ON CLASSES OF MAPS WHICH PRESERVE FINITISTICNESS

AKIRA KOYAMA AND MANUEL A. MORON

(Communicated by Alan Dow)

Abstract. We shall prove the following: (1) Let \( r : X \to Y \) be a refinable map between paracompact spaces. Then \( X \) is finitistic if and only if \( Y \) is finitistic.

(2) Let \( f : X \to Y \) be a hereditary shape equivalence between metric spaces. Then if \( X \) is finitistic, \( Y \) is finitistic.

1. Introduction

To extend the classical cohomological methods in the study of group actions on compact Hausdorff spaces or finite-dimensional paracompact spaces Swan introduced the concept of finitistic spaces and obtained a Smith-type fixed point theorem. Typical results on group actions for finitistic spaces may be found in [B], [De-Si], [De-Si-S], and [De-T].

Definition. A space \( X \) is said to be finitistic if every open cover of \( X \) has an open refinement of finite order.

By the definition, the class of paracompact, finitistic spaces may be considered the natural one combining both compact and finite-dimensional paracompact spaces. Recently from the dimension-theoretical viewpoint several authors investigated finitistic spaces as a kind of infinite-dimensional spaces (cf. [De-P], [De-Si-2], [Dy-M-S], [H1] and [H2]). Moreover, Rubin and Schapiro succeeded to show that the class of paracompact, finitistic spaces has a nice role in cohomological dimension theory. Namely

Theorem (Rubin and Schapiro). Suppose that \( X \) is a paracompact, finitistic space and \( G \) is a finitely generated abelian group. Then:

1. \( \dim_G \beta X = \dim_G X \), where \( \beta X \) is the Stone-\v{C}ech compactification of \( X \).
2. if \( X \) is separable and metrizable, then \( X \) has a metrizable compactification \( kX \) with \( \dim_G kX = \dim_G X \).

On the other hand, since refinable maps were originally introduced by Ford and Rogers to study continuum theory, many authors have found dimension-theoretical properties of refinable maps (cf. [A], [C-V], [G-Ro], [Ka], [Ka-Ko], [Ko1], [Ko2] and [Ko-Sh]). For a refinable map between metric spaces the first
author [Ko2] showed that \( \dim X = \dim Y \) and in [Ko-Sh] that \( \dim_G X = \dim_G Y \) for any finitely generated abelian group \( G \). In case \( r \) is \( c \)-refinable, we also have that \( K \in AE(X) \) if and only if \( K \in AE(Y) \) for any simplicial complex \( K \) (see [Ko2], Theorem 1). Recently Chigogidze and Valov [C-V] generalized those results.

Namely, if \( f : X \to Y \) is a refinable map between metric spaces \( X \) and \( Y \), and \( K \) is a CW-complex, then \( e \cdot \dim X \leq K \) if and only if \( e \cdot \dim Y \leq K \). Note that this result is true for a refinable map between compact Hausdorff spaces.

Similarly, by the definition, we can easily see that if \( f : X \to Y \) is a hereditary shape equivalence between metric spaces \( X \) and \( Y \), and \( e \cdot \dim X \leq K \), where \( K \) is a CW-complex, then \( e \cdot \dim Y \leq K \). Therefore hereditary shape equivalences preserve dimension and cohomological dimension. Moreover, they preserve property \( C \) (see [A]) and small weak infinite-dimensionality (see [Mi]). Recently Dijkstra [Di] and Dijkstra and Mogilski [Di-Mo] gave an interesting example and results about small transfinite inductive dimension and countable dimensionality. Namely, we can say that hereditary shape equivalences have interesting dimension-theoretical properties.

In this paper, first, we shall show that if \( r : X \to Y \) is a refinable map between paracompact spaces, then \( X \) is finitistic if and only if \( Y \) is finitistic. We shall note remarks on the extension property of finitistic spaces. Next, we shall show that if \( r : X \to Y \) is a hereditary shape equivalence between metric spaces and \( X \) is finitistic, \( Y \) is also finitistic. Note that Dydak, Mishra and Shukla [Dy-M-S] discussed several mapping theorems for finitistic spaces.

All spaces considered in this paper are assumed to be normal and maps are continuous.

