PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 10, Pages 3111–3115
S 0002-9939(02)06435-3
Article electronically published on March 12, 2002

G-COINCIDENCES FOR MAPS OF HOMOTOPY SPHERES INTO CW-COMPLEXES

DACIBERG L. GONÇALVES, JAN JAWOROWSKI, AND PEDRO L. Q. PERGHER

(Communicated by Paul Goerss)

Abstract. Let G be a finite group acting freely in a CW-complex Σ^m which is a homotopy m-dimensional sphere and let $f : \Sigma^m \to Y$ be a map of Σ^m to a finite k-dimensional CW-complex Y. We show that if $m \geq |G|k$, then f has an (H, G)-coincidence for some nontrivial subgroup H of G.

1. Introduction

The classical Borsuk-Ulam Theorem has been generalized in many directions; see, for example, [14], [4, Ch. 2, §34] and [10]. More recently (see [7], [8], [9], and [10]), free actions of the cyclic group $\mathbf{Z}_r$ on S^m and certain types of coincidences under maps $f : S^m \to Y$, where Y is a finite-dimensional polyhedron, were studied. It was shown that under certain dimension hypothesis there is at least one (p, r)-coincidence (see below for the definition). In the present work we study a similar problem for an arbitary finite group G which acts freely in a homotopy sphere. It turns out that results similar to those of [7], [8] and [9] remain true in this more general situation. There are many examples of finite groups, besides the cyclic ones, which act freely in homotopy spheres. Suppose that G is a finite group which acts freely in some homotopy sphere Σ^m of dimension m. By [3, Ch. XVI, §9] such groups have periodic cohomology. Further, it was proved in [13, Theorem A, page 267] that if s is the period, then G acts freely on a finite simplicial complex which has the type of homotopy of a sphere of dimension $ds - 1$, where d is the greatest common divisor of $|G|$ and $\phi(|G|)$, $\phi(|G|)$ being the Euler function. Thus each group which has periodic cohomology will provide an example of a free action on a homotopy sphere which is a finite CW-complex. If we do not require the complex to be finite, such groups can act in a complex of the homotopy type of a sphere of dimension $s - 1$ [13, Proposition 4.4, page 277]. There also exist several different actions of G in a fixed homotopy sphere. Such actions can be classified by the number of homotopy types of the orbit spaces. For more details see [6].

Received by the editors December 14, 2000 and, in revised form, May 10, 2001.
1991 *Mathematics Subject Classification.* Primary 55M20; Secondary 55M35.
Key words and phrases. G-coincidence, G-equivariant, polyhedron, G-action, transfer, generalized Gysin sequence.
The first author was partially supported by CNPq and FAPESP and the third author was partially supported by CNPq.

If Σ is a free G-space and H is a subgroup of G, then H acts on the right on each orbit Gx of G as follows: if $y \in Gx$ and $y = gx, g \in G$, then $hy := ghx$. Following [7] and [9], the concept of G-coincidence can be generalized as follows.

Definition. Suppose that Σ is a G-space and H is a subgroup of G. We say that a map $f : \Sigma \to Y$ has a (H, G)-coincidence if there is a point $x \in \Sigma$ such that f sends every orbit of the action of H on the G-orbit of x to a single point.

Of course, if H is the trivial subgroup, then every point of Σ is an (H, G)-coincidence. If $H = G$ this is the usual definition of coincidence. If $G = \mathbf{Z}_r$ and $h = \mathbf{Z}_p$, then an (H, G)-coincidence is a (p, r)-coincidence in the sense of [9]. The purpose of this paper is to prove the following theorem.

Theorem. *Suppose that G is a finite group acting freely on a CW-complex Σ^m which is a homotopy m-dimensional sphere and suppose that $f : \Sigma^m \to Y$ is a map. Suppose one of the following conditions holds:*

a) *Σ^m is a finite m-dimensional CW-complex and Y a k-dimensional CW-complex.*

b) *Y is a finite k-dimensional CW-complex.*

Then, if $m \geq |G|k$, there is a nontrivial subgroup $H \subset G$ and an (H, G)-coincidence for f.

If $G = \mathbf{Z}_2$ this result is known and it is true for every m. In this case the coincidence problem has been completely solved in [8] and [9] (compare also [11]). Furthermore, such a result is proved in [7] and [9] for cyclic groups $\mathbf{Z}_q$ acting on S^{2n+1}. We will restrict ourselves to finite G-CW-complexes which have the homotopy type of an odd-dimensional sphere S^{2n+1}.

