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CONTINUITY AND DIFFERENTIABILITY FOR WEIGHTED SOBOLEV SPACES

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Abstract. Our aim in this paper is to discuss continuity and differentiability of functions in weighted Sobolev spaces in the limiting case of Sobolev's imbedding theorem.

1. Introduction and statement of results

Let X be a metric space with a metric d. We denote by $B(x,r)$ the open ball centered at $x \in X$ with radius $r > 0$. For $\sigma > 0$, we write

$$\sigma B(x,r) = B(x,\sigma r).$$

Let μ be a Borel measure on X. Assume that there exist constants $C > 0$ and $s \geqq 1$ such that

$$\frac{\mu(B')}{\mu(B)} \geqq C\left(\frac{r'}{r}\right)^s \tag{1.1}$$

for all balls $B = B(x,r)$ and $B' = B(x',r')$ with $x' \in B$ and $0 < r' \leqq r$. Note that μ is a doubling measure on X, that is, there exists a constant $C' > 0$ such that

$$\mu(B(x,2r)) \leqq C'\mu(B(x,r)) \tag{1.2}$$

for all $x \in X$ and $r > 0$.

We say that a pair (u,g) of functions in $L^p_{loc}(X;\mu)$ satisfies a p-Poincaré inequality (on rings), $1 \leqq p < \infty$, if for every c_1 and c_2 with $c_2 > c_1 > 1$ there are constants $M > 0$ and $\sigma \geqq 1$ such that

$$\fint_{A(r,r')} |u(y) - u_{A(r,r')}|\, d\mu \leqq Mr\left(\fint_{\sigma A(r,r')} |g|^p\, d\mu\right)^{1/p} \tag{1.3}$$

whenever $c_1 r' < r < c_2 r'$, where $A(r,r') = B(x,r) - B(x,r')$, $\sigma A(r,r') = B(x,\sigma r) - B(x,\sigma^{-1}r')$ and

$$u_G = \fint_G u\, d\mu = \frac{1}{\mu(G)}\int_G u\, d\mu$$

for Borel sets $G \subset X$. If (1.3) holds for $r' = 0$, then the pair (u,g) is said to satisfy the usual p-Poincaré inequality on balls. Under certain assumptions, the

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usual p-Poincaré inequality on balls implies our p-Poincaré inequality; see e.g. [2, Theorem 9.7].

A positive measurable function w on $\mathbf{R}^n$ is called an A_p weight (written as $w \in (A_p)$) if there exists a positive constant C_p such that

$$\fint_B w(x)\ dx \left(\fint_B w(x)^{1/(1-p)}\ dx\right)^{p-1} \leqq C_p \quad (<\infty)$$

for all balls B, where $1 < p < \infty$ and

$$\fint_B w(x)\ dx = \frac{1}{\mathcal{L}^n(B)} \int_B w(x)\ dx$$

with $\mathcal{L}^n$ denoting the n-dimensional Lebesgue measure; we say that w is an A_1 weight if there exists a positive constant C_1 such that

$$\fint_B w(x)\ dx \leqq C_1 \text{ ess inf}_B\ w$$

for all balls B. Note that if w is an A_p weight, then $d\mu = wdx$ satisfies the decay condition (1.1) with $s = np$, due to [3, Section 15.5].

In view of [3, Section 15.26], we can prove that the p-Poincaré inequality is valid for $(u, |\nabla u|)$ and $d\mu = wdx$ with $u \in W^{1,p}(\mathbf{R}^n; \mu)$ and $w \in (A_p)$, where $W^{1,p}(\mathbf{R}^n; \mu)$ denotes the weighted Sobolev space, which is

$$W^{1,p}(\mathbf{R}^n; \mu) = \{u \in L^p(\mathbf{R}^n; \mu) : |\nabla u| \in L^p(\mathbf{R}^n; \mu)\};$$

for this fact, see Section 3 below.

