CONTINUITY AND DIFFERENTIABILITY
FOR WEIGHTED SOBOLEV SPACES

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ABSTRACT. Our aim in this paper is to discuss continuity and differentiabil-
ity of functions in weighted Sobolev spaces in the limiting case of Sobolev’s
imbedding theorem.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $X$ be a metric space with a metric $d$. We denote by $B(x, r)$ the open ball
centered at $x \in X$ with radius $r > 0$. For $\sigma > 0$, we write

$$\sigma B(x, r) = B(x, \sigma r).$$

Let $\mu$ be a Borel measure on $X$. Assume that there exist constants $C > 0$ and
$s \geq 1$ such that

$$\frac{\mu(B')}{\mu(B)} \geq C \left(\frac{r'}{r}\right)^s$$

for all balls $B = B(x, r)$ and $B' = B(x', r')$ with $x' \in B$ and $0 < r' \leq r$. Note that
$\mu$ is a doubling measure on $X$, that is, there exists a constant $C' > 0$ such that

$$\mu(B(x, 2r)) \leq C' \mu(B(x, r))$$

for all $x \in X$ and $r > 0$.

We say that a pair $(u, g)$ of functions in $L^p_{\text{loc}}(X; \mu)$ satisfies a $p$-Poincaré inequality (on rings), $1 \leq p < \infty$, if for every $c_1$ and $c_2$ with $c_2 > c_1 > 1$ there are constants $M > 0$ and $\sigma \geq 1$ such that

$$\int_{A(r, r')} |u(y) - u_{A(r, r')}| \, d\mu \leq M r \left( \int_{\sigma A(r, r')} |g|^p \, d\mu \right)^{1/p}$$

whenever $c_1 r' < r < c_2 r'$, where $A(r, r') = B(x, r) - B(x, r')$, $\sigma A(r, r') = B(x, \sigma r) - B(x, \sigma^{-1} r')$ and

$$u_G = \frac{1}{\mu(G)} \int_G u \, d\mu$$

for Borel sets $G \subset X$. If (1.3) holds for $r' = 0$, then the pair $(u, g)$ is said to
satisfy the usual $p$-Poincaré inequality on balls. Under certain assumptions, the
usual $p$-Poincaré inequality on balls implies our $p$-Poincaré inequality; see e.g. [2 Theorem 9.7].

A positive measurable function $w$ on $\mathbb{R}^n$ is called an $A_p$ weight (written as $w \in (A_p)$) if there exists a positive constant $C_p$ such that

$$
\int_B w(x) \, dx \left( \int_B w(x)^{1/(1-p)} \, dx \right)^{p-1} \leq C_p \quad (\leq \infty)
$$

for all balls $B$, where $1 < p < \infty$ and

$$
\int_B w(x) \, dx = \frac{1}{\mathcal{L}^n(B)} \int_B w(x) \, dx
$$

with $\mathcal{L}^n$ denoting the $n$-dimensional Lebesgue measure; we say that $w$ is an $A_1$ weight if there exists a positive constant $C_1$ such that

$$
\int_B w(x) \, dx \leq C_1 \, \text{ess inf}_B \, w
$$

for all balls $B$. Note that if $w$ is an $A_p$ weight, then $d\mu = w \, dx$ satisfies the decay condition [11] with $s = np$, due to [3] Section 15.5.

In view of [3] Section 15.26, we can prove that the $p$-Poincaré inequality is valid for $(u, |\nabla u|)$ and $d\mu = w \, dx$ with $u \in W^{1,p}(\mathbb{R}^n; \mu)$ and $w \in (A_p)$, where $W^{1,p}(\mathbb{R}^n; \mu)$ denotes the weighted Sobolev space, which is

$$
W^{1,p}(\mathbb{R}^n; \mu) = \{ u \in L^p(\mathbb{R}^n; \mu) : |\nabla u| \in L^p(\mathbb{R}^n; \mu) \};
$$

for this fact, see Section 3 below.