2. Refinable maps and finitisticness

Let \( \mathcal{P} \) be a class of (not necessarily compact) polyhedra. A space \( X \) is said to be \( \mathcal{P} \)-like if for every locally finite open cover \( \mathcal{U} \) of \( X \) there exists a \( \mathcal{U} \)-map \( \varphi : X \to P \in \mathcal{P} \), here a \( \mathcal{U} \)-map means that each point \( z \in P \) has a neighborhood \( O_z \) such that \( \varphi^{-1}(O_z) \) is contained in an element of \( \mathcal{U} \). It is well-known that a paracompact space \( X \) has \( \dim X \leq n \) if and only if \( X \) is \( \mathcal{P}_n \)-like, where \( \mathcal{P}_n \) is the class of all polyhedra of dimension \( \leq n \). In a similar way we can find a similar characterization of paracompact, finitistic spaces as follows:

**Proposition 2.1.** Let \( \mathcal{P}_f \) be the class of all finite-dimensional polyhedra. Then a paracompact space \( X \) is finitistic if and only if \( X \) is \( \mathcal{P}_f \)-like.

A surjective map \( r : X \to Y \) is called refinable if for any open cover \( \mathcal{U} \) of \( X \) and any open cover \( \mathcal{V} \) of \( Y \) there exists a surjective \( \mathcal{U} \)-map \( f : X \to Y \) such that \( r \) and \( f \) are \( \mathcal{V} \)-close, that is, for every point \( x \in X \) there is an element of \( \mathcal{V} \) containing both \( r(x) \) and \( f(x) \), and shortly denoted by \( d(r,f) \leq \mathcal{V} \). The map \( f \) is called a \( (\mathcal{U},\mathcal{V}) \)-refinement of \( r \). When there exists a closed \( (\mathcal{U},\mathcal{V}) \)-refinement of \( r \), we say that \( r \) is \( c \)-refinable.

**Theorem 2.2.** Let \( \mathcal{P} \) be a class of polyhedra and let \( r : X \to Y \) be a refinable map between paracompact spaces. Then \( X \) is \( \mathcal{P} \)-like if and only if \( Y \) is \( \mathcal{P} \)-like.

**Proof.** First we suppose that \( X \) is \( \mathcal{P} \)-like. For a given open cover \( \mathcal{V} \) of \( Y \), let us take a locally finite open refinement \( \mathcal{U} \) of \( r^{-1}(\mathcal{V}) \). By the definition there are a \( P \in \mathcal{P} \) and a \( \mathcal{U} \)-map \( \varphi : X \to P \). Then there is a locally finite open cover \( \tilde{\mathcal{U}} \) of \( P \) such
that
\( (1) \quad \varphi^{-1}(St(U)) \prec r^{-1}(V) \);
here \( St(U) = \{ St(\hat{U}, \hat{U}) \mid \hat{U} \in \mathcal{U} \} \) and \( St(\hat{U}, \hat{U}) = \bigcup \{ \hat{U}_* \in \mathcal{U} \mid \hat{U} \cap \hat{U}_* \neq \emptyset \} \). We choose a sufficiently small triangulation \( T \) of \( P \) such that
\( (2) \quad T = \{ st(v, T) \mid v \in T^{(0)} \} \prec \tilde{U}. \)

Then we can take a \( (\varphi^{-1}(T), V) \)-refinement \( f : X \to Y \) of \( r. \) Since \( f \) is a \( \varphi^{-1}(T) \)-map, there exists a locally finite open cover \( W \) of \( Y \) such that
\( (3) \quad W \prec V \quad \text{and} \quad f^{-1}(W) \prec \varphi^{-1}(T). \)
Moreover we may assume that the cover \( W \) is given by cozero sets of a partition of unity \( \{ \xi_W \}_{W \in \mathcal{W}}. \) Hence we can define the map \( \eta : Y \to N(W) \) of \( Y \) to the nerve \( N(W) \) of \( W \) by \( \eta(y) = \sum_{W \in \mathcal{W}} \xi_W(y) \cdot W. \)

For each \( W \in \mathcal{W} \), by \( (4) \), there exists a \( v_W \in T^{(0)} \) such that
\( (5) \quad f^{-1}(W) \subset \varphi^{-1}(st(v_W, T)). \)
For a finite subset \( \{ W_0, \ldots, W_n \} \) of \( \mathcal{W} \), if \( \bigcap_{i=0}^n W_i \neq \emptyset \), by \( (5) \), \( \bigcap_{i=0}^n st(v_W, T) \neq \emptyset \). Thus, the set of vertices \( \{ v_{W_0}, \ldots, v_{W_n} \} \) spans a simplex of \( T \). Hence the correspondence \( v_W, W \in \mathcal{W} \), induces a map \( \psi : N(W) \to |T| = P. \)

