A PROOF OF WEINBERG’S CONJECTURE
ON LATTICE-ORDERED MATRIX ALGEBRAS

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(Dedicated to Professor Melvin Henriksen on his 75th birthday)

Abstract. Let $F$ be a subfield of the field of real numbers and let $F_n$ (n ≥ 2) be the $n \times n$ matrix algebra over $F$. It is shown that if $F_n$ is a lattice-ordered algebra over $F$ in which the identity matrix 1 is positive, then $F_n$ is isomorphic to the lattice-ordered algebra $F_n$ with the usual lattice order. In particular, Weinberg’s conjecture is true.

Let $L$ be a totally ordered field, and let $L_n$ (n ≥ 2) be the $n \times n$ matrix algebra over $L$. Then $L_n$ may be lattice-ordered by requiring that a matrix in $L_n$ is positive exactly when each of its entries is positive, that is, the positive cone is $(L^+)_n$. This lattice order is called the usual lattice order of $L_n$.

Let $Q$ be the field of rational numbers. In 1966, Weinberg conjectured that $(Q^+)_n$ is the only lattice order of $Q_n$ (up to an isomorphism) such that $Q_n$ is a lattice-ordered algebra ($\ell$-algebra) over $Q$ in which 1 is positive, and he proved his conjecture for $n = 2$ [8]. Recently some conditions have been obtained to ensure an $\ell$-algebra $L_n$, in which 1 is positive, is isomorphic to the $\ell$-algebra $L_n$ with the usual lattice order [3], [7].

In this paper, we show that Weinberg’s conjecture is true for a matrix $\ell$-algebra over any subfield of real numbers. More precisely, suppose that $F$ is a subfield of the field of real numbers; it is shown that an $\ell$-algebra $F_n$ over $F$ in which 1 is positive is isomorphic to the $\ell$-algebra $F_n$ with the usual lattice order.

We begin by collecting some definitions and results we will use later. The reader is referred to Birkhoff & Pierce [2] and Fuchs [4] for the general theory of lattice-ordered rings ($\ell$-rings). A partially ordered ring (po-ring) $R$ is an (associative) ring which is partially ordered, and in which i) $a \geq b$ implies $a + c \geq b + c$, for any $c \in R$, and ii) $a \geq 0$ and $b \geq 0$ imply $ab \geq 0$. Let $R$ be a po-ring. The set $R^+ = \{a \in R : a \geq 0\}$ is called the positive cone of $R$. Clearly $R^+$ is closed under addition and multiplication, and $R^+ \cap -R^+ = \{0\}$. Conversely, if $P$ is a subset of a ring $R$ which is closed under addition and multiplication, and satisfies $P \cap -P = \{0\}$, then the partial order $\geq$ defined by $a \geq b$ if and only if $a - b \in P$ makes $R$ into a po-ring with the positive cone equal to $P$. We will refer to such a

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positive cone as a partial order of a po-ring \( R \), and denote a po-ring \( R \) with positive cone \( P \) by \((R, P)\). If \( R \) is a po-ring and an algebra over a totally ordered field \( L \), then \( R \) is called a partially ordered algebra (po-algebra) over \( L \) if \( L^+ R^+ \subseteq R^+ \). A po-ring (po-algebra) is called an \( \ell \)-ring (\( \ell \)-algebra) if the partial order is a lattice order. Similarly, partially ordered groups (po-groups) and lattice-ordered groups (\( \ell \)-groups) and partially ordered vector spaces (po-vector spaces) and vector lattices over a totally ordered field can be defined. We skip these standard definitions, and refer the reader to \([2]\) and \([1]\).

An element \( a > 0 \) in a po-group \( G \) is called basic if the interval \([0, a] = \{ b \in G : 0 \leq b \leq a \} \) is a chain. A subset \( S \) of an \( \ell \)-group \( G \) is called disjoint if \( S > 0 \) for each \( s \in S \) and \( s \cap t = 0 \) for any \( s, t \in S \), \( s \neq t \). If \( G \) is an \( \ell \)-group and \( a \in G \), then the absolute value of \( a \) is \( |a| = a \lor (-a) \), the positive part of \( a \) is \( a^+ = a \lor 0 \), and the negative part of \( a \) is \( a^- = (-a) \lor 0 \).

