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NORMAL BASES FOR HOPF-GALOIS ALGEBRAS

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ABSTRACT. Let H be a Hopf algebra over a commutative ring R such that H is a finitely generated, projective module over R, let A be a right H-comodule algebra, and let B be the subalgebra of H-coinvariant elements of A. If A is a Galois extension of B and B is a local subalgebra of the center of A, then A is a cleft right H-comodule algebra or, equivalently, there is a normal basis for A over B.

Let R be a commutative ring, and let H be a Hopf algebra over R, which is a finitely generated, projective module over R. When there is no notation to indicate otherwise, the bifunctors $\otimes$ and Hom are applied to the category of R-modules. A right H-comodule algebra A is an algebra over R and a right H-comodule such that the comodule map α of A into $A \otimes H$ is a homomorphism of algebras. Use the sigma notations $\sum_{(h)} h_{(1)} \otimes h_{(2)}$ for the coproduct of an element h of H and $\sum_{(a)} a_{(0)} \otimes a_{(1)}$ for the element $\alpha(a)$ of $A \otimes H$, a in A. An element a of A is called H-coinvariant if $\sum_{(a)} a_{(0)} \otimes a_{(1)} = a \otimes 1$, and the set B of H-coinvariant elements of A is a subalgebra of A. If the extension of α to a left A-module homomorphism of $A \otimes A$ into $A \otimes H$ is surjective, then A is called an H-Galois extension of B [4, Def. 1.4], and an H-Galois extension A of B is said to have a normal basis if there is a left B-module, right H-comodule isomorphism of $B \otimes H$ onto A [4, Def. 2.6]. Y. Doi and M. Takeuchi introduced the notion of a cleft right H-comodule algebra and proved that a right H-comodule algebra is cleft if, and only if, it is a Galois extension with normal basis [2, Thm. 9].

Now assume that B is contained in the center of A. Then $B \otimes H$ is a Hopf algebra and A is a right $B \otimes H$-comodule algebra over the ring B. Thus R may be replaced by B so that the subalgebra of H-coinvariant elements of A is the ground ring R. If A is a Galois extension of R, A is called an H-Galois algebra. In this case, the extension of α is a left A-module isomorphism γ of $A \otimes A$ onto $A \otimes H$ by [4, Thm. 1.7]. Henceforth, assume that A is an H-Galois algebra. Then A is cleft whenever there is a right H-comodule isomorphism of H onto A. But a right H-comodule is a left module over the dual algebra $H^* = \mathrm{Hom}(H, R)$, and because H is a projective module over R, it can be shown that A is cleft exactly when there is a left H^*-module isomorphism of H onto A. Recently, D. Rumynin [5, §2.4] raised the question of whether H-Galois algebras over a local ring R are cleft. In

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a much earlier paper [3], P.M. Cook and H.F. Kreimer proved that for a local ring R and commutative or cocommutative Hopf algebra H, every H-Galois algebra is cleft. The restriction that H be commutative or cocommutative was used to prove that H is a finitely generated, projective left H^*-module, but the restriction is not necessary.

Proposition 1. *H is a finitely generated, projective left module over H^*.*

Proof. H is a left H^*-module with respect to the rule $\varphi \cdot h = \sum_{(h)} \langle \varphi, h_{(2)} \rangle h_{(1)}$ for elements φ of H^* and h of H. The antipode S of H is an antiautomorphism of the algebra H by [4, Prop. 1.1], and H becomes a right H-module by assigning $S(g)h$ to the element $h \otimes g$ of $H \otimes H$. Because H is a finitely generated, projective module over R, this right H-module structure is equivalent to a left H^*-comodule structure for H. Indeed, let $\varphi_i \in H^*$ and $h_i \in H$, $1 \leq i \leq n$, be elements such that $\sum_{i=1}^{n} \langle \varphi_i, h \rangle h_i = h$ for all elements h of H. Then $S(g)h = \sum_{i=1}^{n} \langle \varphi_i, S(g)h \rangle h_i = \sum_{i=1}^{n} \sum_{(\varphi_i)} \langle \varphi_{i,(1)}, S(g) \rangle \langle \varphi_{(2)}, h \rangle h_i$ and the left H^*-comodule structure on H is given by mapping an element h of H to $\sum_{i=1}^{n} \sum_{(\varphi_i)} \langle \varphi_{i,(2)}, h \rangle S^*(\varphi_{i,(1)}) \otimes h_i$, where S^* denotes the adjoint of S. To show that H is a left Hopf module over H^* it is necessary to verify that $S(g)(\varphi \cdot h)$ equals

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{(\varphi_i),(\varphi)} \langle \varphi_{i,(2)}, h \rangle \langle \varphi_{(1)} S^*(\varphi_{i,(1)}), g \rangle \varphi_{(2)} \cdot h_i \\
&\quad = \sum_{i=1}^{n} \sum_{(\varphi_i),(\varphi)} \sum_{(g)} \langle \varphi_{i,(2)}, h \rangle \langle \varphi_{(1)}, g_{(1)} \rangle \langle \varphi_{i,(1)}, S(g_{(2)}) \rangle \varphi_{(2)} \cdot h_i \\
&\quad = \sum_{i=1}^{n} \sum_{(\varphi)} \sum_{(g)} \langle \varphi_i, S(g_{(2)})h \rangle \langle \varphi_{(1)}, g_{(1)} \rangle \varphi_{(2)} \cdot h_i \\
&\quad = \sum_{(\varphi)} \sum_{(g)} \langle \varphi_{(1)}, g_{(1)} \rangle \varphi_{(2)} \cdot (S(g_{(2)})h).
\end{aligned}$$

