G-COINCIDENCES FOR MAPS OF HOMOTOPY SPHERES INTO CW-COMPLEXES

DACIBERG L. GONÇALVES, JAN JAWOROWSKI, AND PEDRO L. Q. PERGHER

Abstract. Let $G$ be a finite group acting freely in a CW-complex $\Sigma^m$ which is a homotopy $m$-dimensional sphere and let $f : \Sigma^m \to Y$ be a map of $\Sigma^m$ to a finite $k$-dimensional CW-complex $Y$. We show that if $m \geq |G|k$, then $f$ has an $(H, G)$-coincidence for some nontrivial subgroup $H$ of $G$.

1. Introduction

The classical Borsuk-Ulam Theorem has been generalized in many directions; see, for example, [4], [1 Ch. 2, §34] and [10]. More recently (see [7], [8], [9], and [10]), free actions of the cyclic group $\mathbb{Z}_r$ on $S^m$ and certain types of coincidences under maps $f : S^m \to Y$, where $Y$ is a finite-dimensional polyhedron, were studied. It was shown that under certain dimension hypothesis there is at least one $(p, r)$-coincidence (see below for the definition). In the present work we study a similar problem for an arbitrary finite group $G$ which acts freely in a homotopy sphere. It turns out that results similar to those of [7], [8] and [9] remain true in this more general situation. There are many examples of finite groups, besides the cyclic ones, which act freely in homotopy spheres. Suppose that $G$ is a finite group which acts freely in some homotopy sphere $\Sigma^m$ of dimension $m$. By [5, Ch. XVI, §9] such groups have periodic cohomology. Further, it was proved in [13, Theorem A, page 267] that if $s$ is the period, then $G$ acts freely on a finite simplicial complex which has the type of homotopy of a sphere of dimension $ds - 1$, where $d$ is the greatest common divisor of $|G|$ and $\phi(|G|)$, $\phi(|G|)$ being the Euler function. Thus each group which has periodic cohomology will provide an example of a free action on a homotopy sphere which is a finite CW-complex. If we do not require the complex to be finite, such groups can act in a complex of the homotopy type of a sphere of dimension $s - 1$ [13 Proposition 4.4, page 277]. There also exist several different actions of $G$ in a fixed homotopy sphere. Such actions can be classified by the number of homotopy types of the orbit spaces. For more details see [6].

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Lemma 1. Let $p$ be a prime which divides the order of the group $G$. Then the cohomology $H^i(B(G), \mathbb{Z}_p)$ contains $\mathbb{Z}_p$ as a summand for $i = 2n + 1, 2n + 2$.

Proof. By [3, Ch. XII, §11], $H^{2n+2}(G, \mathbb{Z}) \cong \mathbb{Z}_{|G|}$, the cyclic group of order $|G|$. Since $G$ is a finite group, either the $\mathbb{Z}$-cohomology or the $\mathbb{Z}$-homology of $B(G)$ has only torsion in dimensions greater than zero. By the universal coefficient formula, $\mathbb{Z}_{|G|} \cong H^{2n+2}(B(G), \mathbb{Z}) = \text{free part of } (H_{2n+2}(B(G), \mathbb{Z})) \oplus \text{torsion } (H_{2n+1}(B(G), \mathbb{Z})) = H_{2n+1}(B(G), \mathbb{Z})$. By the universal coefficient theorem in dimensions $2n + 1$ and $2n + 2$ and the fact that $p$ divides $|G|$, $H^{2n+2}(B(G), \mathbb{Z}_p)$ has a direct summand which is $\text{Ext}(\mathbb{Z}_{|G|}, \mathbb{Z}_p) \cong \mathbb{Z}_p$, and $H^{2n+1}(B(G), \mathbb{Z}_p)$ has a direct summand which is $\text{Hom}(\mathbb{Z}_{|G|}, \mathbb{Z}_p) = \mathbb{Z}_p$. Hence the result follows.
Let \( \pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G \) be the quotient map and let \( c : \Sigma^{2n+1}/G \to B(G) \) be a classifying map for the \( G \)-principal bundle \( \pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G \).

**Lemma 2.** Let \( p \) be a prime which divides \( |G| \). Then the homomorphism \( c^* : H^{2n+1}(B(G), \mathbb{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbb{Z}_p) \) is nontrivial.

**Proof.** Since \( H^{2n+1}(\Sigma^{2n+1}/G, \mathbb{Z}_p) \neq 0 \), it suffices to show that \( c^* : H^{2n+1}(B(G), \mathbb{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbb{Z}_p) \) is surjective.