For the rest of the paper, we always assume that \( F \) is a subfield of the totally ordered field \( R \) of real numbers; \( F_n \) is the \( n \times n \) matrix algebra over \( F \), and \( F^n \) is the vector space \( F_0 \times \cdots \times F_n \) (\( n \) times) over \( F \). We also consider \( F^n \) as a topological subspace of the Euclidean space \( R^n \). If \( (F_n, P) \) is a po-algebra over \( F \) with the positive cone \( P \), then we simply say that \( P \) is a partial order of \( F_n \). The positive cone of a po-vector space \( F^n \) over \( F \) is simply called a cone in \( F^n \).

Let \( P \) be a partial order of \( R_n \). A nonempty subset \( S \) of \( R^n \) is said to be a \( P \)-invariant set if for every \( f \in P \), \( f S \subseteq S \). A cone \( O \) in \( F_n \) is said to be a \( P \)-invariant cone in \( F^n \) if \( O \) is also a \( P \)-invariant set. We remark that in these definitions, a \( P \)-invariant set is not necessarily a subset of \( F^n \), and although \( P \)-invariant cones can be defined more generally, we will only consider \( P \)-invariant cones in \( F^n \) in this paper. Obviously, \( \{0\} \) is a \( P \)-invariant cone for every \( P \). We will refer to this cone as the trivial \( P \)-invariant cone. Other cones will be called nontrivial.

Given a subset \( K \) of \( R^n \), let

\[
\text{cone}_F(K) = \{ \sum \alpha_i v_i : \alpha_i \in F^+, v_i \in K \}.
\]

It is clear that \( \text{cone}_F(K) \subseteq F^n \) exactly when \( K \subseteq F^n \). Also, it is easily verified that \( \text{cone}_F(K) + \text{cone}_F(K) \subseteq \text{cone}_F(K) \) and \( F^+ \text{cone}_F(K) \subseteq \text{cone}_F(K) \). But \( \text{cone}_F(K) \cap -\text{cone}_F(K) \neq \{0\} \) in general. In case \( F = R \), many interesting properties of cones and matrices that leave a cone invariant are discussed in Berman & Plemmons \([1]\) and some papers referenced there. Note that if \( K \) is finite, then \( \text{cone}_R(K) \) is closed in the Euclidean space \( R^n \) \([1]\) Theorem 2.5] and if \( K \) consists of \( n \) linearly independent vectors in \( R^n \), then \( \text{cone}_R(K) \) has a nonempty interior. It actually satisfies definition 2.10 of a proper cone in \([1]\).

**Theorem 1.** Every nontrivial partial order \( P \) of \( F_n \) has a nontrivial \( P \)-invariant cone in \( F^n \).

**Proof.** Consider the collection \( M \) of all the null spaces in \( F^n \) of the nonzero matrices from \( P \). Let \( N \) denote an element of \( M \) with maximum dimension, and let \( u \notin N \).

Define \( O = \{fu : f \in P \text{ and } f N = 0\} \). The set \( O \) is closed under addition and \( F^+ O \subseteq O \). Also, if some \( fu, gu \in O \) and \( fu + gu = 0 \), then \((f + g)u = 0 \) and \((f + g)N = 0 \). Thus \((f + g)(fu + gu) = 0 \). But since \( f + g \in P \) and \( N \subseteq Ffu + N \), it follows from the maximality of the dimension of \( N \) that \( f + g = 0 \). Therefore, \( f = g = 0 \) and \( fu = gu = 0 \). This proves that \( O \) is a cone in \( F^n \). It is obvious that \( O \) is nontrivial and \( P \)-invariant.

\( \square \)
Lemma 2. If $P$ is a directed partial order of $\mathbf{F}_n$ and $S \neq \{0\}$ is a $P$-invariant set, then $S$ contains $n$ linearly independent vectors over $\mathbf{R}$.