Recently Björn ([1] Theorem 1.5 and Theorem 3.1) proved the following theorem:

**Theorem A.** Let $\mu$ be a Borel measure on $\mathbb{R}^n$ satisfying [1.1]. Let $u$ be a function in the Sobolev space $W^{1,p}(\mathbb{R}^n; \mu)$ such that $(u, |\nabla u|)$ satisfies the usual $p$-Poincaré inequality on balls. If $p > s$, then $u$ can be modified on a set of $\mu$-measure zero so that it is locally Hölder continuous in $\mathbb{R}^n$ and totally differentiable $\mu$-a.e. in $\mathbb{R}^n$.

In case $\mu$ is the Lebesgue measure in $\mathbb{R}^n$, the proof of Theorem A is given in Stein [3] Theorem 1, Chap. VIII. Our aim in this paper is to discuss the Hölder continuity and differentiability of Sobolev functions in the limiting case $p = s$.

To do so, consider a positive function $\Phi_p(r)$ on $\mathbb{R}_+ = [0, \infty)$ with the following properties:

1. $\Phi_p(r)$ is of the form $r^p \varphi(r)$, where $1 < p < \infty$ and $\varphi$ is a positive nondecreasing function on $(0, \infty)$. The value $\Phi_p(0)$ is defined to be zero.
2. There exists $c > 1$ such that
   $$
c^{-1} \varphi(r) \leq \varphi(r^2) \leq c \varphi(r) \quad \text{whenever } r > 0.
$$
3. $\varphi^*(1) < \infty$, where
   $$
   \varphi^*(r) = \left( \int_0^r \left[ \varphi(t^{-1}) \right]^{-p'/p} t^{-1} \, dt \right)^{1/p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1.
   $$

Typical examples of $\varphi$ are

$$
[\log(1 + r)]^\delta, \quad [\log(1 + r)]^{p-1}[\log(1 + \log(1 + r))]^\delta, \quad \cdots
$$

for $\delta > p - 1$.

Our first aim is to prove the following result (cf. [1] Theorem 3.1, [2] Theorem 5.1]).
Theorem 1. Let $X$ be a connected metric space and let $\mu$ be a Borel measure on $X$ satisfying the decay condition (1.1) with $s = p$, $1 < p < \infty$. Assume that a pair $(u, g)$ satisfies the $p$-Poincaré inequality in $X$ and

\begin{equation}
\int_X \Phi_p(|g(x)|) \, d\mu(x) < \infty.
\end{equation}

Then $u$ can be modified on a set of $\mu$-measure zero so that it is locally $\varphi^*$-Hölder continuous on $X$. Moreover, $u$ satisfies

$$|u(x) - u(y)| \leq M r_0 \varphi^*(r) \left( \int_{cB_0} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon}$$

for all $x, y \in B = B(x_0, r)$ and $B_0 = B(x_0, r_0)$ with $0 < r < r_0$, where $c = 2^4 + 1 = 17$, $0 < \varepsilon < 1$ and $M$ is a positive constant depending on $\varepsilon$.

Corollary 1. Let $1 < p < \infty$ and $w \in (A_p)$. Assume further that $\mu$ satisfies (1.1) with $s = p$, where $d\mu = wdx$. Let $u$ be a function in $W^{1,p}(\mathbb{R}^n; \mu)$ satisfying

\begin{equation}
\int_{\mathbb{R}^n} \Phi_p(|\nabla u(x)|) \, d\mu(x) < \infty.
\end{equation}

Then $u$ can be modified on a set of measure zero so that it becomes a locally $\varphi^*$-Hölder continuous function on $\mathbb{R}^n$ satisfying

$$|u(x) - u(y)| \leq M r_0 \varphi^*(r) \left( \int_{cB_0} \Phi_p(|\nabla u|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon}$$

for all $x, y \in B = B(x_0, r)$ and $B_0 = B(x_0, r_0)$ with $0 < r < r_0$, where $c = 2^4 + 1 = 17$, $0 < \varepsilon < 1$ and $M$ is a positive constant depending on $\varepsilon$.