Recently Björn ([1, Theorem 1.5 and Theorem 3.1]) proved the following theorem:

Theorem A. *Let μ be a Borel measure on $\mathbf{R}^n$ satisfying (1.1). Let u be a function in the Sobolev space $W^{1,p}(\mathbf{R}^n; \mu)$ such that $(u, |\nabla u|)$ satisfies the usual p-Poincaré inequality on balls. If $p > s$, then u can be modified on a set of μ-measure zero so that it is locally Hölder continuous in $\mathbf{R}^n$ and totally differentiable μ-a.e. in $\mathbf{R}^n$.*

In case μ is the Lebesgue measure in $\mathbf{R}^n$, the proof of Theorem A is given in Stein [9, Theorem 1, Chap. VIII]. Our aim in this paper is to discuss the Hölder continuity and differentiability of Sobolev functions in the limiting case $p = s$. To do so, consider a positive function $\Phi_p(r)$ on $\mathbf{R}_+ = [0, \infty)$ with the following properties:

(φ1) $\Phi_p(r)$ is of the form $r^p\varphi(r)$, where $1 < p < \infty$ and φ is a positive nondecreasing function on $(0, \infty)$. The value $\Phi_p(0)$ is defined to be zero.

(φ2) There exists $c > 1$ such that

$$c^{-1}\varphi(r) \leqq \varphi(r^2) \leqq c\varphi(r) \qquad \text{whenever } r > 0.$$

(φ3) $\varphi^*(1) < \infty$, where

$$\varphi^*(r) = \left(\int_0^r [\varphi(t^{-1})]^{-p'/p} t^{-1} dt\right)^{1/p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Typical examples of φ are

$$[\log(1+r)]^\delta, \quad [\log(1+r)]^{p-1}[\log(1+\log(1+r))]^\delta, \quad \cdots$$

for $\delta > p - 1$.

Our first aim is to prove the following result (cf. [1, Theorem 3.1], [2, Theorem 5.1]).

Theorem 1. *Let X be a connected metric space and let μ be a Borel measure on X satisfying the decay condition (1.1) with $s = p$, $1 < p < \infty$. Assume that a pair (u, g) satisfies the p-Poincaré inequality in X and*

$$\int_X \Phi_p(|g(x)|)\ d\mu(x) < \infty. \tag{1.4}$$

Then u can be modified on a set of μ-measure zero so that it is locally φ^*-Hölder continuous on X. Moreover, u satisfies

$$|u(x) - u(y)| \leqq M r_0\ \varphi^*(r) \left(\fint_{c\sigma B_0} \Phi_p(|g|)\ d\mu \right)^{1/p} + M r^{1-\varepsilon}$$

for all x, $y \in B = B(x_0, r)$ and $B_0 = B(x_0, r_0)$ with $0 < r < r_0$, where $c = 2^4 + 1 = 17$, $0 < \varepsilon < 1$ and M is a positive constant depending on ε.

Corollary 1. *Let $1 < p < \infty$ and $w \in (A_p)$. Assume further that μ satisfies (1.1) with $s = p$, where $d\mu = w dx$. Let u be a function in $W^{1,p}(\mathbf{R}^n; \mu)$ satisfying*

$$\int_{\mathbf{R}^n} \Phi_p(|\nabla u(x)|)\ d\mu(x) < \infty. \tag{1.5}$$

Then u can be modified on a set of measure zero so that it becomes a locally φ^*-Hölder continuous function on $\mathbf{R}^n$ satisfying

$$|u(x) - u(y)| \leqq M r_0\ \varphi^*(r) \left(\fint_{cB_0} \Phi_p(|\nabla u|)\ d\mu \right)^{1/p} + M r^{1-\varepsilon}$$

for all x, $y \in B = B(x_0, r)$ and $B_0 = B(x_0, r_0)$ with $0 < r < r_0$, where $c = 2^4 + 1 = 17$, $0 < \varepsilon < 1$ and M is a positive constant depending on ε.

