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ON CLASSES OF MAPS WHICH PRESERVE FINITISTICNESS

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ABSTRACT. We shall prove the following: (1) Let $r : X \to Y$ be a refinable map between paracompact spaces. Then X is finitistic if and only if Y is finitistic. (2) Let $f : X \to Y$ be a hereditary shape equivalence between metric spaces. Then if X is finitistic, Y is finitistic.

1. Introduction

To extend the classical cohomological methods in the study of group actions on compact Hausdorff spaces or finite-dimensional paracompact spaces Swan [Sw] introduced the concept of finitistic spaces and obtained a Smith-type fixed point theorem. Typical results on group actions for finitistic spaces may be found in [B], [De-Si$_1$],[De-Si-S] and [De-T].

Definition. A space X is said to be finitistic if every open cover of X has an open refinement of finite order.

By the definition, the class of paracompact, finitistic spaces may be considered the natural one combining both compact and finite-dimensional paracompact spaces. Recently from the dimension-theoretical viewpoint several authors investigated finitistic spaces as a kind of infinite-dimensional spaces (cf. [De-P], [De-Si$_2$], [Dy-M-S], [H1] and [H2]). Moreover, Rubin and Schapiro [Ru-Sc$_2$] succeeded to show that the class of paracompact, finitistic spaces has a nice role in cohomological dimension theory. Namely

Theorem (Rubin and Schapiro)**.** *Suppose that X is a paracompact, finitistic space and G is a finitely generated abelian group. Then*:

(1) $\dim_G \beta X = \dim_G X$, *where βX is the Stone-Čech compactification of X,*
(2) *if X is separable and metrizable, then X has a metrizable compactification kX with* $\dim_G kX = \dim_G X$.

On the other hand, since refinable maps were originally introduced by Ford and Rogers [F-R] to study continuum theory, many authors have found dimension-theoretical properties of refinable maps (cf. [A], [C-V], [G-Ro], [Ka], [Ka-Ko], [Ko1], [Ko2] and [Ko-Sh]). For a refinable map between metric spaces the first

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author [Ko2] showed that $\dim X = \dim Y$ and in [Ko-Sh] that $\dim_G X = \dim_G Y$ for any finitely generated abelian group G. In case r is c-refinable, we also have that $K \in AE(X)$ if and only if $K \in AE(Y)$ for any simplicial complex K (see [Ko2], Theorem 1). Recently Chigogidze and Valov [C-V] generalized those results. Namely, *if* $f : X \to Y$ *is a refinable map between metric spaces* X *and* Y*, and* K *is a CW-complex, then* $e\text{-}\dim X \leq K$ *if and only if* $e\text{-}\dim Y \leq K$. Note that this result is true for a refinable map between compact Hausdorff spaces.

Similarly, by the definition, we can easily see that *if* $f : X \to Y$ *is a hereditary shape equivalence between metric spaces* X *and* Y*, and* $e\text{-}\dim X \leq K$*, where* K *is a CW-complex, then* $e\text{-}\dim Y \leq K$. Therefore hereditary shape equivalences preserve dimension and cohomological dimension. Moreover, they preserve property C (see [A]) and small weak infinite-dimensionality (see [Mi]). Recently Dijkstra [Di] and Dijkstra and Mogilski [Di-Mo] gave an interesting example and results about small transfinite inductive dimension and countable dimensionality. Namely, we can say that hereditary shape equivalences have interesting dimension-theoretical properties.

In this paper, first, we shall show that if $r : X \to Y$ is a refinable map between paracompact spaces, then X is finitistic if and only Y is finitistic. We shall note remarks on the extension property of finitistic spaces. Next, we shall show that if $r : X \to Y$ is a hereditary shape equivalence between metric spaces and X is finitistic, Y is also finitistic. Note that Dydak, Mishra and Shukla [Dy-M-S] discussed several mapping theorems for finitistic spaces.