In view of [4, Theorem 1], we know that the Riesz potentials $U_\alpha f$ are locally $\varphi^*$-Hölder continuous on $\mathbb{R}^n$ when $U_\alpha |f| \neq \infty$ and $\int_{\mathbb{R}^n} \Phi_p(|f(y)|) \, dy < \infty$ with $p = n/\alpha > 1$. In particular, if $u \in W^{1,n}(\mathbb{R}^n)$ satisfies (1.1) with $\mu = \mathcal{L}^n$ and $p = n$, then $u$ can be modified on a set of measure zero so that it becomes locally $\varphi^*$-Hölder continuous on $\mathbb{R}^n$. By [7, Remark 3.3], we know that if $\varphi^*(1) = \infty$, then we can find an $f$ satisfying $U_\alpha |f| \neq \infty$ and $\int_{\mathbb{R}^n} \Phi_p(|f(y)|) \, dy < \infty$ with $p = n/\alpha > 1$ such that $U_\alpha |f(0)| = \infty$, so that $U_\alpha f$ is not continuous at the origin in the usual sense.

We say that a function $u$ on $\mathbb{R}^n$ is totally differentiable at $x_0$ if

$$\lim_{x \to x_0} \frac{|u(x) - u(x_0) - a \cdot (x - x_0)|}{|x - x_0|} = 0$$

for some $a \in \mathbb{R}^n$.

Theorem 2. Let $1 < p < \infty$ and $w \in (A_p)$. Assume further that $\mu$ satisfies (1.1) with $s = p$, where $d\mu = wdx$. Let $u$ be a function in $W^{1,p}(\mathbb{R}^n; \mu)$ satisfying (1.1). Then $u$ can be modified on a set of measure zero so that it becomes totally differentiable a.e. on $\mathbb{R}^n$.

Theorem 2 was proved by the first author [6, Theorem 3.2] in the case $\mu = \mathcal{L}^n$ and $p = n$ (see also [4, Theorem 2] and [7, Theorem 3.2]).
2. Lemmas

Throughout this paper, let $M$ denote various positive constants independent of the variables in question and $M(\varepsilon)$ denote a positive constant which depends on $\varepsilon$.

Let $u \in L^1_{\text{loc}}(X; \mu)$. We say that $x \in X$ is a Lebesgue point of $u$ if

$$
\lim_{r \to 0} \int_{B(x,r)} |u(y) - u(x)| \, d\mu(y) = 0.
$$

**Lemma 1.** Let $\mu$ be a Borel measure on a connected metric space $X$ with the decay condition (1.1), and $u \in L^1_{\text{loc}}(X; \mu)$. If $x \in X$ is a Lebesgue point of $u$ and $0 < c < 1$, then

$$
\lim_{r \to 0} \int_{A(r)} |u(y) - u(x)| \, d\mu(y) = 0,
$$

where $A(r) = B(r) - B(cr)$ with $B(r) = B(x, r)$.

**Proof.** Since $X$ is connected, we can find $y$ such that $d(x, y) = (r + cr)/2 = (1 + c)r/2$. Let $B' = B(y, (1 - c)r/2)$. By the lower bound (1.1), we have

$$
\frac{\mu(B')}{\mu(B(r))} \geq C \left( \frac{(1 - c)r/2}{r} \right)^s = C((1 - c)^2/r)^s.
$$

Since $B' \subset A(r) \subset B(r)$, we have

$$
\int_{A(r)} |u(y) - u(x)| \, d\mu(y) \leq M \int_{B(r)} |u(y) - u(x)| \, d\mu(y) \to 0
$$

as $r \to 0$, by the assumption that $x$ is a Lebesgue point of $u$. Thus the present lemma is obtained.

**Lemma 2.** Let $\mu$ be a Borel measure on $X$. Let $A$ and $B$ be Borel sets such that $A \subset B$ and $0 < \mu(B) \leq M\mu(A)$. Then

$$
|u_A - u_B| \leq M \int_B |u - u_B| \, d\mu.
$$

**Proof.** Since $A \subset B$ and $\mu(B) \leq M\mu(A)$, we have

$$
|u_A - u_B| \leq \int_A |u - u_B| \, d\mu \leq M \int_B |u - u_B| \, d\mu,
$$

as required.