Then we shall show that \( \psi \circ \eta : Y \to P \) is a \( St(V) \)-map. First we note the following:
\( (6) \quad d(\varphi, \psi \circ \eta \circ f) \leq T. \)
For any \( x \in X \), let us take a \( W_0 \in \mathcal{W} \) such that \( f(x) \in W_0 \). Then \( \xi_{W_0}(f(x)) > 0 \) and \( \psi \circ \eta \circ f(x) \in st(v_{W_0}, T). \) Hence, by \( (5) \), \( \varphi(x), \psi \circ \eta \circ f(x) \in st(v_{W_0}, T). \)

Let us fix an arbitrary vertex \( v_0 \) of \( T \). Take a given point \( y \in (\psi \circ \eta)^{-1}(st(v_0, T)) \) and a point \( x \in f^{-1}(y) \). Then, by \( (6) \), there exists a vertex \( v_x \in T \) such that \( \varphi(x), \psi \circ \eta(y) \in st(v_x, T) \). Hence when we take \( \tilde{U}_0, \hat{U}_x \in \tilde{U} \) such that \( st(v_0, T) \subset \tilde{U}_0 \) and \( st(v_x, T) \subset \hat{U}_x \), \( \tilde{U}_x \cap \hat{U}_0 \neq \emptyset \) and \( \varphi(x) \in St(\tilde{U}_0, \hat{U}). \) Hence \( f^{-1}(y) \subset \varphi^{-1}(St(\tilde{U}_0, \hat{U})). \) Therefore, by \( (1) \), \( f^{-1}(y) \subset r^{-1}(V_0) \) for some \( V_0 \in \mathcal{V}. \) Since \( d(f, r) \leq \nu, \ y \in f(r^{-1}(V_0)) \subset St(V_0, V). \) Note that choosing \( V_0 \) depends on only the vertex \( v_0. \) It follows that \( (\psi \circ \eta)^{-1}(st(v_0, T)) \subset St(V_0, V). \) Namely \( Y \) is \( \mathcal{P} \)-like.

Next suppose that \( Y \) is \( \mathcal{P} \)-like. For a locally finite open cover \( U \) of \( X \) there exists a \( U \)-map \( f : X \to Y \) and an open cover \( V \) of \( Y \) such that \( f^{-1}(V) \prec U. \) Since \( Y \) is \( \mathcal{P} \)-like, there exist a \( P \in \mathcal{P} \) and a \( V \)-map \( \psi : Y \to P. \) Then the composition \( \psi \circ f : X \to P \) is a \( U \)-map. Therefore \( X \) is \( \mathcal{P} \)-like.

By Theorem 2.2 and Proposition 2.1 we can see the following:

**Corollary 2.3.** Let \( r : X \to Y \) be a refinable map between paracompact spaces. Then \( X \) is finitistic if and only if \( Y \) is finitistic.

Dranishnikov [Dr] gave a remarkable example of a separable metric space \( X \) such that \( \dim_2 X \leq 4 \) but \( \dim_3 \beta X = \infty \) and Dydak-Walsh [Dy-W], for any abelian group \( G \), constructed a separable metric space \( Y \) such that \( \dim_G Y \leq 3 \) but \( \dim_G \alpha Y > 3 \) for any compactification \( \alpha Y \) of \( Y \). In spite of these examples, finitistic spaces, by Rubin-Schaprio theorem, still give a large class of spaces whose Stone-Čech compactifications keep cohomological dimension with respect to any
finitely generated abelian groups. A current movement of cohomological dimension theory is shifting to a more general notation called extension theory. Namely, for a CW-complex $K$ the extension dimension of a space $X$ is equal or less than $K$, shorty $e\text{dim} X \leq K$, if every map $f : A \to K$ of a closed subset $A$ of $X$ to $K$ admits a continuous extension $F : X \to K$. Corresponding and improved examples of Dranishnikov's and Dydak-Walsh's examples to extension dimension theory were obtained by Levin [L]. Here we state a corresponding result with Rubin-Schapiro theorem as follows:

**Theorem 2.4 ([Dy-M-S], Theorem 4.1).** Suppose that $X$ is a finitistic, paracompact space and $K$ is a CW-complex of finite type, that is, each skeleton of $K$ is a finite subcomplex. If $e\text{dim} X \leq K$, then $e\text{dim} \beta X \leq K$.