All spaces considered in this paper are assumed to be *normal* and maps are *continuous.*

2. Refinable maps and finitisticness

Let $\mathcal{P}$ be a class of (not necessarily compact) polyhedra. A space X is said to be $\mathcal{P}$-like if for every locally finite open cover $\mathcal{U}$ of X there exists a $\mathcal{U}$-map $\varphi : X \to P \in \mathcal{P}$, here a $\mathcal{U}$-map means that each point $z \in P$ has a neighborhood O_z such that $\varphi^{-1}(O_z)$ is contained in an element of $\mathcal{U}$. It is well-known that a paracompact space X has $\dim X \leq n$ if and only if X is $\mathcal{P}_n$-like, where $\mathcal{P}_n$ is the class of all polyhedra of dimension $\leq n$. In a similar way we can find a similar characterization of paracompact, finitistic spaces as follows:

Proposition 2.1. *Let* $\mathcal{P}_f$ *be the class of all finite-dimensional polyhedra. Then a paracompact space* X *is finitistic if and only if* X *is* $\mathcal{P}_f$*-like.*

A surjective map $r : X \to Y$ is called *refinable* if for any open cover $\mathcal{U}$ of X and any open cover $\mathcal{V}$ of Y there exists a surjective $\mathcal{U}$-map $f : X \to Y$ such that r and f are $\mathcal{V}$-close, that is, for every point $x \in X$ there is an element of $\mathcal{V}$ containing both $r(x)$ and $f(x)$, and shortly denoted by $d(r, f) \leq \mathcal{V}$. The map f is called a $(\mathcal{U}, \mathcal{V})$*-refinement* of r. When there exists a closed $(\mathcal{U}, \mathcal{V})$-refinement of r, we say that r is *c-refinable.*

Theorem 2.2. *Let* $\mathcal{P}$ *be a class of polyhedra and let* $r : X \to Y$ *be a refinable map between paracompact spaces. Then* X *is* $\mathcal{P}$*-like if and only if* Y *is* $\mathcal{P}$*-like.*

Proof. First we suppose that X is $\mathcal{P}$-like. For a given open cover $\mathcal{V}$ of Y, let us take a locally finite open refinement $\mathcal{U}$ of $r^{-1}(\mathcal{V})$. By the definition there are a $P \in \mathcal{P}$ and a $\mathcal{U}$- map $\varphi : X \to P$. Then there is a locally finite open cover $\tilde{\mathcal{U}}$ of P such that

$$\varphi^{-1}(St(\tilde{\mathcal{U}})) \prec r^{-1}(\mathcal{V}); \tag{1}$$

here $St(\tilde{\mathcal{U}}) = \{St(\tilde{U},\tilde{\mathcal{U}}) \mid \tilde{U} \in \tilde{\mathcal{U}}\}$ and $St(\tilde{U},\tilde{\mathcal{U}}) = \cup\{\tilde{U}_* \in \mathcal{U} \mid \tilde{U} \cap \tilde{U}_* \neq \emptyset\}$. We choose a sufficiently small triangulation T of P such that

$$\mathcal{T} = \{st(v,T) \mid v \in T^{(0)}\} \prec \tilde{\mathcal{U}}. \tag{2}$$

Then we can take a $(\varphi^{-1}(\mathcal{T}),\mathcal{V})$-refinement $f : X \to Y$ of r. Since f is a $\varphi^{-1}(\mathcal{T})$-map, there exists a locally finite open cover $\mathcal{W}$ of Y such that

$$\mathcal{W} \prec \mathcal{V} \text{ and} \tag{3}$$

$$f^{-1}(\mathcal{W}) \prec \varphi^{-1}(\mathcal{T}). \tag{4}$$

Moreover we may assume that the cover $\mathcal{W}$ is given by cozero sets of a partition of unity $\{\xi_W\}_{W\in\mathcal{W}}$. Hence we can define the map $\eta : Y \to N(\mathcal{W})$ of Y to the nerve $N(\mathcal{W})$ of $\mathcal{W}$ by $\eta(y) = \sum_{W\in\mathcal{W}} \xi_W(y) \cdot W$.

For each $W \in \mathcal{W}$, by (4), there exists a $v_W \in T^{(0)}$ such that

$$f^{-1}(W) \subset \varphi^{-1}(st(v_W,T)). \tag{5}$$

For a finite subset $\{W_0,\cdots,W_n\}$ of $\mathcal{W}$, if $\bigcap_{i=0}^n W_i \neq \emptyset$, by (5), $\bigcap_{i=0}^n st(v_{W_i},T) \neq \emptyset$. Thus, the set of vertices $\{v_{W_0},\cdots,v_{W_n}\}$ spans a simplex of T. Hence the correspondence $v_W, W \in \mathcal{W}$, induces a map $\psi : N(\mathcal{W}) \to |T| = P$.