In view of [4, Theorem 1], we know that the Riesz potentials $U_\alpha f$ are locally φ^*-Hölder continuous on R^n when $U_\alpha |f| \not\equiv \infty$ and $\displaystyle\int_{\mathbf{R}^n} \Phi_p(|f(y)|)\ dy < \infty$ with $p = n/\alpha > 1$. In particular, if $u \in W^{1,n}(\mathbf{R}^n)$ satisfies (1.5) with $\mu = \mathcal{L}^n$ and $p = n$, then u can be modified on a set of measure zero so that it becomes locally φ^*-Hölder continuous on $\mathbf{R}^n$. By [7, Remark 3.3], we know that if $\varphi^*(1) = \infty$, then we can find an f satisfying $U_\alpha |f| \not\equiv \infty$ and $\displaystyle\int_{\mathbf{R}^n} \Phi_p(|f(y)|)\ dy < \infty$ with $p = n/\alpha > 1$ such that $U_\alpha |f|(0) = \infty$, so that $U_\alpha f$ is not continuous at the origin in the usual sense.

We say that a function u on $\mathbf{R}^n$ is totally differentiable at x_0 if

$$\lim_{x \to x_0} \frac{|u(x) - u(x_0) - a \cdot (x - x_0)|}{|x - x_0|} = 0$$

for some $a \in \mathbf{R}^n$.

Theorem 2. *Let $1 < p < \infty$ and $w \in (A_p)$. Assume further that μ satisfies (1.1) with $s = p$, where $d\mu = w dx$. Let u be a function in $W^{1,p}(\mathbf{R}^n; \mu)$ satisfying (1.5). Then u can be modified on a set of measure zero so that it becomes totally differentiable a.e. on $\mathbf{R}^n$.*

Theorem 2 was proved by the first author [6, Theorem 3.2] in the case $\mu = \mathcal{L}^n$ and $p = n$ (see also [4, Theorem 2] and [7, Theorem 3.2]).

2. Lemmas

Throughout this paper, let M denote various positive constants independent of the variables in question and $M(\varepsilon)$ denote a positive constant which depends on ε.

Let $u \in L^1_{loc}(X;\mu)$. We say that $x \in X$ is a Lebesgue point of u if

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y)-u(x)|\, d\mu(y) = 0.$$

Lemma 1. *Let μ be a Borel measure on a connected metric space X with the decay condition (1.1), and $u \in L^1_{loc}(X;\mu)$. If $x \in X$ is a Lebesgue point of u and $0 < c < 1$, then*

$$\lim_{r\to 0} \fint_{A(r)} |u(y)-u(x)|\, d\mu(y) = 0,$$

where $A(r) = B(r) - B(cr)$ with $B(r) = B(x,r)$.

Proof. Since X is connected, we can find y such that $d(x,y) = (r+cr)/2 = (1+c)r/2$. Let $B' = B(y,(1-c)r/2)$. By the lower bound (1.1), we have

$$\frac{\mu(B')}{\mu(B(r))} \geqq C\left(\frac{(1-c)r/2}{r}\right)^s = C((1-c)/2)^s.$$

Since $B' \subset A(r) \subset B(r)$, we have

$$\fint_{A(r)} |u(y)-u(x)|\, d\mu(y) \leqq M \fint_{B(r)} |u(y)-u(x)|\, d\mu(y) \to 0$$

as $r \to 0$, by the assumption that x is a Lebesgue point of u. Thus the present lemma is obtained.

Lemma 2. *Let μ be a Borel measure on X. Let A and B be Borel sets such that $A \subset B$ and $0 < \mu(B) \leqq M\mu(A)$. Then*

$$|u_A - u_B| \leqq M \fint_B |u - u_B|\, d\mu.$$

Proof. Since $A \subset B$ and $\mu(B) \leqq M\mu(A)$, we have

$$|u_A - u_B| \quad \leqq \quad \fint_A |u-u_B|\, d\mu \leqq M \fint_B |u-u_B|\, d\mu,$$

as required.