**Corollary 2.** Let $\mu$ be a Borel measure on $X$. Let $A$, $B$ and $C$ be Borel sets such that $A \subset C$, $B \subset C$, $0 < \mu(C) \leq M\mu(A)$ and $0 < \mu(C) \leq M\mu(B)$. Then

$$
|u_A - u_B| \leq 2M \int_C |u - u_C| \, d\mu.
$$

In fact, by Lemma 2, we see that

$$
|u_A - u_B| \leq |u_A - u_C| + |u_B - u_C| \leq 2M \int_C |u - u_C| \, d\mu.
$$

For every ball $B = B(x_0, r)$, $x \in B$, and $\sigma \geq 1$, put $A_i = A_i(x) = B(x, 2^i) - B(x, 2^{i-j_0})$, where $j_0$ is the integer such that $2^{j_0+1} > 4\sigma^2 \geq 2^{j_0}$. Let $i_0$ be the integer such that $\sigma 2^{j_0+1} > 2^{j_0+2}r \geq \sigma 2^{j_0}$.
Lemma 3. Let $\mu$ be a doubling measure on a connected metric space $X$. If a pair $(u, g)$ satisfies the $p$-Poincaré inequality in $X$, then

$$|u(x) - u_{\sigma^{-1}A_{i_0}}| \leq M \sum_{i = -\infty}^{i_0} 2^i \left( \int_{\tilde{A}_i} |g|^p \, d\mu \right)^{1/p}$$

holds for almost every $x \in B$, where $\tilde{A}_i = A_i \cup A_{i-1} = B(x, 2^i) - B(x, 2^{i-j_0-1})$.

Proof. Let $x \in B$ be a Lebesgue point of $u$. Note that $u_{\sigma^{-1}A_i} \to u(x)$ as $i \to -\infty$ by Lemma 1. Using Lemma 2 and the doubling property of $\mu$, we obtain

$$|u(x) - u_{\sigma^{-1}A_{i_0}}| \leq \sum_{i = -\infty}^{i_0} |u_{\sigma^{-1}A_i} - u_{\sigma^{-1}A_{i-1}}| \leq M \sum_{i = -\infty}^{i_0} \int_{\tilde{A}_i} |u - u_{\sigma^{-1}A_i}| \, d\mu.$$ 

By the $p$-Poincaré inequality, we obtain

$$|u(x) - u_{\sigma^{-1}A_{i_0}}| \leq M \sum_{i = -\infty}^{i_0} \sigma^{-1} 2^i \left( \int_{\tilde{A}_i} |g|^p \, d\mu \right)^{1/p},$$

as required.

Lemma 4. Let $\mu$ and $(u, g)$ be as above. Then

$$|u_{\sigma^{-1}A_{i_0}} - u_{\tilde{A}}| \leq M 2^{i_0} \left( \int_{\sigma \tilde{A}} |g|^p \, d\mu \right)^{1/p}$$

holds for almost every $x \in B$ and $\tilde{A} = B(x_0, cr) - B(x_0, r)$ with $c = 2^4 + 1$.

In fact, since $\sigma^{-1}A_{i_0} \subset \tilde{A}$, Lemma 4 follows readily from Lemma 2 and the $p$-Poincaré inequality.

Lemma 5. Let $g$ be a measurable function on $X$ satisfying

$$\int_X \Phi_p(g(y)) \, d\mu(y) < \infty.$$ 

Then

$$\lim_{r \to 0} \int_{B(x, r)} \Phi_p(|g(y) - g(x)|) \, d\mu(y) = 0$$

for $\mu$-a.e. $x \in X$.

To prove this, it suffices to note that

$$\Phi_p(a) \leq \Phi_p(a + b) - \Phi_p(b)$$

for all $a \geq 0$ and $b \geq 0$.

3. Proofs of Theorems 1 and 2

First we collect properties which follow from conditions (ϕ1) and (ϕ2) (see [5 Section 2] and [8 Section 2]).