Then we shall show that $\psi\circ\eta : Y \to P$ is a $St(\mathcal{V})$-map. First we note the following:

$$d(\varphi,\psi\circ\eta\circ f) \leq \mathcal{T}. \tag{6}$$

For any $x \in X$, let us take a $W_0 \in \mathcal{W}$ such that $f(x) \in W_0$. Then $\xi_{W_0}(f(x)) > 0$ and $\psi\circ\eta\circ f(x) \in st(v_{W_0},T)$. Hence, by (5), $\varphi(x), \psi\circ\eta\circ f(x) \in st(v_{W_0},T)$.

Let us fix an arbitrary vertex v_0 of T. Take a given point $y \in (\psi\circ\eta)^{-1}(st(v_0,T))$ and a point $x \in f^{-1}(y)$. Then, by (6), there exists a vertex $v_x \in T$ such that $\varphi(x), \psi\circ\eta(y) \in st(v_x,T)$. Hence when we take $\tilde{U}_0, \tilde{U}_x \in \tilde{\mathcal{U}}$ such that $st(v_0,T) \subset \tilde{U}_0$ and $st(v_x,T) \subset \tilde{U}_x$, $\tilde{U}_x \cap \tilde{U}_0 \neq \emptyset$ and $\varphi(x) \in St(\tilde{U}_0,\tilde{\mathcal{U}})$. Hence $f^{-1}(y) \subset \varphi^{-1}(St(\tilde{U}_0,\tilde{\mathcal{U}}))$. Therefore, by (1), $f^{-1}(y) \subset r^{-1}(V_0)$ for some $V_0 \in \mathcal{V}$. Since $d(f,r) \leq \mathcal{V}$, $y \in f(r^{-1}(V_0)) \subset St(V_0,\mathcal{V})$. Note that choosing V_0 depends on only the vertex v_0. It follows that $(\psi\circ\eta)^{-1}(st(v_0,T)) \subset St(V_0,\mathcal{V})$. Namely Y is $\mathcal{P}$-like.

Next suppose that Y is $\mathcal{P}$-like. For a locally finite open cover $\mathcal{U}$ of X there exists a $\mathcal{U}$-map $f : X \to Y$ and an open cover $\mathcal{V}$ of Y such that $f^{-1}(\mathcal{V}) \prec \mathcal{U}$. Since Y is $\mathcal{P}$-like, there exist a $P \in \mathcal{P}$ and a $\mathcal{V}$-map $\psi : Y \to P$. Then the composition $\psi\circ f : X \to P$ is a $\mathcal{U}$-map. Therefore X is $\mathcal{P}$-like. □

By Theorem 2.2 and Proposition 2.1 we can see the following:

Corollary 2.3. *Let $r : X \to Y$ be a refinable map between paracompact spaces. Then X is finitistic if and only if Y is finitistic.*

Dranishnikov [Dr] gave a remarkable example of a separable metric space X such that $\dim_{\mathbb{Z}} X \leq 4$ but $\dim_{\mathbb{Z}} \beta X = \infty$ and Dydak-Walsh [Dy-W], for any abelian group G, constructed a separable metric space Y such that $\dim_G Y \leq 3$ but $\dim_G \alpha Y > 3$ for any compactification αY of Y. In spite of these examples, finitistic spaces, by Rubin-Schapiro theorem, still give a large class of spaces whose Stone-Čech compactifications keep cohomological dimension with respect to any finitely generated abelian groups. A current movement of cohomological dimension theory is shifting to a more general notation called *extension theory*. Namely, for a CW-complex K the *extension dimension of a space* X *is equal or less than* K, shortly $e\text{-}\dim X \leq K$, if every map $f : A \to K$ of a closed subset A of X to K admits a continuous extension $F : X \to K$. Corresponding and improved examples of Dranishnikov's and Dydak-Walsh's examples to extension dimension theory were obtained by Levin [L]. Here we state a corresponding result with Rubin-Schapiro theorem as follows:

Theorem 2.4 ([Dy-M-S],Theorem 4.1)**.** *Suppose that* X *is a finitistic, paracompact space and* K *is a CW-complex of finite type, that is, each skeleton of* K *is a finite subcomplex. If* $e\text{-}\dim X \leq K$*, then* $e\text{-}\dim \beta X \leq K$.

