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GLOBAL EXISTENCE FROM SINGLE-COMPONENT L_p ESTIMATES IN A SEMILINEAR REACTION-DIFFUSION SYSTEM

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Abstract. For a system of two reaction-diffusion equations coupled by power nonlinearities, we prove that an L_p bound on a single component for suitable p is enough to guarantee global existence. Also we provide a strong indication that our condition on p is the best possible. Moreover, this continuation result is in contrast with the corresponding necessary and sufficient conditions for local existence obtained earlier by the authors.

1. Introduction and main result

Let us consider the following reaction-diffusion system:

$$
(1.1)\qquad \begin{cases} \dfrac{\partial u_1}{\partial t} - \Delta u_1 = |u_2|^{p_1-1}u_2, & x \in \Omega, \quad 0 < t < T, \\ \dfrac{\partial u_2}{\partial t} - \Delta u_2 = |u_1|^{p_2-1}u_1, & x \in \Omega, \quad 0 < t < T, \\ \quad u_1(x,t) = u_2(x,t) = 0, & x \in \partial\Omega, \quad 0 < t < T, \\ \quad u_1(x,0) = \phi_1(x), \quad u_2(x,0) = \phi_2(x), & x \in \Omega, \end{cases}
$$

where Ω is a smoothly bounded domain of $\mathbb{R}^N$, $p_1,\, p_2 \geq 1$, $p_1 p_2 > 1$ and, e.g., $\phi_1,\, \phi_2 \in L_\infty(\Omega)$. We denote by $T = T(\phi_1,\, \phi_2) \in (0, \infty]$ the maximal existence time of (u_1, u_2).

It is well-known that if (u_1, u_2) does not exist globally in time, then *both* components have to blow up in L_∞ norm, in the sense that

$$
\limsup_{t\to T} |u_i(t)|_\infty = \infty, \quad i = 1,\, 2
$$

(where $|\cdot|_r$ denotes the norm in $L_r(\Omega)$). Indeed if u_2, say, was uniformly bounded on $[0, T)$, the first equation would then imply a uniform bound for u_1 on $[0, T)$ and it would be possible to extend the solution after T by standard arguments.

In this paper, we address the following question: given $i \in \{1,\, 2\}$, for what values of $r_i \in [1, \infty)$ does a bound on the single component u_i in L_{r_i} imply global existence?

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We shall prove the following

Theorem 1. *Let $N \geq 2$ and assume that there exists $i \in \{1, 2\}$ such that*

$$r_i > \frac{N(p_1p_2-1)}{2(p_i+1)} \tag{1.2}$$

and $\sup_{(0,T)} |u_i(t)|_{r_i} < \infty$. *Then $T = \infty$.*

There is a strong indication that the values of r_i in the theorem above are the best possible (except perhaps for the equality case). Indeed it is known (see [4], [6], [3], [5], [18]) that for large classes of initial data, the blow-up rate of nonglobal solutions of (1.1) is of the order

$$|u_i(t)|_\infty \sim (T-t)^{-\alpha_i}, \quad \text{where } \alpha_i = \frac{p_i+1}{p_1p_2-1}, \quad i = 1, 2.$$

In analogy to known results from the scalar case (see [10], [16], [17]), the parabolicity of the problem thus suggests that, in case of blow-up at a single-point x_0 (see [9]), the blow-up profile in the x–variable should be generically of the order

$$u_i(x,T) \sim |x-x_0|^{-2\alpha_i}, \quad i = 1,\, 2$$

(up to a power ε), so that

$$|u_i(\cdot,T)|_{r_i} < \infty \quad \text{if} \quad r_i < \frac{N(p_1p_2-1)}{2(p_i+1)}.$$

The present work is in some respect a sequel to our previous paper [14], where we studied local existence/nonexistence and continuation properties for parabolic systems (including (1.1)) in product spaces $L_{r_1} \times L_{r_2} := L_{r_1}(\Omega) \times L_{r_2}(\Omega)$. It was proved there that for the system (1.1), there is no coincidence between the properties of continuation and of local existence in $L_{r_1} \times L_{r_2}$.