Corollary 2. *Let μ be a Borel measure on X. Let A, B and C be Borel sets such that $A \subset C$, $B \subset C$, $0 < \mu(C) \leqq M\mu(A)$ and $0 < \mu(C) \leqq M\mu(B)$. Then*

$$|u_A - u_B| \leqq 2M \fint_C |u - u_C|\, d\mu.$$

In fact, by Lemma 2, we see that

$$\begin{aligned} |u_A - u_B| &\leqq |u_A - u_C| + |u_B - u_C| \\ &\leqq 2M \fint_C |u - u_C|\, d\mu. \end{aligned}$$

For every ball $B = B(x_0,r)$, $x \in B$, and $\sigma \geqq 1$, put $A_i = A_i(x) = B(x,2^i) - B(x,2^{i-j_0})$, where j_0 is the integer such that $2^{j_0+1} > 4\sigma^2 \geqq 2^{j_0}$. Let i_0 be the integer such that $\sigma 2^{i_0+1} > 2^{j_0+2} r \geqq \sigma 2^{i_0}$.

Lemma 3. *Let μ be a doubling measure on a connected metric space X. If a pair (u, g) satisfies the p-Poincaré inequality in X, then*

$$|u(x) - u_{\sigma^{-1}A_{i_0}}| \leqq M \sum_{i=-\infty}^{i_0} 2^i \left(\fint_{\tilde{A}_i} |g|^p \, d\mu \right)^{1/p}$$

holds for almost every $x \in B$, where $\tilde{A}_i = A_i \cup A_{i-1} = B(x, 2^i) - B(x, 2^{i-j_0-1})$.

Proof. Let $x \in B$ be a Lebesgue point of u. Note that $u_{\sigma^{-1}A_i} \to u(x)$ as $i \to -\infty$ by Lemma 1. Using Lemma 2 and the doubling property of μ, we obtain

$$|u(x) - u_{\sigma^{-1}A_{i_0}}| \leqq \sum_{i=-\infty}^{i_0} |u_{\sigma^{-1}A_i} - u_{\sigma^{-1}A_{i-1}}| \leqq M \sum_{i=-\infty}^{i_0} \fint_{\sigma^{-1}\tilde{A}_i} |u - u_{\sigma^{-1}\tilde{A}_i}| \, d\mu.$$

By the p-Poincaré inequality, we obtain

$$|u(x) - u_{\sigma^{-1}A_{i_0}}| \quad \leqq \quad M \sum_{i=-\infty}^{i_0} \sigma^{-1} 2^i \left(\fint_{\tilde{A}_i} |g|^p \, d\mu \right)^{1/p},$$

as required.

Lemma 4. *Let μ and (u, g) be as above. Then*

$$|u_{\sigma^{-1}A_{i_0}} - u_{\tilde{A}}| \leqq M 2^{i_0} \left(\fint_{\sigma\tilde{A}} |g|^p \, d\mu \right)^{1/p}$$

holds for almost every $x \in B$ and $\tilde{A} = B(x_0, cr) - B(x_0, r)$ with $c = 2^4 + 1$.

In fact, since $\sigma^{-1}A_{i_0} \subset \tilde{A}$, Lemma 4 follows readily from Lemma 2 and the p-Poincaré inequality.

Lemma 5. *Let g be a measurable function on X satisfying*

$$\int_X \Phi_p(g(y)) \, d\mu(y) < \infty.$$

Then

$$\lim_{r \to 0} \fint_{B(x,r)} \Phi_p(|g(y) - g(x)|) \, d\mu(y) = 0$$

for μ-a.e. $x \in X$.

To prove this, it suffices to note that

$$\Phi_p(a) \leqq \Phi_p(a + b) - \Phi_p(b)$$

for all $a \geqq 0$ and $b \geqq 0$.

3. Proofs of Theorems 1 and 2

First we collect properties which follow from conditions (φ1) and (φ2) (see [5, Section 2] and [8, Section 2]).