Proof. Let $M$ be the subspace of $\mathbf{R}^n$ generated by $S$. Then for every $f \in P$, $fM \subseteq M$. But then, since $P$ is directed, every matrix $g \in \mathbf{F}_n$ satisfies $g = f_1 - f_2$ with $f_1, f_2 \in P$, so $gM = (f_1 - f_2)M \subseteq M$, and hence $M$ is a nontrivial invariant subspace for every matrix in $\mathbf{F}_n$. Thus $M = \mathbf{R}^n$, so $S$ contains $n$ linearly independent vectors over $\mathbf{R}$. □

Let $S$ be a subset of $\mathbf{R}^n$. The topological closure of $S$ in the Euclidean space $\mathbf{R}^n$ will be denoted by $\overline{S}$.

Lemma 3. If $P$ is a directed partial order of $\mathbf{F}_n$ and $O$ is a $P$-invariant cone in $\mathbf{F}^n$, then $\overline{O} \cap -\overline{O} = \{0\}$.

Proof. The set $\overline{O} \cap -\overline{O}$ is a $P$-invariant set. Moreover, since both $\overline{O}$ and $-\overline{O}$ are closed under addition and scalar multiplication by nonnegative real numbers, so is $\overline{O} \cap -\overline{O}$. If we suppose that $\overline{O} \cap -\overline{O} \neq \{0\}$, then by Lemma 2, $\overline{O} \cap -\overline{O}$ contains a set $K$ of $n$ linearly independent vectors of $\mathbf{R}^n$ over $\mathbf{R}$. Therefore, $\text{cone}_R(K) \subseteq \overline{O} \cap -\overline{O}$. Let $U$ be the interior of $\text{cone}_R(K)$. Note that $0 \notin U$. Since $U \cap -U \neq \emptyset$, $U \cap -U \neq \{0\}$, and hence by Lemma 2 it contains a set $K_1$ of $n$ linearly independent vectors of $\mathbf{R}^n$ over $\mathbf{R}$. Let $U_1$ be the interior of $\text{cone}_R(K_1)$. Then $U_1 \cap O \neq \emptyset$, since $U_1 \subseteq \text{cone}_R(K_1) \subseteq \overline{O}$. Since $K_1 \subseteq -O \subseteq \mathbf{F}^n$, $U_1 \cap O \subseteq U_1 \cap \mathbf{F}^n \subseteq \text{cone}_R(K_1) \cap \mathbf{F}^n = \text{cone}_F(K_1) \subseteq -O$, and hence $U_1 \cap O \subseteq O \cap -O$; so $O \cap -O \neq \{0\}$, which is a contradiction. □

Note that we have just proven that if $P$ is a directed partial order of $\mathbf{F}_n$ and $O$ is a $P$-invariant cone in $\mathbf{F}^n$, then $(\mathbf{R}^n, \overline{O})$ is a po-vector space over $\mathbf{R}$ with the positive cone $\overline{O}$, and $\overline{O}$ is $P$-invariant.

Lemma 4. Let $T$ be a subspace of $\mathbf{F}^m$ ($m > 1$) over $\mathbf{F}$ which is totally ordered. If $\dim_{\mathbf{F}}(T) > 1$, then $\overline{T} \cap -\overline{T} \neq \{0\}$.

Proof. First notice that $\overline{T}$ is a vector subspace of $\mathbf{R}^m$ over $\mathbf{R}$. Suppose that $\overline{T} \cap -\overline{T} = \{0\}$. Since $T = T^+ \cup -T^+$, we have that $\overline{T} = T^+ \cup -T^+$. If we write $X = \overline{T} \setminus \{0\}$, $A = T^+ \setminus \{0\} = T^+ \setminus -T^-$ and $B = -T^+ \setminus \{0\} = T^- \setminus -T^+$, then we have that $X = A \cup B$ and $A \cap B = \emptyset$, so $X$ is a union of two open, disjoint, nonempty subsets of $\overline{T}$, proving that $X$ is disconnected as a subset of $\overline{T}$. Since $\overline{T}$ is a Euclidean space, this is true only if $\dim_{\mathbf{R}}(T) = 1$. Finally, since $T \subseteq \mathbf{F}^m$, we have $\dim_{\mathbf{R}}(T) = 1$. □

Corollary 5. Let $\mathbf{F}^m$ ($m > 1$) be a po-vector space over $\mathbf{F}$ such that $(\mathbf{F}^m)^+ \subseteq (\mathbf{F}^+)^m$, and let $T \neq 0$ be a totally ordered subspace of $\mathbf{F}^m$. Then $T$ is one-dimensional.