Moreover, if K *is complete and* $e\text{-}\dim \beta X \leq K$*, then* $e\text{-}\dim X \leq K$.

We note that for any finitely generated abelian group G and $n \geq 1$ we can have an Eilenberg-MacLane complete complex $K(G, n)$ of finite type.

We state the following fact about Stone-Čech extension of c-refinable maps:

Theorem 2.5 ([Ko1],Theorem 3.1)**.** *Let* $r : X \to Y$ *be a c-refinable map between normal spaces. Then the extension* $\beta f : \beta X \to \beta Y$ *is refinable.*

Therefore Theorems 2.4 and 2.5 induce the following result related to extension dimension:

Corollary 2.6. *Let* $r : X \to Y$ *be a c-refinable map between paracompact spaces. If one of* X *or* Y *is finitistic, then another is finitistic, and* $e\text{-}\dim X \leq K$ *for a complete CW-complex* K *of finite type if and only if* $e\text{-}\dim Y \leq K$.

Remark 1. To investigate extension property of noncompact or nonmetrizable spaces, the notation $\alpha(K)$ introduced by Kuz'minov [Ku] may be useful. The author essentially used the property in [Ko2], and Chigogidze and Valov [C-V] succeeded to characterize extension dimension by using the notation "$\alpha(K)$-like spaces".

3. Hereditary shape equivalences and finitisticness

A map between metric spaces is called *proper* if the preimage of every compact subset is compact, or equivalently the map is closed and has compact fibers. A proper map f from X onto Y is a *hereditary shape equivalence* if for every closed subset B of Y the restriction $f|_A : A \to B, A = f^{-1}(B)$, is a shape equivalence.

Theorem 3.1. *Let* $f : X \to Y$ *be a hereditary shape equivalence. If* X *is finitistic, then* Y *is also finitistic.*

For the proof we shall use the following characterization by [H1] and [Dy-M-S].

Proposition 3.2. *A paracompact space* X *is finitistic if and only if there exists a compact subspace* K *of* X *such that* $\dim F < \infty$ *for every closed subspace* F *with* $F \cap K = \emptyset$.

Proof of Theorem 3.1. Let $f : X \to Y$ be a hereditary shape equivalence on a finitistic space X. By Proposition 3.2, we take a compact subspace K of X satisfying the desired property. We shall show the compact subspace $f(K) = L$ of Y has the property in Proposition 3.2.

Let us take a closed subset F of Y with $F \cap L = \emptyset$. Then $f^{-1}(F) \cap K = \emptyset$. Hence $\dim f^{-1}(F) < \infty$. Now the restriction $f|_{f^{-1}(F)} : f^{-1}(F) \to F$ is a hereditary shape equivalence. Therefore $\dim F \leq \dim f^{-1}(F) < \infty$. Thus, L has the required property. □

Remark 2. We recall that a proper map f from X onto Y is *cell-like* if for every $y \in Y$, $f^{-1}(y)$ has the trivial shape. Namely, the notation of hereditary shape equivalences is a strengthening of cell-like maps. Now let us consider Dranishnikov's separable metric space X in [Dr] again. Then, by Rubin and Schapiro's cell-like resolution theorem [Ru-Sc$_1$], there can exist a cell-like map from a metric space Z with $\dim Z = \dim_{\mathbb{Z}} X \leq 4$ onto X. Thus, a cell-like image of a finitistic space, even a finite-dimensional metric space, is not finitistic.

References

[A] F. Ancel, *Proper hereditary shape equivalence property C*, Topology and its Appl., **19**(1985), 71–74. MR **86g:**54026

[B] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, 1972, New York. MR **54:**1265

[C-V] A. Chigogidze and V. Valov, *Extension dimension and refinable maps*, preprint(1999).