(φ4) The function φ satisfies the doubling condition, that is, there exists $c > 1$ such that

$$c^{-1}\varphi(r) \leqq \varphi(2r) \leqq c\varphi(r) \qquad \text{whenever } r > 0.$$

(φ5) For any $\gamma > 0$, there exists $c = c(\gamma) \geqq 1$ such that

$$c^{-1}\varphi(r) \leqq \varphi(r^\gamma) \leqq c\varphi(r) \qquad \text{whenever } r > 0.$$

(φ6) If $\gamma > 0$, then there exists $c = c(\gamma) \geqq 1$ such that

$$s^\gamma \varphi(s^{-1}) \leqq c t^\gamma \varphi(t^{-1}) \qquad \text{whenever } 0 < s < t.$$

Proof of Theorem 1. By Lemma 3, we have for every ball B with radius r, $0 < \varepsilon < 1$ and $\delta > 0$

$$\begin{aligned}
|u(x) - u_{\sigma^{-1}A_{i_0}}| &\leqq M \sum_{i=-\infty}^{i_0} 2^i \mu(\tilde{A}_i)^{-1/p} \left(\int_{\tilde{A}_i} |g(y)|^p \, d\mu(y) \right)^{1/p} \\
&= M \sum_{i=-\infty}^{i_0} 2^i \mu(\tilde{A}_i)^{-1/p} \left(\int_{\{y \in \tilde{A}_i : |g(y)| > (\delta 2^{-i})^\varepsilon\}} |g(y)|^p \, d\mu \right)^{1/p} \\
&\quad + M \sum_{i=-\infty}^{i_0} 2^i \mu(\tilde{A}_i)^{-1/p} \left(\int_{\{y \in \tilde{A}_i : |g(y)| \leqq (\delta 2^{-i})^\varepsilon\}} |g(y)|^p \, d\mu \right)^{1/p} \\
&= M u_1 + M u_2.
\end{aligned}$$

By condition (φ5), we see that if $|g(y)| > (\delta 2^{-i})^\varepsilon$, then

$$\varphi(|g|) \geqq \varphi((\delta 2^{-i})^\varepsilon) \geqq M(\varepsilon) \varphi(\delta 2^{-i}).$$

By the lower bound (1.1) with $s = p$, we have

$$\mu(\tilde{A}_i) \geqq M \mu(B) \left(\frac{2^i}{r} \right)^p,$$

so that

$$\begin{aligned}
u_1 &\leqq M \sum_{i=-\infty}^{i_0} 2^i \mu(\tilde{A}_i)^{-1/p} [\varphi(\delta 2^{-i})]^{-1/p} \left(\int_{\tilde{A}_i} \Phi_p(|g|) \, d\mu \right)^{1/p} \\
&\leqq M \sum_{i=-\infty}^{i_0} 2^i \mu(B)^{-1/p} \left(\frac{2^i}{r} \right)^{-1} [\varphi(\delta 2^{-i})]^{-1/p} \left(\int_{\tilde{A}_i} \Phi_p(|g|) \, d\mu \right)^{1/p} \\
&\leqq M r \mu(B)^{-1/p} \sum_{i=-\infty}^{i_0} [\varphi(\delta 2^{-i})]^{-1/p} \left(\int_{\tilde{A}_i} \Phi_p(|g|) \, d\mu \right)^{1/p}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
u_2 &\leqq M \sum_{i=-\infty}^{i_0} 2^i \mu(\tilde{A}_i)^{-1/p} (\delta 2^{-i})^\varepsilon \mu(\tilde{A}_i)^{1/p} \\
&= M \sum_{i=-\infty}^{i_0} (2^i)^{1-\varepsilon} \delta^\varepsilon \\
&\leqq M r^{1-\varepsilon} \delta^\varepsilon.
\end{aligned}$$

Hence we establish

$$\begin{aligned}
|u(x) - u_{\sigma^{-1}A_{i_0}}| \leqq{}& M r \mu(B)^{-1/p} \sum_{i=-\infty}^{i_0} [\varphi(\delta 2^{-i})]^{-1/p} \left(\int_{\tilde{A}_i} \Phi_p(|g|) \, d\mu \right)^{1/p} \\
&+ M r^{1-\varepsilon} \delta^\varepsilon,
\end{aligned}$$

where $M = M(\varepsilon)$ is a positive constant independent of δ, r and x.