Proof. Since $T^+ \subseteq (\mathbf{F}^+)^m$, $\overline{T^+} \subseteq (\mathbf{F}^+)^m = (\mathbf{R}^+)^m$. Similarly, $-\overline{T^+} \subseteq (-\mathbf{R}^+)^m$. Since $(\mathbf{R}^+)^m \cap (-\mathbf{R}^+)^m = \{0\}$, $\overline{T^+} \cap -\overline{T^+} = \{0\}$, and hence $T$ is one-dimensional by Lemma 4. □

A vector lattice over a totally ordered field $L$ is called Archimedean over $L$ if it has no nonzero bounded subspaces.

Corollary 6. Let $(\mathbf{F}_n, P)$ be an $\ell$-algebra over $\mathbf{F}$ with the positive cone $P$. If $P$ is contained in the usual lattice order $(\mathbf{F}^+)^n$, then $(\mathbf{F}_n, P)$ contains a set of exactly $n^2$ disjoint basic elements.
Proof. By [2, Corollary 1, p.51], $F_n$ is Archimedean over $F$. So $F_n = \mathbb{F}^{n^2}$ as a vector lattice over $F$ is a vector lattice direct sum of finitely many totally ordered subspaces of $F_n$ (see [3, p.27] or [6, Theorem 2.12]). Each of these subspaces is one-dimensional by Corollary. Therefore, there exist $n^2$ disjoint basic elements.

Theorem 7. Let $(F_n, P)$ be an $\ell$-algebra over $F$. If for some vector $v \in F^n$, $O = Pv$ is a $P$-invariant cone in $F^n$, then there exists a finite subset $K \subseteq F^n$ such that $O = \text{cone}_F(K)$.

Proof. We know that $F_n$ is a vector lattice direct sum of finitely many totally ordered subspaces, say, $F_n = \bigoplus_{i=1}^n T_i$. For every $i$, $T_i v$ is a totally ordered subspace of $(F^n, O)$ with the positive cone $(T_i)_+ v$. Since $(T_i v)^+ \subseteq \mathcal{O}$ and $-(T_i v)^+ \subseteq -O$, $(T_i v)^+ \cap -(T_i v)^+ \subseteq \mathcal{O} \cap -O = \{0\}$ by Lemma 8. So by Lemma 4, $\dim_F(T_i v) \leq 1$. This means that for every $i = 1, \ldots, m$, there exists $f_i \in T_i^+$ such that $T_i v = F f_i v$. Therefore $O = P v = (\sum_{i=1}^m T_i^+) v = \sum_{i=1}^m (T_i^+) v = \sum_{i=1}^m F^+ f_i v = \text{cone}_F(K)$, where $K = \{ f_i v : i = 1, \ldots, m \}$.

Let $O$ be a nontrivial cone in $F^n$. If $O = \text{cone}_F(K)$ for some finite subset $K$ of $F^n$, then $O$ will be called a polyhedral cone. If there exists such a $K$ which has precisely $n$ elements, and they are linearly independent, then $O$ will be called a simplicial cone. In the case $F = \mathbb{R}$, these notions coincide with definitions 2.4 and 2.13 from [1]. If $K$ is a minimal finite set with the property that $O = \text{cone}_F(K)$ for a polyhedral cone $O$, then the vectors from $K$ will be called the edges of $O$. Let $P$ be a directed partial order of $F_n$, and $O$ a nontrivial $P$-invariant cone in $F^n$. Then, by Lemma 2, $O$ contains $n$ linearly independent vectors; therefore, if $O = \text{cone}_F(K)$ for some $K \subseteq F^n$, then $K$ contains at least $n$ elements.

Theorem 8. Let $P$ be a lattice order of $F_n$. Then every nontrivial $P$-invariant cone in $F^n$ contains a minimal nontrivial $P$-invariant cone in $F^n$. In particular, there exists a minimal nontrivial $P$-invariant cone.