Namely, we showed that local existence in $L_{r_1} \times L_{r_2}$ ($1 < r_1,\, r_2 < \infty$) holds for system (1.1) if and only if

$$\mathcal{P}_1 := N\Big(\frac{p_1}{r_2} - \frac{1}{r_1}\Big) \leq 2 \quad \text{and} \quad \mathcal{P}_2 := N\Big(\frac{p_2}{r_1} - \frac{1}{r_2}\Big) \leq 2,$$

whereas the continuation property is true under the weaker condition

$$\mathcal{P}_1 + \mathcal{P}_2 = \frac{N(p_1-1)}{r_2} + \frac{N(p_2-1)}{r_1} < 4 \tag{1.3}$$

(plus an extra condition if $N = 1$). As we remarked in [14], this stands in sharp contrast with the case of the corresponding scalar equation $u_t - \Delta u = |u|^{p-1}u$, for which both properties are known to be essentially equivalent. It is easily seen that the assumption of Theorem 1 is much weaker than (1.3), so that the result of the present paper strongly improves the continuation results of [14] for system (1.1).

While the strategy of [14] was the use of suitable interpolation spaces to estimate both components at the same time, the basic idea of the present proof is to carry out a bootstrap argument in Lebesgue norms, *alternatively on each component.* This works more easily in dimensions $N \geq 4$. In lower dimensions, some technical complications arise, and one also has to use intermediate Sobolev spaces, together with Sobolev and interpolation inequalities.

Finally, to give some references related to our work, let us mention that system (1.1) has received a lot of attention from the point of view of blowup in the recent past (see, e.g. [9], [8], [4], [6], [3], [5], [18], [7] and the references therein). On the other hand, the question of boundedness in a strong (L_∞ or Sobolev) norm, in terms of boundedness of a weaker norm, was studied for systems before, e.g., in [15], [1], [2], [12], [13]. The results there apply to large classes of parabolic systems, including systems with nonlinear boundary conditions. Also, negative results in this direction were obtained in [11], where blow-up solutions were constructed for some systems with dissipation of mass, for which the L^1 norm of the solution is controlled.

2. Proof of Theorem 1

Remark. In dimension $N = 1$, we do not know if Theorem 1 remains true. However, the same conclusion is valid under the stronger assumption that there exists $i \in \{1, 2\}$ such that

$$r_i > \frac{p_1 p_2 - 1}{p_i + 2}.$$

This follows from Cases 1 and 2 of the proof below.

Throughout this section, we denote $|u_i|_r$ for $\sup_{(0,T)} |u_i(t)|_r$ and $|u_i|_{s,r}$ for $\sup_{(0,T)} |u_i(t)|_{s,r}$, where $|u_i(t)|_{s,r}$ denotes the norm of $u_i(t)$ in the Sobolev space $W^s_r(\Omega)$. By C we denote positive constants which may vary from line to line. Also, for a given real number s, we denote by s^+ (resp. s^-) a real $> s$ (resp. $< s$), suitably close to s. Finally, we put $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$.

The solution of (1.1) satisfies the variation-of-constants formula

$$\begin{cases} u_1(t) = e^{t\Delta}\phi_1 + \displaystyle\int_0^t e^{(t-s)\Delta}|u_2(s)|^{p_1-1}u_2(s)\,ds, & 0 \le t < T, \\ u_2(t) = e^{t\Delta}\phi_2 + \displaystyle\int_0^t e^{(t-s)\Delta}|u_1(s)|^{p_2-1}u_1(s)\,ds, & 0 \le t < T, \end{cases} \tag{2.1}$$

where Δ denotes the Laplace operator on Ω with homogeneous Dirichlet boundary conditions. It is known that there exists $\omega > 0$ such that

$$\begin{cases} \|e^{t\Delta}\|_{\mathcal{L}(L_p,L_q)} \le Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\omega t}, & 1 \le p \le q \le \infty, \\ \|e^{t\Delta}\|_{\mathcal{L}(L_p,W^m_p)} \le Ct^{-\frac{m}{2}}e^{-\omega t}, & 1 < p < \infty,\ 0 \le m \le 2. \end{cases} \tag{2.2}$$