From Hölder's inequality, we obtain

$$|u(x)-u_{\sigma^{-1}A_{i_0}}|$$
$$\leqq Mr\mu(B)^{-1/p}\left(\sum_{i=-\infty}^{i_0}[\varphi(\delta 2^{-i})]^{-p'/p}\right)^{1/p'}\left(\sum_{i=-\infty}^{i_0}\int_{\tilde{A}_i}\Phi_p(|g|)\,d\mu\right)^{1/p}+Mr^{1-\varepsilon}\delta^{\varepsilon},$$

where $1/p+1/p'=1$. By condition ($\varphi 4$), we see that

$$\left(\sum_{i=-\infty}^{i_0}[\varphi(\delta 2^{-i})]^{-p'/p}\right)^{1/p'}\leqq M\varphi^*(2^4\sigma r\delta^{-1})\leqq M\varphi^*(r\delta^{-1}).$$

Since $\tilde{A}_i\subset c\sigma B$ for $i\leqq i_0$ and $c=2^4+1$, it follows that

$$\begin{aligned}|u(x)-u_{\sigma^{-1}A_{i_0}}| &\leqq Mr\mu(B)^{-1/p}\varphi^*(r\delta^{-1})\left(\int_{\cup A_i}\Phi_p(|g|)\,d\mu\right)^{1/p}+Mr^{1-\varepsilon}\delta^{\varepsilon}\\ &\leqq Mr\mu(B)^{-1/p}\varphi^*(r\delta^{-1})\left(\int_{c\sigma B}\Phi_p(|g|)\,d\mu\right)^{1/p}+Mr^{1-\varepsilon}\delta^{\varepsilon}\\ &\leqq Mr\varphi^*(r\delta^{-1})\left(\fint_{c\sigma B}\Phi_p(|g|)\,d\mu\right)^{1/p}+Mr^{1-\varepsilon}\delta^{\varepsilon}.\end{aligned}$$

Next we estimate $|u_{\sigma^{-1}A_{i_0}}-u_{\tilde{A}}|$ by Lemma 4. Noting that by condition ($\varphi 4$)

$$\varphi^*(\delta^{-1}r)\geqq\left(\int_{\frac{1}{2}\delta^{-1}r}^{\delta^{-1}r}[\varphi(t^{-1})]^{-p'/p}t^{-1}dt\right)^{1/p'}\geqq M[\varphi(\delta r^{-1})]^{-1/p},$$

we have

$$\begin{aligned}|u_{\sigma^{-1}A_{i_0}}-u_{\tilde{A}}| &\leqq M2^{i_0}\left(\fint_{\sigma\tilde{A}}|g|^p\,d\mu\right)^{1/p}\\ &\leqq Mr[\varphi(\delta r^{-1})]^{-1/p}\left(\fint_{c\sigma B}\Phi_p(|g|)\,d\mu\right)^{1/p}+Mr^{1-\varepsilon}\delta^{\varepsilon}\\ &\leqq Mr\varphi^*(\delta^{-1}r)\left(\fint_{c\sigma B}\Phi_p(|g|)\,d\mu\right)^{1/p}+Mr^{1-\varepsilon}\delta^{\varepsilon}.\end{aligned}$$