Proof. First, we consider a chain $\{ P v_\alpha \}$ of nontrivial $P$-invariant cones $P v_\alpha$ and show that $\bigcap \{ P v_\alpha \}$ is a nontrivial $P$-invariant cone. Let $S$ denote the unit sphere in $\mathbb{R}^n$. The family $\{ P v_\alpha \cap S \}$ is a chain of closed nonempty sets in the space $S$, and therefore their intersection $\bigcap_\alpha (P v_\alpha \cap S) \neq \emptyset$ since $S$ is compact. Therefore, $\bigcap_\alpha P v_\alpha \neq \{0\}$. Since this set is $P$-invariant, it has a subset $K$ of $n$ linearly independent vectors from $\mathbb{R}^n$ over $\mathbb{R}$ by Lemma 2. By Theorem 7, for each $\alpha$, $P v_\alpha = \text{cone}_F(K_\alpha)$ for some finite subset $K_\alpha \subseteq F^n$. It is clear that $P v_\alpha = \text{cone}_F(K_\alpha) = \text{cone}_\mathbb{R}(K_\alpha)$ and $P v_\alpha \cap \mathbb{R}^n = \text{cone}_\mathbb{R}(K_\alpha) \cap \mathbb{R}^n = \text{cone}_F(K_\alpha)$ since a polyhedral cone containing $n$ linearly independent vectors is a union of simplicial cones and the equality is clearly true for a simplicial cone. Thus $\text{cone}_\mathbb{R}(K) \cap \mathbb{R}^n \subseteq P v_\alpha \cap \mathbb{R}^n = \text{cone}_F(K_\alpha) = P v_\alpha$, for every $P v_\alpha$, and hence $\text{cone}_\mathbb{R}(K) \cap \mathbb{R}^n \subseteq \bigcap_\alpha P v_\alpha$, so $\bigcap_\alpha P v_\alpha \neq \{0\}$.

Now let $O$ be a nontrivial $P$-invariant cone in $F^n$. Let $0 \neq v \in O$. Then $P v$ is contained in $O$, so $P v$ is a nontrivial $P$-invariant cone. By the above argument and Zorn’s Lemma, $P v$ contains a minimal nontrivial $P$-invariant cone, so $O$ contains a minimal nontrivial $P$-invariant cone. Finally, because of Theorem 1, there exists a minimal nontrivial $P$-invariant cone in $F^n$. 

\[ \square \]
Lemma 9. Let $O$ be a polyhedral cone in $\mathbb{F}^n$, and let $k$ be an edge of $O$. If $0 < u < k$ for some $u \in \mathbb{F}^n$, then $u = ak$ for some $a \in \mathbb{F}^+$. In particular, the edges of $O$ are basic elements of the partial order $O$ in $\mathbb{F}^n$.

Proof. Let $K = \{k, k_1, \ldots, k_m\}$ be minimal with the property $O = \text{cone}_\mathbb{F}(K)$. Then, $u = ak + \sum_{j=1}^m \alpha_j k_j$ and $k - u = \beta k + \sum_{j=1}^m \beta_j k_j$, where $\alpha, \beta, \alpha_j, \beta_j \in \mathbb{F}^+$ for $j = 1, \ldots, m$. Therefore, $k = (\alpha + \beta)k + \sum_{j=1}^m (\alpha_j + \beta_j)k_j$. Since $K$ is minimal, we must have $\alpha + \beta = 1$, and $\alpha_j = \beta_j = 0$ for $j = 1, \ldots, m$. So $u = ak$. □

Theorem 10. Let $(\mathbb{F}_n, P)$ be an $\ell$-algebra over $\mathbb{F}$ and let $O$ be a minimal nontrivial $P$-invariant cone in $\mathbb{F}^n$. Then:

1. $O$ is a lattice order in $\mathbb{F}^n$.
2. $O$ is a simplicial cone in $\mathbb{F}^n$, so $O$ has exactly $n$ disjoint edges.

Proof. (1) By Lemma 2, $O$ contains $n$ linearly independent vectors $v_i$, $i = 1, \ldots, n$, over $\mathbb{F}$. Let $v = \sum v_i$. Then $Pv = 0$ from the minimality of $O$, and hence $O$ is a polyhedral cone by Theorem 7. Let $k$ be an edge of $O$. There exists a matrix $f_k \in P$ such that $k = f_k v = \sum f_k v_i$. Since $k$ is an edge of $O$, by Lemma 9, we have that for all $i = 1, \ldots, n$, $f_k v_i = \alpha_i k$ for some scalars $\alpha_i \geq 0$. Consequently, $\text{rank}(f_k) = 1$. Thus the null space $N$ of $f_k$ has dimension $n - 1$. By the proof of Theorem 11, $O' = \{fv : f \in P \text{ and } fN = 0\}$ is a nontrivial $P$-invariant cone. Obviously, $O' \subseteq P v$, and we conclude that $O = O' = P v$.