Consider the \( G \)-universal bundle \( E(G) \to B(G) \). By [3, Ch. III, §4, page 208] we have the bundle \( \Sigma^{2n+1} \to E(G) \times_G \Sigma^{2n+1} \to B(G) \) with base space \( B(G) \) and fiber \( \Sigma^{2n+1} \) (see also [11, Ch. IV, §6, Lemma 6.2, page 146]). Since \( G \) is a finite group and acts freely on \( \Sigma^{2n+1} \), then \( E(G) \times_G \Sigma^{2n+1} \) is homotopy equivalent to \( \Sigma^{2n+1}/G \). If \( t : \Sigma^{2n+1}/G \to E(G) \times_G \Sigma^{2n+1} \) is a homotopy equivalence, \( \rho : \Sigma^{2n+1}/G \to B(G) \) also classifies the \( G \)-principal bundle \( \pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G \), and so it is homotopic to \( c \). Hence it suffices to show that

\[
\rho^* : H^{2n+1}(B(G), \mathbb{Z}_p) \to H^{2n+1}(E(G) \times_G \Sigma^{2n+1}, \mathbb{Z}_p)
\]

is surjective. To do this, consider the generalized Gysin cohomology sequence associated to the fibration \( \rho : E(G) \times_G \Sigma^{2n+1} \to B(G) \) (see [12, Ch. 9, §5, Theorem 2]):

\[
H^{2n+1}(B(G), \mathbb{Z}_p) \xrightarrow{\rho^*} H^{2n+1}(E(G) \times_G \Sigma^{2n+1}, \mathbb{Z}_p) \xrightarrow{\Psi} H^{2n+2}(B(G), \mathbb{Z}_p) \xrightarrow{\rho^*} H^{2n+2}(E(G) \times_G \Sigma^{2n+1}, \mathbb{Z}_p).
\]

Since \( H^{2n+2}(E(G) \times_G \Sigma^{2n+1}, \mathbb{Z}_p) = 0 \), \( \Psi \) is surjective; but \( H^0(B(G), \mathbb{Z}_p) \cong \mathbb{Z}_p \), and by Lemma 2.1 \( H^{2n+2}(B(G), \mathbb{Z}_p) \neq 0 \), hence the only possibility is that \( H^{2n+2}(B(G), \mathbb{Z}_p) = \mathbb{Z}_p \), which implies that \( \Psi \) is an isomorphism. Thus \( \rho^* \) is surjective and the fact is proved. \( \square \)

### 3. Proof of the main result

Let \( G = \{g_1, \ldots, g_r\} \) be a fixed enumeration of elements of \( G \), where \( r \) is the order of \( G \). We construct a map \( G \times Y^r \to Y^r \), where \( Y^r = Y \times \ldots \times Y \) is the \( r \)-fold product, as follows. For each \( g \in G \) and \((y_1, \ldots, y_r) \in Y^r \), let \( g(y_1, \ldots, y_r) = (g_{\sigma_g(1)}, \ldots, g_{\sigma_g(r)}) \), where the permutation \( \sigma_g \) is defined by \( g, y \mapsto g_y \). It is straightforward to verify that the above map is a left \( G \)-action on \( Y^r \). For a subgroup \( H \subset G \), let \((Y^r)^H \) be the fixed point set of \( H \) and \( F = \bigcup_H (Y^r)^H \), where \( H \) runs over all nontrivial subgroups of \( G \). Let \( Y_0^{(r)} := Y^r - F \); it is precisely the part of \( Y^r \) where the \( G \)-action is free. If \( X \) is any space with a \( G \)-action, then a map \( f : X \to Y \) induces an equivariant map \( \phi : X \to Y^r \), \( \phi(x) = (f(g_1x), \ldots, f(g_rx)) \).

Suppose that \( f : \Sigma^{2n+1} \to Y \) has no \((H, G)\)-coincidence points for any nontrivial subgroup \( H \subset G \). Then \( \phi(\Sigma^{2n+1}) \subset Y_0^{(r)} \), so \( \phi \) factors through \( Y_0^{(r)} \). Let \( \phi_0 : \Sigma^{2n+1} \to Y_0^{(r)} \) be this factorization; it is an equivariant map.