By the lower bound (1.1), we have

$$\begin{aligned}|u(x)-u(y)| &\leqq |u(x)-u_{\sigma^{-1}A_{i_0}(x)}|+|u_{\sigma^{-1}A_{i_0}(x)}-u_{\tilde{A}}|\\ &\quad+|u_{\tilde{A}}-u_{\sigma^{-1}A_{i_0}(y)}|+|u_{\sigma^{-1}A_{i_0}(y)}-u(y)|\\ &\leqq Mr\varphi^*(\delta^{-1}r)\left(\fint_{c\sigma B}\Phi_p(|g|)\,d\mu\right)^{1/p}+Mr^{1-\varepsilon}\delta^{\varepsilon}\\ &\leqq Mr\varphi^*(\delta^{-1}r)\left(\frac{\mu(c\sigma B)}{\mu(c\sigma B_0)}\right)^{-1/p}\left(\fint_{c\sigma B_0}\Phi_p(|g|)\,d\mu\right)^{1/p}+Mr^{1-\varepsilon}\delta^{\varepsilon}\\ &\leqq Mr\varphi^*(\delta^{-1}r)\left(\frac{r}{r_0}\right)^{-1}\left(\fint_{c\sigma B_0}\Phi_p(|g|)\,d\mu\right)^{1/p}+Mr^{1-\varepsilon}\delta^{\varepsilon}\\ &\leqq Mr_0\varphi^*(\delta^{-1}r)\left(\fint_{c\sigma B_0}\Phi_p(|g|)\,d\mu\right)^{1/p}+Mr^{1-\varepsilon}\delta^{\varepsilon}. \qquad (3.1)\end{aligned}$$

The proof of Theorem 1 is completed by taking $\delta=1$.

To prove Corollary 1 in case $n \geqq 2$, with the aid of [3, Section 15.26], we first see that if $u \in W^{1,p}(\mathbf{R}^n;\mu)$ and $B' = B(x',r') \subset A = B(x_0,r) - B(x_0,r')$, then

$$\begin{aligned} |u(x)-b| &\leqq \frac{1}{\mathcal{L}^n(B')}\int_{B'} |u(x)-u(y)|\,dy \\ &\leqq M\int_A |x-y|^{1-n}|\nabla u(y)|\,dy \end{aligned} \tag{3.2}$$

for a.e. $x \in A$, where $b = (1/\mathcal{L}^n(B'))\displaystyle\int_{B'} u(y)\,dy$. If $B' = B(x',r')$ with $x' \in A = B(x_0,r) - B(x_0,\sqrt{2}r')$, then we can also show that

$$\begin{aligned} |u(x)-b'| &\leqq \frac{1}{\mathcal{L}^n(B'\cap A)}\int_{B'\cap A} |u(x)-u(y)|\,dy \\ &\leqq M\int_A |x-y|^{1-n}|\nabla u(y)|\,dy \end{aligned}$$

for a.e. $x \in B(x',r') - B(x_0,\sqrt{5}r')$, where $b' = (1/\mathcal{L}^n(B'\cap A))\displaystyle\int_{B'\cap A} u(y)\,dy$. Hence it follows from (3.2) that

$$\begin{aligned} |u(x)-b'| &\leqq \frac{1}{\mathcal{L}^n(B'\cap A)}\int_{B'\cap A} |u(x)-u(y)|\,dy \\ &\leqq M\int_A |x-y|^{1-n}|\nabla u(y)|\,dy \end{aligned}$$

for a.e. $x \in A$. Taking a finite covering of A by balls $B(x',r')$ with $x' \in A = B(x_0,r) - B(x_0,\sqrt{2}r')$, we have

$$\begin{aligned} |u(x)-a| &\leqq \frac{1}{\mathcal{L}^n(A)}\int_A |u(x)-u(y)|\,dy \\ &\leqq M\int_A |x-y|^{1-n}|\nabla u(y)|\,dy \end{aligned} \tag{3.3}$$

for a.e. $x \in A$, where $a = (1/\mathcal{L}^n(A))\displaystyle\int_A u(y)\,dy$. In view of [3, Section 15.23], we establish

$$\begin{aligned} \fint_A |u-u_A|\,d\mu &\leqq 2\fint_A |u(x)-a|\,d\mu(x) \\ &\leqq Mr\left(\fint_A |\nabla u|^p\,d\mu\right)^{1/p}, \end{aligned} \tag{3.4}$$

which implies the p-Poincaré inequality in our sense. Thus the case $n \geqq 2$ follows from Theorem 1.