Since $O$ contains $n$ linearly independent vectors, it is a directed partial order in $\mathbb{F}^n$. Let $u \in \mathbb{F}^n$. There exist $u_1, u_2 \in O$ such that $u = u_1 - u_2$. Let $u_1 = f_1 v$ and $u_2 = f_2 v$ for some $f_1, f_2 \in P$ and $f_1 N = f_2 N = 0$. Then $u = (f_1 - f_2) v$. Let $f = f_1 - f_2$. Then $f^+ v \in O$ since $f^+ \in P$. Similarly $(f^+ - f)v = f^- v \in O$. Thus $f^+ v$ is an upper bound for both $0$ and $u$. Below we show that $f^+ v$ is the least upper bound of $0$ and $u$.

Let $0, u \leq w$. Then $w \in O$, so $w = gv$ for some $g \in P$ with $gN = 0$. Similarly $w - u \in O$ implies $w - u = hv$ for some $h \in P$ with $hN = 0$. Thus $w - u = gw - fv = hv$, so $(g - f)v = hv$ and also $(g - f)N = hN = 0$. Therefore, $g - f = h$ since $v \not\in N$ and $N$ has dimension $n - 1$. Since $h \in P$, we have $g = h + f \geq f$, and since $g \in P$, we obtain $g \geq f^+$. Thus $g - f^+ \in P$, and hence $(g - f^+)v \in O$; so $gv - f^+ v \geq 0$; i.e., $w \geq f^+ v$.

(2) By (1) $O$ is a lattice order of $\mathbb{F}^n$. By Lemma 3 the edges of $O$ are basic elements in this lattice order, and therefore they are linearly independent over $\mathbb{F}$. Therefore there are at most $n$ disjoint edges, and thus $O$ has exactly $n$ disjoint edges. □

Theorem 11. Every $\ell$-algebra $(\mathbb{F}_n, P_1)$ is isomorphic to an $\ell$-algebra $(\mathbb{F}_n, P_2)$ with $P_2 \subseteq (\mathbb{F}^+)_n$, and $(\mathbb{F}^+)^n$ is a minimal $P_2$-invariant cone.

Proof. Let $O_1$ be a nontrivial $P_1$-invariant cone in $\mathbb{F}^n$. By Theorem 11, $O_1$ is simplicial. Let $k_1, \ldots, k_n$ be the edges of $O_1$. Let $\psi$ be the $n \times n$ matrix with the columns $k_1, \ldots, k_n$. Since the edges form a linearly independent set, $\psi$ is nonsingular and yields both an algebra isomorphism $\mathbb{F}^n \to \psi^{-1} \mathbb{F}_n \psi$ and a vector space isomorphism $\mathbb{F}^n \to \psi^{-1} \mathbb{F}^n$. Let $P_2 = \psi^{-1} P_1 \psi$. Then $(\mathbb{F}_n, P_2)$ becomes an $\ell$-algebra which is isomorphic to $(\mathbb{F}_n, P_1)$. If we let $O_2 = \psi^{-1} O_1$, then $O_2 = \psi^{-1} (F^+ k_1 + \ldots + F^+ k_n) = \psi^{-1} (F^+)^n = (F^+)^n$, and it is clear that $O_2$ is a minimal $P_2$-invariant cone. Finally, $P_2 \subseteq (F^+)_n$ since $(F^+)^n$ is $P_2$-invariant. □
Given $f \in \mathbb{F}_n$, the transpose of $f$ is denoted by $f^T$. It is well-known that if $P$ is a partial order (lattice order) of $\mathbb{F}_n$, then so is $P^T = \{f^T : f \in P\}$.

