nonnegative integer (when γ is a nonnegative integer, (4.7) follows easily from (4.4) and the fact that $D^{\mathbf{k}}\phi_1, \dots, D^{\mathbf{k}}\phi_M$ have compact support for any $|\mathbf{k}| \le \gamma$).

Denote the pairing action between a distribution in $\mathcal{D}'$ and a function in $\mathcal{D}$ by $\langle \cdot, \cdot \rangle$, and the convolution $f*g$ between f in $\mathcal{D}$ and g in $\mathcal{D}'$ by $f*g(x) = \langle f(x-\cdot), \overline{g}\rangle$. To prove Theorem 4.3, we need a result in [3, 20], and a characterization of the fractional Sobolev space $L^{p,\gamma}$ in the time domain ([18], p.15).

Lemma 4.6. *Let $\phi_1, \dots, \phi_M$ be compactly supported distributions having linearly independent shifts. Then there exist $\psi_1, \dots, \psi_M \in \mathcal{D}$ such that*

$$\langle \phi_s, \psi_t(\cdot - \mathbf{j})\rangle = \delta_{st}\delta_{\mathbf{j}0}.$$

Here δ is the Kronecker symbol.

Lemma 4.7. *Let $1 < p < \infty, -\infty < \gamma < \infty$, and let Ψ_{-1} and $\Psi_0 \in \mathcal{D}$ be chosen so that $|\widehat{\Psi}_0(\xi)| \le C_1|\xi|^{|\gamma|+1}$ and*

$$|\widehat{\Psi}_{-1}(\xi)| + \sum_{l\ge 0} |\widehat{\Psi}_0(2^{-l}\xi)| \ge C_1 \quad \textit{for all} \quad \xi \in \mathbf{R}^d,$$

where C_1 is a positive constant independent of ξ. Set $\Psi_l = 2^{ld}\Psi_0(2^l\cdot)$ for $l \ge 0$. Then there exists $C > 0$ such that

$$C^{-1}\|f\|_{p,\gamma} \le \Big\| \Big(\sum_{l\ge -1} 2^{2l\gamma}|\Psi_l * f|^2\Big)^{1/2}\Big\|_p \le C\|f\|_{p,\gamma}.$$

Proof of Theorem 4.3. Let g be any distribution in $S(\phi_1, \dots, \phi_M) \cap L^{p,\gamma}$. Then $g \in L^{p,\gamma}$ and $g = \sum_{s=1}^M \sum_{\mathbf{j} \in \mathbf{Z}^d} d_s(\mathbf{j}) \phi_s(\cdot - \mathbf{j})$ for some sequences $D_s = (d_s(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d}, 1 \le s \le M$. Let γ_0 be the smallest nonnegative integer bigger than $-\gamma/2$, and let $\psi_1, \dots, \psi_M \in \mathcal{D}$ be the functions in Lemma 4.6. Then, by Lemma 4.6,

$$d_s(\mathbf{j}) = \langle g, \psi_s(\cdot - \mathbf{j})\rangle = \langle (1-\triangle)^{-\gamma_0} g, (1-\triangle)^{\gamma_0} \psi_s(\cdot - \mathbf{j})\rangle$$

for all $\mathbf{j} \in \mathbf{Z}^d$ and $1 \le s \le M$. This leads to

$$\Big(\sum_{\mathbf{j} \in \mathbf{Z}^d} |d_s(\mathbf{j})|^p\Big)^{1/p} \le \|(1-\triangle)^{-\gamma_0} g\|_p \|(1-\triangle)^{\gamma_0} \psi_s\|_{\mathcal{L}^\infty} < \infty \quad \forall\ 1 \le s \le M,$$

where we have used (4.5) and the fact that $(1-\triangle)^{\gamma_0}\psi_s \in \mathcal{D}$ for all $1 \le s \le M$. Thus, $D_s \in \ell^p, 1 \le s \le M$, and $g \in V_p(\phi_1, \dots, \phi_M)$. This proves (4.6).

Let $\phi \in L^{p,\gamma}$ and $D = (d(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d} \in \ell^p, 1 \le s \le M$. Let $\Psi_l, l \ge -1$, be as in Lemma 4.7. Then $\Psi_l * \phi$ are supported in a compact set K independent of $l \ge -1$. Therefore by Lemma 4.7, and using Hölder's inequality,

$$\begin{aligned}
\Big\|\sum_{\mathbf{j} \in \mathbf{Z}^d} d(\mathbf{j})\phi(\cdot - \mathbf{j})\Big\|_{p,\gamma} &\le C_1 \Big\|\Big(\sum_{l \ge -1} 2^{2l\gamma}\Big|\sum_{\mathbf{j} \in \mathbf{Z}^d} d(\mathbf{j}) \Psi_l * \phi(\cdot - \mathbf{j})\Big|^2\Big)^{1/2}\Big\|_p \\
&\le C_2 \Big\|\sum_{\mathbf{j} \in \mathbf{Z}^d} |d(\mathbf{j})| \Big(\sum_{l \ge -1} 2^{2l\gamma}\Big|\Psi_l * \phi(\cdot - \mathbf{j})\Big|^2\Big)^{1/2}\Big\|_p \\
&\le C_3 \Big(\sum_{\mathbf{j} \in \mathbf{Z}^d} |d(\mathbf{j})|^p\Big)^{1/p} \times \Big\|\Big(\sum_{l \ge -1} 2^{2l\gamma} |\Psi_l * \phi(\cdot)|^2\Big)^{1/2}\Big\|_p \\
&\le C_4 \|\phi\|_{p,\gamma} \|(d(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d}\|_{\ell^p},
\end{aligned}$$

where C_1, C_2, C_3, C_4 are positive constants independent of the sequence $(d(\mathbf{j}))_{\mathbf{j} \in \mathbf{Z}^d} \in \ell^p$. This proves (4.7), and hence

$$V_p(\phi_1, \dots, \phi_M) \subset S(\phi_1, \dots, \phi_M) \cap L^{p,\gamma}. \tag{4.8}$$

Combining (4.6) and (4.8) leads to the desired result. □

Remark. A shift-invariant space V is said to be *injectable* if there exist finitely many compactly supported distributions such that they have linearly independent shifts, and such that V is a subspace of the space spanned by their shifts. In particular, as a consequence of Theorem 3.1, every locally finite dimensional shift-invariant space of distributions is injectable. The concept of injectability was recently introduced by Ron in [15], and we became aware of this definition after we submitted this manuscript. In his recent preprint [15], Ron states that "At the time this article is written, ... we do not know of a general technique for smoothness-preserving injection." He also states that " ... local FSI spaces that are generated by compactly supported *functions* are injectable as well. It is safe to conjecture that the results here are valid for spaces generated by compactly supported distributions, and it would be nice to find a neat way to close this small gap." But one of the main results of our paper is to give an injection that preserves smoothness. Moreover, this smoothness-preserving injection is also valid in shift-invariant spaces of compactly supported distributions, and even in locally finite dimensional shift-invariant spaces.

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