In the case $n = 1$, we apply the usual Poincaré inequality on intervals. We have

$$|u(x)-u_{I_{i_0}}| \leqq M\sum_{i=-\infty}^{i_0} 2^i\mu(\tilde{I}_i)^{-1/p}\left(\int_{\tilde{I}_i} |\nabla u|^p\,d\mu\right)^{1/p},$$

where I_i denotes the interval $(x+2^{i-1}, x+2^i)$ or $(x-2^i, x-2^{i-1})$. In the same way as in the proof of Theorem 1, we finally obtain

$$|u(x)-u(y)| \leqq Mr_0\varphi^*(\delta^{-1}r)\left(\fint_{B_0} \Phi_p(|\nabla u|)\,d\mu\right)^{1/p} + Mr^{1-\varepsilon}\delta^{\varepsilon}$$

whenever $\delta > 0$ and x, $y \in I = (x_0 - r, x_0 + r) \subset (x_0 - r_0, x_0 + r_0) = B_0$, where $0 < \varepsilon < 1$ and M is a positive constant (which may depend on ε). Hence the case $n = 1$ is proved similarly.

Proof of Theorem 2. Modify u as in Theorem 1 so that it is locally φ^*-Hölder continuous on $\mathbf{R}^n$. Let x_0 be a point such that

$$\lim_{r\to 0} \fint_{B(x_0,r)} \Phi_p(|\nabla u(x) - \nabla u(x_0)|)\, d\mu(x) = 0. \tag{3.5}$$

By Lemma 5, almost every $x \in \mathbf{R}^n$ has this property. Set $v(x) = u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)$. Then $\nabla v(x) = \nabla u(x) - \nabla u(x_0)$, and (3.5) yields

$$\lim_{r\to 0} \fint_{B(x_0,r)} \Phi_p(|\nabla v(x)|)\, d\mu(x) = 0. \tag{3.6}$$

Applying (3.1) in the proof of Theorem 1 with v, for $x \in B(x_0, r)$, $0 < \varepsilon < 1$ and $\delta > 0$, we have

$$\frac{|v(x)|}{r} = \frac{|v(x) - v(x_0)|}{r} \leqq M\varphi^*(\delta^{-1}r) \left(\fint_{c\sigma B} \Phi_p(|\nabla v|)\, d\mu \right)^{1/p} + Mr^{-\varepsilon}\delta^{\varepsilon},$$

where $c = 2^4 + 1$ and $M = M(\varepsilon)$ does not depend on δ, r and x. Here, taking $\delta^{-1}r = N$ with a positive integer N, we obtain

$$\frac{|v(x)|}{r} \leqq M\varphi^*(N) \left(\fint_{c\sigma B} \Phi_p(|\nabla v|)\, d\mu \right)^{1/p} + MN^{-\varepsilon}.$$

Hence it follows from (3.6) that

$$\limsup_{x\to x_0} \frac{|v(x)|}{|x - x_0|} \leqq MN^{-\varepsilon}.$$

Letting $N \to \infty$, we see that the left side is equal to zero, which implies that v is totally differentiable at x_0.

Remark. The proof of Theorem 1 implies that

$$|u(x) - u(x_0)| \leqq M|x - x_0| \left(\sup_{0<r<c\sigma r_0} \fint_{B(x_0,r)} \Phi_p(|g|)\, d\mu \right)^{1/p} + M|x - x_0|$$

for all $x \in B(x_0, r_0)$. This fact completes the proof of Theorem 2 with the aid of Rademacher-Stepanov theorem (cf. [9, Theorem 3, Chap. VIII]).

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