Denote $u := u_1$, $v := u_2$ and $r := r_2$. By symmetry, we may assume that

$$|v|_r \le C \tag{2.3}$$

and

$$r > \frac{N(p_1p_2 - 1)}{2(p_2 + 1)}. \tag{2.4}$$

Let s_1, k satisfy

$$s_1 \ge k \ge 1, \quad k \ge \frac{r}{p_1} \quad \text{and} \quad \frac{1}{k} - \frac{1}{s_1} < \frac{2}{N}. \tag{2.5}$$

By $(2.1)_1$ and the smoothing property $(2.2)_1$ with $p = k$, $q = s_1$ it follows that

$$|u|_{s_1} \le C(1 + ||v|^{p_1}|_k) = C(1 + |v|^{p_1}_{kp_1})$$

hence, by (2.3) and interpolation,

$$|u|_{s_1} \le C(1 + |v|^{p_1-(r/k)}_\infty). \tag{2.6}$$

We then consider separately the following cases.

Case 1. Assume $N \geq 4$, or ($N = 3$ and $p_2 \geq 2$), or ($N \leq 3$ and $r > p_1 - (1/p_2)$). If in addition

$$s_1 > \frac{Np_2}{2},$$

then $(2.1)_2$ and $(2.2)_1$ with $p = s_1/p_2$, $q = \infty$ imply that

$$|v|_\infty \leq C(1 + ||u|^{p_2}|_{s_1/p_2}) = C(1 + |u|_{s_1}^{p_2});$$

hence, by (2.6),

$$|v|_\infty \leq C(1 + |v|_\infty^{p_2(p_1-(r/k))}).$$

It follows that $|v|_\infty \leq C$ if

$$\frac{p_1p_2 - 1}{p_2 r} < \frac{1}{k}.$$

The sufficient conditions are thus

$$0 \vee \Big(\frac{1}{k} - \frac{2}{N}\Big) < \frac{1}{s_1} < \frac{2}{Np_2} \wedge \frac{1}{k} \tag{2.7}$$

and

$$\frac{p_1p_2 - 1}{p_2 r} < \frac{1}{k} < 1 \wedge \frac{p_1}{r}. \tag{2.8}$$

The condition (2.7) can be solved in s_1 if

$$\frac{1}{k} - \frac{2}{N} < \frac{2}{Np_2},$$

i.e.,

$$\frac{1}{k} < \frac{2(p_2+1)}{Np_2}.$$

Since

$$\frac{p_1p_2 - 1}{p_2 r} < \frac{p_1}{r},$$

it then suffices to satisfy

$$\frac{p_1p_2 - 1}{p_2 r} < \frac{2(p_2+1)}{Np_2} \quad \text{and} \quad \frac{p_1p_2 - 1}{p_2 r} < 1,$$

that is,

$$r > \frac{N(p_1p_2 - 1)}{2(p_2+1)} \quad \text{and} \quad r > p_1 - \frac{1}{p_2}.$$

Finally, note that

$$\frac{N(p_1p_2 - 1)}{2(p_2+1)} \geq p_1 - \frac{1}{p_2} \quad \text{if} \quad (N-2)p_2 \geq 2,$$

which is true for all $p_2 \geq 1$ if $N \geq 4$, and for $p_2 \geq 2$ if $N = 3$. Under the assumptions of Case 1, the hypothesis (2.4) thus implies the solvability of (2.7)-(2.8). Consequently, $|v|_\infty \leq C$, hence $|u|_\infty \leq C$ by $(2.1)_1$. It follows that $T = \infty$.

Case 2. Assume $r \le p_1 - (1/p_2)$ and either $N \le 2$ or ($N = 3$, $p_2 < 2$ and $rp_2 < 3$).

