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A COUNTEREXAMPLE CONCERNING WHITNEY REVERSIBLE PROPERTIES

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Abstract. Two properties of continua are shown which are strong Whitney reversible while not sequential strong Whitney reversible. This answers an old question of S. B. Nadler, Jr.

A *continuum* means a compact, connected metric space, and a *mapping* means a continuous transformation. By an n-od (for an integer $n > 2$) we understand a continuum X containing a subcontinuum Z such that $X \setminus Z$ has at least n components. Given a continuum X with a metric d, we denote by $C(X)$ the hyperspace of all subcontinua of X equipped with the Hausdorff metric (see e.g. [2, (0.1), p. 1 and (0.12), p. 10]). A *Whitney map* for $C(X)$ is a mapping

$$\mu : C(X) \to [0, \infty)$$

such that:

(0.1) $\mu(A) < \mu(B)$ for every two $A, B \in C(X)$ such that $A \subset B$ and $A \neq B$;
(0.2) $\mu(A) = 0$ if and only if $A \in F_1(X)$.

For the concept and existence of a Whitney map see [1, Section 13, pp. 105-110]. For each $t \in [0, \mu(X)]$ the preimage $\mu^{-1}(t)$ is called a *Whitney level*. It is known that each Whitney level is a continuum; see [1, p. 159].

Let $\mathcal{P}$ be a topological property. We write $X \in \mathcal{P}$ to denote that a space X has the property $\mathcal{P}$. A property $\mathcal{P}$ is said to be:

— a *strong Whitney-reversible property* (briefly (*sWrp*)) provided that for each continuum X the following implication holds:

(0.3) if there is a Whitney map $\mu : C(X) \to [0, \infty)$ such that for all $t \in (0, \mu(X))$ we have $\mu^{-1}(t) \in \mathcal{P}$, then $X \in \mathcal{P}$;

— a *sequential strong Whitney-reversible property* (briefly (*ssWrp*)) provided that for each continuum X the following implication holds:

(0.4) if there are a Whitney map $\mu : C(X) \to [0, \infty)$ and a sequence $\{t_n : n \in \mathbb{N}\}$ in $(0, \mu(X))$ such that $\lim t_n = 0$ and for each $n \in \mathbb{N}$ we have $\mu^{-1}(t_n) \in \mathcal{P}$, then $X \in \mathcal{P}$.

The definitions are taken from [3]; compare [1, Definitions 27.1 (c) and (d), pp. 232 and 233].

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S. B. Nadler, Jr. asks in [2, Remark and question 14.55.1, p. 463] if there is an (sWrp) which is not an (ssWrp). The same question is repeated 21 years later in [1, Question 27.2, p. 233]. In this note we answer the question in the affirmative.

Let M_3 be the union of a 2-cell D and of three mutually disjoint arcs A_1, A_2, A_3 such that $A_i \cap D$ is an end point of A_i for each $i \in \{1, 2, 3\}$ (see [1, Figure 56 (b), p. 431] for a picture). Further, let $\mathcal{P}_1$ be the property "to be homeomorphic to M_3". We start with the following proposition.

Proposition 1. *There is no continuum X such that there exists a Whitney map $\mu : C(X) \to [0, \infty)$ all positive Whitney levels of which have property $\mathcal{P}_1$.*

Proof. Suppose on the contary that there are a continuum X and a Whitney map $\mu : C(X) \to [0, \infty)$ such that for each $t \in (0, \mu(X))$ the condition $\mu^{-1}(t) \in \mathcal{P}_1$ holds. Note that local connectedness is an $(sWrp)$, [1, 52.2, p. 281], and that some Whitney levels of a 4-od contain 3-cubes, [2, (14.33), p. 430]. Thus:

(1.1) X is locally connected;
(1.2) X does not contain any 4-od.

It follows from (1.1) and (1.2) that the continuum X is

(1.3) either an arc, or a simple closed curve, or a simple triod, or a noose.

(A noose is the union of a simple closed curve and an arc whose intersection is one of the end points of the arc; see [1, p. 36]). For none of these four continua in (1.3) do all Whitney levels have property $\mathcal{P}_1$; in particular, large Whitney levels of a simple triod are 2-cells [1, Theorem 65.10 (c), p. 309]. The proof is complete. □

Theorem 2. *Property $\mathcal{P}_1$ is an $(sWrp)$ and it is not an $(ssWrp)$.*

Proof. It follows from Proposition 1 that $\mathcal{P}_1$ satisfies the definition of an $(sWrp)$ (in an empty way, since no continuum having the needed property exists). To see that $\mathcal{P}_1 \notin (ssWrp)$ note that if T is a simple triod, and $\mu : C(T) \to [0, \infty)$ is a Whitney map, then the Whitney levels $\mu^{-1}(t)$ for small positive numbers t have property $\mathcal{P}_1$, [1, Example 65.4, pp. 307-308], while $T \notin \mathcal{P}_1$. □

As it is said in the beginning of the proof of Theorem 2, property $\mathcal{P}_1$ is an $(sWrp)$ in an empty way. Since this can be seen as defect of our solution of the Nadler problem mentioned above, we will consider another property, $\mathcal{P}_2$, for which this imperfection is eliminated.

Let $\mathcal{P}_2$ be the property "to be homeomorphic either to an arc or to M_3".

Theorem 3. *Property $\mathcal{P}_2$ is an $(sWrp)$ and it is not an $(ssWrp)$.*

Proof. As in the proof of Theorem 2, a simple triod T shows that $\mathcal{P}_2 \notin (ssWrp)$. Further, arguing as previously in the proof of Proposition 1, we see that of the four continua in (1.3) only the arc has all Whitney levels with $\mathcal{P}_2$. Since obviously $\mathcal{P}_2$ holds for an arc, it follows that $\mathcal{P}_2 \in (sWrp)$. The proof is complete. □

Remark 4. The reader can consider other continua to answer Nadler's question: for example the continuum M_n instead of M_3, for $n > 3$ (see [1, Figure 56 (a), p. 431]) and/or a simple closed curve in place of an arc for $\mathcal{P}_2$.

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