Let \( \gamma : Y_0^{(r)} \to Y_0^{(r)}/G \) be the quotient map and \( \overline{\phi}_0 : \Sigma^{2n+1}/G \to Y_0^{(r)}/G \) be the map induced by \( \phi_0 \). Let \( c_G : Y_0^{(r)}/G \to B(G) \) be a classifying map for the \( G \)-principal bundle \( \gamma : Y_0^{(r)} \to Y_0^{(r)}/G \). Then \( c = c_G \overline{\phi}_0 : \Sigma^{2n+1}/G \to B(G) \) is a classifying map for the \( G \)-principal bundle \( \pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G \) of the previous section.
First, if $2n + 1 > rk$, then the topological dimension of $Y_0^{(r)}/G$ is less than $2n + 1$, hence $c_0^2 : H^{2n+1}(B(G), Z_p) \to H^{2n+1}(Y_0^{(r)}/G, Z_p)$ is zero and $c^* = \phi_0^*c_0^2 = 0$ which contradicts Lemma 2.2. Thus we can assume that $2n + 1 = rk$. We will again obtain a contradiction to Lemma 2.2 by showing that $\overline{\phi_0} : H^{2n+1}(Y_0^{(r)}/G, Z_p) \to H^{2n+1}(\Sigma^{2n+1}/G, Z_p)$ is zero. First, if $\Sigma^{2n+1}$ is a finite CW-complex, then $f(\Sigma^{2n+1})$ is compact and so it is contained in a finite subcomplex of $Y$. Thus without loss of generality we can assume that $Y$ is a finite CW-complex.

The map $i^* : H^{2n+1}(Y^r, Z_p) \to H^{2n+1}(Y_0^{(r)}, Z_p)$ is a part of the cohomology sequence of the pair $(Y^r, Y_0^{(r)})$ and $H^{2n+1}(Y^r, Y_0^{(r)}) = 0$, so $i^*$ is surjective. On the other hand, by the Künneth formula, any class of $H^{2n+1}(Y^r, Z_p)$ is a cup product of classes of lower dimension, and $H^s(\Sigma^{2n+1}, Z_p) = 0$ for $0 < s < 2n + 1$ which implies that $\phi^* : H^{2n+1}(Y^r, Z_p) \to H^{2n+1}(\Sigma^{2n+1}, Z_p)$ is zero. Since $\phi_0i = \phi$, it follows that $\phi_0^* : H^{2n+1}(Y_0^{(r)}, Z_p) \to H^{2n+1}(\Sigma^{2n+1}, Z_p)$ is zero. Now since $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$ and $\sigma : Y_0^{(r)} \to Y_0^{(r)}/G$ are covering projections, there are transfer homomorphisms $\tau : H^{2n+1}(\Sigma^{2n+1}, Z_p) \to H^{2n+1}(\Sigma^{2n+1}/G, Z_p)$ and $\tau_0 : H^{2n+1}(Y_0^{(r)}, Z_p) \to H^{2n+1}(Y_0^{(r)}/G, Z_p)$ [2] Chapter III, Section 2, page 118. Further we have the following commutative rectangle:

\[
\begin{array}{ccc}
H^{2n+1}(\Sigma^{2n+1}, Z_p) & \xrightarrow{\tau} & H^{2n+1}(\Sigma^{2n+1}/G, Z_p) \\
\uparrow{\phi_0^*} & & \uparrow{\sigma_0^*} \\
H^{2n+1}(Y_0^{(r)}, Z_p) & \xrightarrow{\tau_0} & H^{2n+1}(Y_0^{(r)}/G, Z_p) \\
\end{array}
\]

and $\tau{\phi_0^*} = \overline{\phi_0^*}{\tau_0}$ implies that $\overline{\phi_0^*}{\tau_0} : H^{2n+1}(Y_0^{(r)}, Z_p) \to H^{2n+1}(\Sigma^{2n+1}/G, Z_p)$ is zero.

Since $Y_0^{(r)}$ is a $(2n + 1)$-dimensional CW-complex, the transfer homomorphism $\tau_0 : H^{2n+1}(Y_0^{(r)}, Z_p) \to H^{2n+1}(Y_0^{(r)}/G, Z_p)$ is surjective. This implies that $(\overline{\phi_0^*})^* : H^{2n+1}(Y_0^{(r)}/G, Z_p) \to H^{2n+1}(\Sigma^{2n+1}/G, Z_p)$ is the zero homomorphism, and the result follows.

Remark. The result of the Theorem is the best possible if all finite groups are considered. In fact, for each $k > 1$ and $m < 2k$, an example of a $k$-dimensional polyhedron $Y^k$ and a map $f : S^r \to Y^k$ without antipodal coincidences was constructed in [8] (compare also [11]). It is an interesting question whether a similar example can be constructed for $G = Z_r$, $r > 2$.

References


Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, Agência Jardim Paulistano, Caixa Postal 66281, CEP 05315-970, São Paulo, SP, Brasil.

E-mail address: digonal@ime.usp.br.

Department of Mathematics, Indiana University, Bloomington, Indiana 47405-5701

E-mail address: jaworows@indiana.edu.

Departamento de Matemática, Universidade Federal de São Carlos, Rodovia Washington Luiz, km 235, Caixa Postal 676, CEP 13.565-905, São Carlos, SP, Brasil.

E-mail address: pergher@dm.ufscar.br.