[De-P] S. Deo and A. R. Pears, *A completely finitistic space is finite-dimensional*, Bull. London Math. Soc., **17**(1985), 49–51. MR **85k:**54041

[De-Si$_1$] S. Deo and T. B. Singh, *On the converse of some theorems about orbit spaces*, J. London Math. Soc., **25**(1982), 162–170. MR **83k:**54039

[De-Si$_2$] ———, *On certain constructions in finitistic spaces*, Internat. J. Math. Math. Sci., **6**(1983), 477–482. MR **85c:**54058

[De-Si-S] S. Deo, T. B. Singh, and R. A. Shukra, *On an extension of localization theorem and generalized Conner conjecture*, Trans. Amer. Math. Soc., **269**(1982), 395–402. MR **83a:**57051

[De-T] S. Deo and H. S. Tripathi, *Compact Lie group actions on finitistic spaces*, Topology, **21**(1982), 393–399. MR **83k:**54042

[Di] J. J. Dijkstra, *A dimension raising hereditary shape equivalence*, Fund. Math., **149**(1996), 265–274. MR **97f:**54040

[Di-Mo] J. J. Dijkstra and J. Mogilski, *Countable dimensionality and dimension raising cell-like maps*, **80**(1997), 73–79. MR **98i:**54017

[Dr] A. N. Dranishnikov, *Cohomological dimension is not preserved under Stone-Čech compactification*, Comptes Rendus Bulgarian Acad. of Sci., **41**(1988), 9–10. MR **90e:**55002

[Dy-M-S] J. Dydak, S. N. Mishra and R. A. Shukla, *On finitistic spaces*, Topology and its Appl., **97**(1999), 217–229. MR **2000i:**55003

[Dy-W] J. Dydak and J. Walsh, *Spaces without cohomological dimension preserving compactifications*, Proc. Amer. Math. Soc., **113**(1991), 1155–1162. MR **92c:**54039

[F-R] Jo Ford and J. Rogers, Jr., *Refinable maps*, Colloq. Math., **39**(1978), 263–269. MR **80d:**54009

[G-Ro] D. Garity and D. Rohm, *Property C, refinable maps and dimension raising maps*, Proc. Amer. Math. Soc., **98**(1986), 336–340. MR **87i:**54077

[H1] Y. Hattori, *A note on finitistic spaces*, Questions and Answers Gen. Topology, **3**(1985), 47–55. MR **86m:**54025

[H2] ———, *Finitistic spaces and dimension*, Houston J. Math., **25**(1999), 687–696.

[Ka] H. Kato, *A note on infinite dimension under refinable maps*, Proc. Amer. Math. Soc., **88**(1983), 177–180. MR **84c:**54064

[Ka-Ko] H. Kato and A. Koyama, *A role of refinable maps - a survey*, Topology Proc., **11**(1986), 317–348. MR **89f:**54030

[Ko1] A. Koyama, *Refinable maps in dimension theory*, Topology and its Appl., **17**(1984), 247–255. MR **86d:**54056

[Ko2] ———, *Refinable maps in dimension theory* II, Bull. Pol. Math. Soc., **42**(1994), 255–261. CMP 2001:07

[Ko-Sh] A. Koyama and R. Sher, *Approximable dimension and acyclic dimension*, Fund. Math., **152**(1997), 43–53.

[Ku] V. I. Kuz'minov, *Homological dimension theory*, Russian Math. Surveys, **23**(1968), 1–45. MR **39:**2158

[L] M. Levin, *Some examples in cohomological dimension theory*, preprint(1999).

[Mi] R. Millspaugh, *Proper hereditary shape equivalences preserve small weak infinite dimensionality*, Proc. Amer. Math. Soc., **90**(1984), 1055–1061. MR **91d:**54041

[Ru-Sc$_1$] L. Rubin and P. Schapiro, *Cell-like maps onto non-compact spaces of finite cohomological dimension*, Topology and its Appl., **27**(1987), 221–244. MR **89b:**55002

[Ru-Sc$_2$] ———, *Compactifications which preserve cohomological dimension*, Glasnik Mat., **28(48)**(1993), 155–165. MR **95g:**54029

[Sw] R. G. Swan, *A new method in fixed point theory*, Comment. Math. Helv., **34**(1960), 1–16. MR **22:**5978

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