Moreover, if $K$ is complete and $e\text{dim} \beta X \leq K$, then $e\text{dim} X \leq K$.

We note that for any finitely generated abelian group $G$ and $n \geq 1$ we can have an Eilenberg-MacLane complete complex $K(G, n)$ of finite type.

We state the following fact about Stone-Ščech extension of c-refinable maps:

**Theorem 2.5 ([Ko1], Theorem 3.1).** Let $r : X \to Y$ be a c-refinable map between normal spaces. Then the extension $\beta f : \beta X \to \beta Y$ is refinable.

Therefore Theorems 2.4 and 2.5 induce the following result related to extension dimension:

**Corollary 2.6.** Let $r : X \to Y$ be a c-refinable map between paracompact spaces. If one of $X$ or $Y$ is finitistic, then another is finitistic, and $e\text{dim} X \leq K$ for a complete CW-complex $K$ of finite type if and only if $e\text{dim} Y \leq K$.

**Remark 1.** To investigate extension property of noncompact or nonmetrizable spaces, the notation $\alpha(K)$ introduced by Kuz’minov [Ku] may be useful. The author essentially used the property in [Ko2], and Chigogidze and Valov [C-V] succeeded to characterize extension dimension by using the notation “$\alpha(K)$-like spaces”.

3. HEREDITARY SHAPE EQUIVALENCES AND FINITISTICNESS

A map between metric spaces is called proper if the preimage of every compact subset is compact, or equivalently the map is closed and has compact fibers. A proper map $f$ from $X$ onto $Y$ is a hereditary shape equivalence if for every closed subset $B$ of $Y$ the restriction $f|_{A} : A \to B, A = f^{-1}(B)$, is a shape equivalence.

**Theorem 3.1.** Let $f : X \to Y$ be a hereditary shape equivalence. If $X$ is finitistic, then $Y$ is also finitistic.

For the proof we shall use the following characterization by [H1] and [Dy-M-S].

**Proposition 3.2.** A paracompact space $X$ is finitistic if and only if there exists a compact subspace $K$ of $X$ such that $\dim F < \infty$ for every closed subspace $F$ with $F \cap K = \emptyset$.

**Proof of Theorem 3.1.** Let $f : X \to Y$ be a hereditary shape equivalence on a finitistic space $X$. By Proposition 3.2, we take a compact subspace $K$ of $X$ satisfying the desired property. We shall show the compact subspace $f(K) = L$ of $Y$ has the property in Proposition 3.2.
Let us take a closed subset $F$ of $Y$ with $F \cap L = \emptyset$. Then $f^{-1}(F) \cap K = \emptyset$. Hence $\dim f^{-1}(F) < \infty$. Now the restriction $f|_{f^{-1}(F)} : f^{-1}(F) \to F$ is a hereditary shape equivalence. Therefore $\dim F \leq \dim f^{-1}(F) < \infty$. Thus, $L$ has the required property.

Remark 2. We recall that a proper map $f$ from $X$ onto $Y$ is cell-like if for every $y \in Y$, $f^{-1}(y)$ has the trivial shape. Namely, the notation of hereditary shape equivalences is a strengthening of cell-like maps. Now let us consider Dranishnikov’s separable metric space $X$ in [Dr] again. Then, by Rubin and Schapiro’s cell-like resolution theorem [Ru-Sc], there can exist a cell-like map from a metric space $Z$ with $\dim Z = \dim_2 \bar{X} \leq 4$ onto $X$. Thus, a cell-like image of a finitistic space, even a finite-dimensional metric space, is not finitistic.

References


L. Rubin and P. Schapiro, Compactifications which preserve cohomological dimension, Glasnik Mat., 28 (1993), 155–165. MR 95g:54029


Division of Mathematical Sciences, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan
E-mail address: koyama@cc.osaka-kyoiku.ac.jp

Unidad Dovente de Matematicas, E. T. S. I. Montes, Universidad Politécnica, 28040, Madrid, Spain
E-mail address: mam@montes.upm.es