This paper is divided into two sections besides the Introduction. In section 2 we prove some results about G and the orbit space of an action of G in a homotopy sphere. In section 3 we prove the main result. It would be interesting to know if an (H, G)-coincidence exists for every nontrivial cyclic subgroup $H \subset G$ of a prime order. Our main result asserts only the existence of such an (H, G)-coincidence for some subgroup $H \subset G$.

2. Background

From now on we will assume that m is odd, $m = 2n + 1$, Σ^{2n+1} is a $(2n + 1)$-dimensional CW-complex (not necessarily finite) which has the homotopy type of the sphere S^{2n+1}, and G is a finite group which acts freely in Σ^{2n+1}. We denote by $B(G)$ the classifying space of G.

Lemma 1. *Let p be a prime which divides the order of the group G. Then the cohomology $H^i(B(G), \mathbf{Z}_p)$ contains $\mathbf{Z}_p$ as a summand for $i = 2n + 1,\ 2n + 2$.*

Proof. By [3, Ch. XII, §11], $H^{2n+2}(G, \mathbf{Z}) \approx \mathbf{Z}_{|G|}$, the cyclic group of order $|G|$. Since G is a finite group, either the $\mathbf{Z}$-cohomology or the $\mathbf{Z}$-homology of $B(G)$ has only torsion in dimensions greater than zero. By the universal coefficient formula, $\mathbf{Z}_{|G|} \cong H^{2n+2}(B(G), \mathbf{Z}) =$ free part of $(H_{2n+2}(B(G), \mathbf{Z})) \oplus$ torsion $(H_{2n+1}(B(G), \mathbf{Z})) = H_{2n+1}(B(G), \mathbf{Z})$. By the universal coefficient theorem in dimensions $2n + 1$ and $2n + 2$ and the fact that p divides $|G|$, $H^{2n+2}(B(G), \mathbf{Z}_p)$ has a direct summand which is $Ext(\mathbf{Z}_{|G|}, \mathbf{Z}_p) \cong \mathbf{Z}_p$, and $H^{2n+1}(B(G), \mathbf{Z}_p)$ has a direct summand which is $Hom(\mathbf{Z}_{|G|}, \mathbf{Z}_p) = \mathbf{Z}_p$. Hence the result follows. $\square$

Let $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$ be the quotient map and let $c : \Sigma^{2n+1}/G \to B(G)$ be a classifying map for the G-principal bundle $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$.

Lemma 2. *Let p be a prime which divides $|G|$. Then the homomorphism $c^* : H^{2n+1}(B(G), \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ is nontrivial.*

Proof. Since $H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p) \neq 0$, it suffices to show that $c^* : H^{2n+1}(B(G), \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ is surjective.

Consider the G-universal bundle $E(G) \to B(G)$. By [5, Ch. III, §4, page 208] we have the bundle $\Sigma^{2n+1} \to E(G) \times_G \Sigma^{2n+1} \xrightarrow{\rho} B(G)$ with base space $B(G)$ and fiber Σ^{2n+1} (see also [1, Ch. IV, §6, Lemma 6.2, page 146]). Since G is a finite group and acts freely on Σ^{2n+1}, then $E(G) \times_G \Sigma^{2n+1}$ is homotopy equivalent to Σ^{2n+1}/G. If $t : \Sigma^{2n+1}/G \to E(G) \times_G \Sigma^{2n+1}$ is a homotopy equivalence, $\rho t : \Sigma^{2n+1}/G \to B(G)$ also classifies the G-principal bundle $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$, and so it is homotopic to c. Hence it suffices to show that

$$\rho^* : H^{2n+1}(B(G), \mathbf{Z}_p) \to H^{2n+1}(E(G) \times_G \Sigma^{2n+1}, \mathbf{Z}_p)$$

is surjective. To do this, consider the generalized Gysin cohomology sequence associated to the fibration $\rho : E(G) \times_G \Sigma^{2n+1} \to B(G)$ (see [12, Ch. 9, §5, Theorem 2]):

$$\begin{aligned} H^{2n+1}(B(G), \mathbf{Z}_p) &\xrightarrow{\rho^*} H^{2n+1}(E(G) \times_G \Sigma^{2n+1}, \mathbf{Z}_p) \to H^0(B(G), \mathbf{Z}_p) \\ &\xrightarrow{\Psi} H^{2n+2}(B(G), \mathbf{Z}_p) \xrightarrow{\rho^*} H^{2n+2}(E(G) \times_G \Sigma^{2n+1}, \mathbf{Z}_p). \end{aligned}$$