Choose $k = 1$, $s_1 \gg 1$ if $N \le 2$, $s_1 = (N/(N-2))^- = 3^-$ if $N = 3$. Since (2.5) is satisfied, (2.6) holds, i.e.

$$|u|_{s_1} \le C(1 + |v|_\infty^{p_1 - r}).$$

Since $s_1 > Np_2/2$ (which follows from $p_2 < 2$ if $N = 3$) we can choose $\theta = (Np_2/(2s_1))^+ \in (0,1)$. It follows from $(2.1)_2$, $(2.2)_2$ and $s_1 > p_2$ that

$$|v|_{2^-, s_1/p_2} \le C(1 + ||u|^{p_2}|_{s_1/p_2}) = C(1 + |u|_{s_1}^{p_2}).$$

Now using Sobolev ($2\theta s_1/p_2 > N$) and interpolation inequalities, together with (2.3), we get

$$\begin{aligned} |v|_\infty &\le C|v|_{2\theta, s_1/p_2} \le C|v|_{2^-, s_1/p_2}^{\theta^+} |v|_{s_1/p_2}^{1-\theta^+} \\ &\le C(1 + |u|_{s_1}^{\theta^+ p_2}) |v|_r^{(1-\theta^+)\eta} |v|_\infty^{(1-\theta^+)(1-\eta)} \le C(1 + |v|_\infty^{\theta^+ p_2 (p_1 - r) + (1-\theta^+)(1-\eta)}), \end{aligned}$$

where $\eta := rp_2/s_1 \in (0,1)$ (due to $rp_2 < 3$ if $N = 3$). The last inequality implies a bound for $|v|_\infty$ provided

$$1 > \theta^+ p_2 (p_1 - r) + (1 - \theta^+)(1 - \eta)$$

or

$$1 > \frac{Np_2^2}{2s_1}(p_1 - r) + \frac{2s_1 - Np_2}{2s_1} \frac{s_1 - rp_2}{s_1},$$

which is equivalent to

$$s_1[2r + Nrp_2 - N(p_1p_2 - 1)] > Nrp_2.$$

If $N = 3$ (and $s_1 = 3^-$) this condition reduces to (2.4). If $N \le 2$ (and s_1 is big), then we obtain the condition $2r + Nrp_2 - N(p_1p_2 - 1) > 0$ which is equivalent to (2.4) for $N = 2$.

Case 3. Assume $N = 3$, $p_2 < 2$, $rp_2 \ge 3$ and $r \le p_1 - (1/p_2)$.

Since $p_2 < 2$ we have $r > 3/2$. Choose

$$k = \frac{3rp_2}{3 + 2rp_2} \quad \text{and} \quad s_1 = \Big(\frac{3k}{3 - 2k}\Big)^-.$$

Then

$$k \in \Big[1, \frac{3}{2}\Big), \quad rp_2 = \frac{3k}{3 - 2k}, \quad \frac{1}{s_1} > \frac{1}{k} - \frac{2}{N} \quad \text{and} \quad \frac{3}{2} < r^- < \frac{s_1}{p_2} < r.$$

Choose also

$$\theta = \Big(\frac{3p_2}{2s_1}\Big)^+ = \Big(\frac{p_2(3 - 2k)}{2k}\Big)^+ \in (0,1).$$

Since (2.5) is satisfied, (2.6) holds and, similarly as in Case 2, we obtain

$$\begin{aligned} |v|_\infty &\le C|v|_{2\theta, s_1/p_2} \le C|v|_{2^-, s_1/p_2}^{\theta^+} |v|_{s_1/p_2}^{1-\theta^+} \\ &\le C(1 + |u|_{s_1}^{\theta^+ p_2}) |v|_r^{1-\theta^+} \le C(1 + |v|_\infty^{\theta^+ p_2 (p_1 - r/k)}). \end{aligned}$$

The last inequality implies a bound for $|v|_\infty$ provided

$$1 > \theta^+ p_2 \Big(p_1 - \frac{r}{k}\Big) \quad \text{or} \quad 1 > p_2^2 \Big(\frac{3}{2k} - 1\Big)\Big(p_1 - \frac{r}{k}\Big)$$

which is (due to $1/k = 1/(rp_2) + 2/3$) equivalent to (2.4) with $N = 3$.

The proof of Theorem 1 is complete.

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