(ϕ4) The function $\varphi$ satisfies the doubling condition, that is, there exists $c > 1$ such that

$$c^{-1} \varphi(r) \leq \varphi(2r) \leq c \varphi(r) \quad \text{whenever } r > 0.$$

(ϕ5) For any $\gamma > 0$, there exists $c = c(\gamma) \geq 1$ such that

$$c^{-1} \varphi(r) \leq \varphi(r^\gamma) \leq c \varphi(r) \quad \text{whenever } r > 0.$$
(φ6) If γ > 0, then there exists c = c(γ) ≥ 1 such that
\[ s^γ \varphi(s^{-1}) \leq ct^γ \varphi(t^{-1}) \quad \text{whenever } 0 < s < t. \]

Proof of Theorem 1. By Lemma 3, we have for every ball B with radius r, 0 < ε < 1 and δ > 0
\[
|u(x) - u_{σ^{-1}A_0}| \leq M \sum_{i=σ}^{∞} 2^i μ(\tilde{A}_i)^{-1/p} \left( \int_{\tilde{A}_i} |g(y)|^p \, dμ(y) \right)^{1/p} 
\]
By condition (φ5), we see that if |g(y)| > (δ2^{-i})^γ, then
\[ \varphi(|g|) \leq ϕ((δ2^{-i})^γ) \geq M(ε)ϕ(δ2^{-i}). \]
By the lower bound (1.1) with s = p, we have
\[ μ(\tilde{A}_i) \geq Mμ(B) \left( \frac{2^i}{r} \right)^p, \]
so that
\[
u_1 \leq M \sum_{i=σ}^{∞} 2^i μ(\tilde{A}_i)^{-1/p} [ϕ(δ2^{-i})]^{-1/p} \left( \int_{\tilde{A}_i} ϕ_p(|g|) \, dμ \right) 
\]
\[
\leq M \sum_{i=σ}^{∞} 2^i μ(B)^{-1/p} \left( \frac{2^i}{r} \right)^{-1} [ϕ(δ2^{-i})]^{-1/p} \left( \int_{\tilde{A}_i} ϕ_p(|g|) \, dμ \right) 
\]
\[
\leq Mrμ(B)^{-1/p} \sum_{i=σ}^{∞} [ϕ(δ2^{-i})]^{-1/p} \left( \int_{\tilde{A}_i} ϕ_p(|g|) \, dμ \right) 
\]
On the other hand, we have
\[
u_2 \leq M \sum_{i=σ}^{∞} 2^i μ(\tilde{A}_i)^{-1/p} (δ2^{-i})^γ μ(\tilde{A}_i) \]
\[
\leq M \sum_{i=σ}^{∞} (2^i)^{1−ε} δ^ε 
\]
Hence we establish
\[
|u(x) - u_{σ^{-1}A_0}| \leq M rμ(B)^{-1/p} \sum_{i=σ}^{∞} [ϕ(δ2^{-i})]^{-1/p} \left( \int_{\tilde{A}_i} ϕ_p(|g|) \, dμ \right) 
\]
\[ + M r^{1−ε} δ^ε, \]
where M = M(ε) is a positive constant independent of δ, r and x.
From Hölder’s inequality, we obtain
\[ |u(x) - u_{\sigma^{-1}A_{i_0}}| \]
\[ \leq Mr \mu(B)^{-1/p} \left( \sum_{i = -\infty}^{i_0} |\varphi(\delta 2^{-i})|^{-p'/p} \right)^{1/p'} \left( \sum_{i = -\infty}^{i_0} \int_{A_i} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon} \delta^\varepsilon, \]
where \( 1/p + 1/p' = 1 \). By condition (\( \varphi 4 \)), we see that
\[ \left( \sum_{i = -\infty}^{i_0} |\varphi(\delta 2^{-i})|^{-p'/p} \right)^{1/p'} \leq M \varphi^*(2^4 \sigma r \delta^{-1}) \leq M \varphi^*(r \delta^{-1}). \]