**Theorem 12.** Let $(\mathbb{F}_n, P)$ be an $\ell$-algebra over $\mathbb{F}$ such that $1 \in P$. Then $(\mathbb{F}_n, P)$ is isomorphic to the $\ell$-algebra $(\mathbb{F}_n^\ell, (\mathbb{F}^\ell)^n)$.

**Proof.** By Theorem 11 we may assume that $P \subseteq (\mathbb{F}^\ell)_n$ and $O = (\mathbb{F}^\ell)^n$ is a minimal nontrivial $P$-invariant cone. We show that $P = (\mathbb{F}^\ell)_n$. Let $v \in O$ be any vector with all components strictly positive. Then $Pv \neq \{0\}$, and since $O$ is minimal, $Pv = O$. Hence there are $n$ matrices $f_i \in P$ such that $f_i v = e_i$, $i = 1, \ldots, n$, where $e_i$ denotes the vector in which the $i^{th}$ component is 1 and the other components are zero. Since every $f_i$, $i = 1, \ldots, n$, has all its entries nonnegative, all rows of $f_i$ except the $i^{th}$ one are zero rows. It follows that $f_i^T e_j = 0$ for $i \neq j$, and $f_i^T v \neq 0$, $i, j = 1, \ldots, n$. Let $N_i$ denote the subspace of $\mathbb{F}^n$ spanned by $\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\}$. Then for every $i = 1, \ldots, n$, $f_i^T N_i = 0$. Now $(\mathbb{F}_n, P^T)$ is an $\ell$-algebra over $\mathbb{F}$. It follows from the proof of Theorem 11 that the sets $O_i = \{f^T v : f_i^T N_i = 0, f \in P\}$, $i = 1, \ldots, n$, are nontrivial $P^T$-invariant cones. By Lemma 2 each $O_i$ contains $n$ linearly independent vectors. For each $1 \leq i \leq n$, suppose $f^T_i v, j = 1, \ldots, n$, are linearly independent vectors in $O_i$, where $f^T_i \in P$ and $f_i^T N_i = 0$. Then it is clear that for each $i$, $f_i^T$, $j = 1, \ldots, n$, are linearly independent over $\mathbb{F}$, and all rows of $f^T_i$ except the $i^{th}$ one are zero rows. Then it follows that the whole set $\{f^T_i : 1 \leq i, j \leq n\}$ is linearly independent over $\mathbb{F}$. Since $P \subseteq (\mathbb{F}^\ell)_n$, by Corollary 1 in (\mathbb{F}, P) contains a set $G$ of $n^2$ disjoint basic elements. Then $P = \sum_{g \in G} \mathbb{F}^\ell g$.

Since $\{f^T_i : 1 \leq i, j \leq n\} \subseteq P$, each $f^T_i$ is a nonnegative linear combination of elements in $G$. Since $\{f^T_i : 1 \leq i, j \leq n\}$ is linearly independent, each $g \in G$ must appear in at least one such linear combination with a nonzero coefficient from $\mathbb{F}^\ell$. Keeping in mind that all entries of the matrices $g \in G$ are nonnegative numbers, we conclude that each $g \in G$ has all but one row zero.

Now let $G = \{g_{ij} : 1 \leq i, j \leq n\}$. Then for every pair of indices $i, j = 1, \ldots, n$, there exists a matrix from $G$, say, $g_{ij}$, whose $ij^{th}$ entry is not 0. Moreover, since $1 \in P$, for each $i = 1, \ldots, n$ there is a matrix from $G$ which we will, without loss of generality, call $g_{ii}$, such that it has only a nonzero $ii^{th}$ entry. Let $e_{ij}, i, j = 1, \ldots, n$, denote the usual matrix units in $\mathbb{F}_n$. Then $g_{ii} = \lambda_{ii} e_{ii}$ for some $0 < \lambda_{ii} \in \mathbb{F}$, for every $i = 1, \ldots, n$. Therefore, for every $i = 1, \ldots, n$, the matrices $e_{ii} \in P$. Now for every $i, j = 1, \ldots, n$, $g_{ij} e_{jj} = \lambda_{ij} e_{ij}$ for some $0 < \lambda_{ij} \in \mathbb{F}$. Thus $\{e_{ij} : i, j = 1, \ldots, n\} \subseteq P$, and hence $P = (\mathbb{F}^\ell)_n$.  

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