Since $H^{2n+2}(E(G) \times_G \Sigma^{2n+1}, \mathbf{Z}_p) = 0$, Ψ is surjective; but $H^0(B(G), \mathbf{Z}_p) \cong \mathbf{Z}_p$, and by Lemma 2.1 $H^{2n+2}(B(G), \mathbf{Z}_p) \neq 0$, hence the only possibility is that $H^{2n+2}(B(G), \mathbf{Z}_p) = \mathbf{Z}_p$, which implies that Ψ is an isomorphism. Thus ρ^* is surjective and the fact is proved. $\square$

3. Proof of the main result

Let $G = \{g_1, ..., g_r\}$ be a fixed enumeration of elements of G, where r is the order of G. We construct a map $G \times Y^r \to Y^r$, where $Y^r = Y \times ... \times Y$ is the r-fold product, as follows. For each $g \in G$ and $(y_1, ..., y_r) \in Y^r$, let $g.(y_1, ..., y_r) = (y_{\sigma_g(1)}, ..., y_{\sigma_g(r)})$, where the permutation σ_g is defined by $g_i g = g_{\sigma_g(i)}$. It is straightforward to verify that the above map is a left G-action on Y^r. For a subgroup $H \subset G$, let $(Y^r)^H$ be the fixed point set of H and $F = \bigcup_H (Y^r)^H$, where H runs over all nontrivial subgroups of G. Let $Y_0^{(r)} := Y^r - F$; it is precisely the part of Y^r where the G-action is free. If X is any space with a G-action, then a map $f : X \to Y$ induces an equivariant map $\phi : X \to Y^r$, $\phi(x) = (f(g_1 x), ..., f(g_r x))$.

Suppose that $f : \Sigma^{2n+1} \to Y$ has no (H, G)-coincidence points for any nontrivial subgroup $H \subset G$. Then $\phi(\Sigma^{2n+1}) \subset Y_0^{(r)}$, so ϕ factors through $Y_0^{(r)}$. Let $\phi_0 : \Sigma^{2n+1} \to Y_0^{(r)}$ be this factorization; it is an equivariant map.

Let $\gamma : Y_0^{(r)} \to Y_0^{(r)}/G$ be the quotient map and $\overline{\phi}_0 : \Sigma^{2n+1}/G \to Y_0^{(r)}/G$ be the map induced by ϕ_0. Let $c_Y : Y_0^{(r)}/G \to B(G)$ be a classifying map for the G-principal bundle $\gamma : Y_0^{(r)} \to Y_0^{(r)}/G$. Then $c = c_Y \overline{\phi}_0 : \Sigma^{2n+1}/G \to B(G)$ is a classifying map for the G-principal bundle $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$ of the previous section.

First, if $2n+1 > rk$, then the topological dimension of $Y_0^{(r)}/G$ is less than $2n+1$, hence $c_Y^* : H^{2n+1}(B(G), \mathbf{Z}_p) \to H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p)$ is zero and $c^* = \overline{\phi}_0^* c_Y^* = 0$ which contradicts Lemma 2.2. Thus we can assume that $2n+1 = rk$. We will again obtain a contradiction to Lemma 2.2 by showing that $\overline{\phi}_0^* : H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ is zero. First, if Σ^{2n+1} is a finite CW-complex, then $f(\Sigma^{2n+1})$ is compact and so it is contained in a finite subcomplex of Y. Thus without loss of generality we can assume that Y is a finite CW-complex.

The map $i^* : H^{2n+1}(Y^r, \mathbf{Z}_p) \to H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p)$ is a part of the cohomology sequence of the pair $(Y^r, Y_0^{(r)})$ and $H^{2n+1}(Y^r, Y_0^{(r)}) = 0$, so i^* is surjective. On the other hand, by the Künneth formula, any class of $H^{2n+1}(Y^r, \mathbf{Z}_p)$ is a cup product of classes of lower dimension, and $H^s(\Sigma^{2n+1}, \mathbf{Z}_p) = 0$ for $0 < s < 2n+1$ which implies that $\phi^* : H^{2n+1}(Y^r, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}, \mathbf{Z}_p)$ is zero. Since $\phi_0 i = \phi$, it follows that $\phi_0^* : H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}, \mathbf{Z}_p)$ is zero. Now since $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$ and $\sigma : Y_0^{(r)} \to Y_0^{(r)}/G$ are covering projections, there are transfer homomorphisms $\tau : H^{2n+1}(\Sigma^{2n+1}, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ and $\tau_0 : H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p) \to H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p)$ [2, Chapter III, Section 2, page 118]. Further we have the following commutative rectangle:

$$\begin{tikzcd}
H^{2n+1}(\Sigma^{2n+1}, \mathbf{Z}_p) \arrow[r, "\tau"] & H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p) & \\
H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p) \arrow[u, "\phi_0^*"'] \arrow[r, "\tau_0"] & H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p) \arrow[u, "\overline{\phi}_0^*"'] \arrow[r] & 0
\end{tikzcd}$$

and $\tau\phi_0^* = \overline{\phi}_0^*\tau_0$ implies that $\overline{\phi}_0^*\tau_0 : H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ is zero.

Since $Y_0^{(r)}$ is a $(2n+1)$-dimensional CW-complex, the transfer homomorphism $\tau_0 : H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p) \to H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p)$ is surjective. This implies that $(\overline{\phi}_0)^* : H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ is the zero homomorphism, and the result follows.

Remark. The result of the Theorem is the best possible if all finite groups are considered. In fact, for each $k > 1$ and $m < 2k$, an example of a k-dimensional polyhedron Y^k and a map $f : S^r \to Y^k$ without antipodal coincidences was constructed in [8] (compare also [11]). It is an interesting question whether a similar example can be constructed for $G = \mathbf{Z}_r$, $r > 2$.

References

1. A. Adem and J. Milgram, *Cohomology of groups*, Springer-Verlag, New York-Heidelberg-Berlin (1982).
2. G. E. Bredon, *Introduction to Compact Transformation Groups*, Pure and Applied Mathematics - Academic Press, New York and London (1972). MR **54:**1265
3. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton, New Jersey, 1956. MR **17:**1040e
4. P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete 33, Springer-Verlag, Berlin, 1964. MR **31:**750
5. T. tom Dieck, *Transformation Groups*, Walter de Gruyter, Berlin-New York, 1987. MR **89c:**57048
6. M. Golasiński and D. L. Gonçalves, *Homotopy spherical space forms - a numerical bound for homotopy types*, Hiroshima Mathematical Journal **31** (2001), 107–116. CMP 2001:10
7. D. L. Gonçalves and P. L. Q. Pergher, $\mathbf{Z}_p$*-coincidence for maps of spheres into CW complexes*, Kobe Journal of Math. **15** (1998), 191–195. MR **2000a:**55004

8. Marek Izydorek and Jan Jaworowski, *Antipodal coincidence for maps of spheres into complexes*, Proc. Amer. Math. Soc. **123(6)** (1995), 1947–1950. MR **96c:**55002
9. Jan Jaworowski, *Periodic Coincidence for Maps of Spheres*, Kobe Journal of Math. **17** (2000), 21–26. MR **2001k:**55007
10. Neza Mramor-Kosta, *Coincidence points of maps on Z_{p^α}-spaces*, Rendiconti dell'Instituto di Mat. Trieste **XXV** (1993), 379–389. MR **96e:**55004
11. E.V. Ščepin, *On a Problem of L. A. Tumarkin*, Soviet Math. Dokl. **15** (1974), 1024–1026.
12. E. Spanier, *Algebraic Topology*, Academic Press, 1988.
13. R. G. Swan, *Periodic resolutions for finite groups*, Ann. of Math. **72(2)**, 267–291. MR **23:**A2205
14. C. T. Yang, *On theorems of Borsuk-Ulam Kakutani-Yamabe-Yujobô and Dyson I*, Ann. of Math (2) **60** (1954), 262–282. MR **16:**502d

Instituto de Matemática e Estatísca, Universidade de São Paulo, Rua do Matão, 1010, Agência Jardim Paulistano, Caixa Postal 66281, CEP 05315-970, São Paulo, SP, Brasil

E-mail address: dlgoncal@ime.usp.br.

Department of Mathematics, Indiana University, Bloomington, Indiana 47405-5701

E-mail address: jaworows@indiana.edu.

Departamento de Matemática, Universidade Federal de São Carlos, Rodovia Washington Luiz, km 235, Caixa Postal 676, CEP 13.565-905, São Carlos, SP, Brasil

E-mail address: pergher@dm.ufscar.br.