Since \( \tilde{A}_i \subset \sigma B \) for \( i \leq i_0 \) and \( c = 2^4 + 1 \), it follows that
\[ |u(x) - u_{\sigma^{-1}A_{i_0}}| \leq Mr \mu(B)^{-1/p} \varphi^*(r \delta^{-1}) \left( \int_{\cup A_i} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon} \delta^\varepsilon \]
\[ \leq Mr \mu(B)^{-1/p} \varphi^*(r \delta^{-1}) \left( \int_{\cup \sigma B} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon} \delta^\varepsilon \]
\[ \leq Mr \varphi^*(r \delta^{-1}) \left( \int_{\cup \sigma B} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon} \delta^\varepsilon. \]

Next we estimate \( |u_{\sigma^{-1}A_{i_0}} - u_{\tilde{A}}| \) by Lemma 4. Noting that by condition (\( \varphi 4 \))
\[ \varphi^*(\delta^{-1}r) \geq \left( \int_{\frac{\delta^{-1}}{4}}^{\delta^{-1}r} |\varphi(t^{-1})|^{-\sigma r^{-1}} dt \right)^{1/p'} \geq M[\varphi(\delta r^{-1})]^{-1/p}, \]
we have
\[ |u_{\sigma^{-1}A_{i_0}} - u_{\tilde{A}}| \leq M2^{i_0} \left( \int_{\sigma \tilde{A}} |g|^p \, d\mu \right)^{1/p} \]
\[ \leq Mr[\varphi(\delta r^{-1})]^{-1/p} \left( \int_{\cup \sigma B} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon} \delta^\varepsilon \]
\[ \leq Mr \varphi^*(\delta^{-1}r) \left( \int_{\cup \sigma B} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon} \delta^\varepsilon. \]

By the lower bound (1.1), we have
\[ |u(x) - u(y)| \leq |u(x) - u_{\sigma^{-1}A_{i_0}(x)}| + |u_{\sigma^{-1}A_{i_0}(x)} - u_{\tilde{A}}| + |u_{\tilde{A}} - u_{\sigma^{-1}A_{i_0}(y)}| + |u_{\sigma^{-1}A_{i_0}(y)} - u(y)| \]
\[ \leq Mr \varphi^*(\delta^{-1}r) \left( \int_{\cup \sigma B} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon} \delta^\varepsilon \]
\[ \leq Mr \varphi^*(\delta^{-1}r) \left( \frac{\mu(\sigma B)}{\mu(c \sigma B_0)} \right)^{-1/p} \left( \int_{\cup \sigma B_0} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon} \delta^\varepsilon \]
\[ \leq Mr \varphi^*(\delta^{-1}r) \left( \frac{r}{r_0} \right)^{-1} \left( \int_{\cup \sigma B_0} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon} \delta^\varepsilon \]
\[ \leq Mr_0 \varphi^*(\delta^{-1}r) \left( \int_{\cup \sigma B_0} \Phi_p(|g|) \, d\mu \right)^{1/p} + Mr^{1-\varepsilon} \delta^\varepsilon. \]

(3.1)

The proof of Theorem 1 is completed by taking \( \delta = 1 \).
Hence it follows from (3.2) that

\[ |u(x) - b| \leq \frac{1}{E^n(B')} \int_{B'} |u(x) - u(y)| \, dy \]

(3.2)

\[ \leq M \int_A |x - y|^{1-n} |\nabla u(y)| \, dy \]

for a.e. \( x \in A \), where \( b = (1/E^n(B')) \int_{B'} u(y) \, dy \). If \( B' = B(x', r') \) with \( x' \in A = B(x_0, r) - B(x_0, \sqrt{2}r') \), then we can also show that

\[ |u(x) - b'| \leq \frac{1}{E^n(B' \cap A)} \int_{B' \cap A} |u(x) - u(y)| \, dy \]

\[ \leq M \int_A |x - y|^{1-n} |\nabla u(y)| \, dy \]

for a.e. \( x \in B(x', r') - B(x_0, \sqrt{5}r') \), where \( b' = (1/E^n(B' \cap A)) \int_{B' \cap A} u(y) \, dy \). Hence it follows from (3.2) that

\[ |u(x) - b'| \leq \frac{1}{E^n(A)} \int_A |u(x) - u(y)| \, dy \]

\[ \leq M \int_A |x - y|^{1-n} |\nabla u(y)| \, dy \]

for a.e. \( x \in A \), where \( a = (1/E^n(A)) \int_A u(y) \, dy \). In view of [3] Section 15.23, we establish

\[ \int_A |u - u_A| \, d\mu \leq 2 \int_A |u(x) - a| \, d\mu(x) \]

(3.3)

\[ \leq Mr \left( \int_A |\nabla u|^p \, d\mu \right)^{1/p}, \]

which implies the \( p \)-Poincaré inequality in our sense. Thus the case \( n \geq 2 \) follows from Theorem 1.

In the case \( n = 1 \), we apply the usual Poincaré inequality on intervals. We have

\[ |u(x) - u_{I_i}| \leq M \sum_{i=-\infty}^{i_0} 2^i \mu(I_i)^{-1/p} \left( \int_{I_i} |\nabla u|^p \, d\mu \right)^{1/p}, \]

where \( I_i \) denotes the interval \((x + 2^{i-1}, x + 2^i)\) or \((x - 2^i, x - 2^{i-1})\). In the same way as in the proof of Theorem 1, we finally obtain

\[ |u(x) - u(y)| \leq Mr_0 \varphi^*(\delta^{-1}r) \left( \int_{B_0} \Phi_p(|\nabla u|) \, d\mu \right)^{1/p} + Mr_1 \delta^\varphi \]
whenever $\delta > 0$ and $x, y \in I = (x_0 - r, x_0 + r) \subset (x_0 - r_0, x_0 + r_0) = B_0$, where $0 < \varepsilon < 1$ and $M$ is a positive constant (which may depend on $\varepsilon$). Hence the case $n = 1$ is proved similarly.

**Proof of Theorem 2.** Modify $u$ as in Theorem 1 so that it is locally $\varphi^*$-Hölder continuous on $\mathbb{R}^n$. Let $x_0$ be a point such that

$$
\lim_{r \to 0} \frac{1}{r} \int_{B(x_0, r)} \Phi_p(|\nabla u(x)|) \, d\mu(x) = 0.
$$

By Lemma 5, almost every $x \in \mathbb{R}^n$ has this property. Set $v(x) = u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)$. Then $\nabla v(x) = \nabla u(x) - \nabla u(x_0)$, and (3.5) yields

$$
\lim_{r \to 0} \frac{1}{r} \int_{B(x_0, r)} \Phi_p(|\nabla v(x)|) \, d\mu(x) = 0.
$$

Applying (3.1) in the proof of Theorem 1 with $v$, for $x \in B(x_0, r)$, $0 < \varepsilon < 1$ and $\delta > 0$, we have

$$
\frac{|v(x)|}{r} \leq M \varphi^*(\delta^{-1} r) \left( \int_{c r B} \Phi_p(|\nabla v|) \, d\mu \right)^{1/p} + M r^{-\varepsilon} \delta^\varepsilon,
$$

where $c = 2^d + 1$ and $M = M(\varepsilon)$ does not depend on $\delta$, $r$ and $x$. Here, taking $\delta^{-1} r = N$ with a positive integer $N$, we obtain

$$
\frac{|v(x)|}{r} \leq M \varphi^*(N) \left( \int_{c r B} \Phi_p(|\nabla v|) \, d\mu \right)^{1/p} + M N^{-\varepsilon}.
$$

Hence it follows from (3.6) that

$$
\limsup_{x \to x_0} \frac{|v(x)|}{|x - x_0|} \leq M N^{-\varepsilon}.
$$

Letting $N \to \infty$, we see that the left side is equal to zero, which implies that $v$ is totally differentiable at $x_0$.

**Remark.** The proof of Theorem 1 implies that

$$
|u(x) - u(x_0)| \leq M |x - x_0| \left( \sup_{0 < r < \varepsilon r_0} \int_{B(x_0, r)} \Phi_p(|g|) \, d\mu \right)^{1/p} + M |x - x_0|
$$

for all $x \in B(x_0, r_0)$. This fact completes the proof of Theorem 2 with the aid of Rademacher-Stepanov theorem (cf. [9, Theorem 3, Chap. VIII]).

**References**


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