But

$$\begin{aligned}
&\sum_{(\varphi)} \sum_{(g)} \langle \varphi_{(1)}, g_{(1)} \rangle \varphi_{(2)} \cdot (S(g_{(2)})h) \\
&\quad = \sum_{(\varphi)} \sum_{(g),(h)} \langle \varphi_{(1)}, g_{(1)} \rangle \langle \varphi_{(2)}, S(g_{(2)})h_{(2)} \rangle S(g_{(3)})h_{(1)} \\
&\quad = \sum_{(g).(h)} \langle \varphi, g_{(1)} S(g_{(2)})h_{(2)} \rangle S(g_{(3)})h_{(1)} \\
&\quad = \sum_{(h)} \langle \varphi, h_{(2)} \rangle S(g)h_{(1)} = S(g)(\varphi \cdot h).
\end{aligned}$$

Letting I be the set of elements h of H such that $S(g)h = \langle \varepsilon, g \rangle h = \langle \varepsilon, S(g) \rangle h$, I is the ideal of left integrals in H, and by the theory of Hopf modules, I is a direct summand of H and the H^*-Hopf module H is isomorphic to $H^* \otimes I$. In fact the projection of H onto I is given by mapping an element h of H to

$\sum_{i=1}^{n} \sum_{(\varphi_i)} \langle \varphi_{i,(2)}, h \rangle S^* S^*(\varphi_{i,(1)}) \cdot h_i$ and the H^*-Hopf module isomorphism of H onto $H^* \otimes I$ is given by mapping h to $\sum_{i=1}^{n} \sum_{(\varphi_i)} \langle \varphi_{i,(3)}, h \rangle S^*(\varphi_{i,(2)}) \otimes S^* S^*(\varphi_{i,(1)}) \cdot h_i$. Then I is a finitely generated, projective module over R, since I is a direct summand of H, and H is a finitely generated, projective left module over H^*, since it is isomorphic to $H^* \otimes I$. □

Now the program in [3] can be carried out and complete proofs of the following results can be found there.

Lemma 2. *If A and H are free modules over R, then A and H have the same rank n and the direct sum of n copies of A is isomorphic as a left module over H^* to the direct sum of n copies of H.*

Note that an H-Galois algebra is a finitely generated and projective module over R [4, Thm. 1.7]. To prove the lemma, let n be the rank of the free module A and use the left A-module isomorphism γ of $A \otimes A$ onto $A \otimes H$, which is induced by the comodule map α of A into $A \otimes H$.

Proposition 3. *If R is a field, then A and H are isomorphic left modules over H^*.*

The direct sum of n copies of A and the direct sum of n copies of H are finite dimensional vector spaces over a field R, and so they satisfy the ascending and descending chain conditions for left H^*-submodules. Apply the Krull-Schmidt theorem to prove this proposition.

Lemma 4. *Let J be an ideal in R, let $\bar{R} = R/J$, and assume that $\bar{\omega} : \bar{R} \otimes H \longrightarrow \bar{R} \otimes A$ is a homomorphism of left modules over $\bar{R} \otimes H^*$. There exists a left H^*-module homomorphism $\omega : H \longrightarrow A$ such that $\bar{\omega} = 1 \otimes \omega$. Moreover, if $\bar{\omega}$ is an isomorphism and J is contained in the Jacobson radical of R, then ω is an isomorphism.*

Since H is a projective left H^*-module, the map of H to $\bar{R} \otimes A$, obtained by composing $\bar{\omega}$ with the canonical map of H onto $\bar{R} \otimes H$, can be lifted to a left H^*-module homomorphism ω of H into A. Assume that J is contained in the Jacobson radical of R and $\bar{\omega}$ is an isomorphism. Then coker ω is a finitely generated module over R and $\bar{R} \otimes$ coker $\omega =$ coker $\bar{\omega} = 0$. By Nakayama's Lemma, coker $\omega = 0$. Since A is a projective module over R, the sequence of R-modules ker $\omega \longrightarrow H \xrightarrow{\omega} A$ is split, ker ω is a finitely generated module over R, and $\bar{R} \otimes$ ker $\omega =$ ker $\bar{\omega} = 0$. Again by Nakayama's Lemma, ker $\omega = 0$.

The fact that every H-Galois algebra over a local ring is cleft follows easily from Lemma 4 and Proposition 3. Or, one can follow the argument in [3] to prove the following theorem.

Theorem 5. *If there is a basis of sets which are both open and closed for the Zariski topology on the set of maximal ideals of R, then A and H are isomorphic left modules over H^*.*

The following corollary is the only claim in [3] left to be verified.

Corollary 6. *Any H-Galois algebra is a finitely generated, projective left module over H^*.*

Proof. Let A be an H-Galois algebra. First it will be shown that A is a finitely presented left module over H^*. Since A is a finitely generated module over R, it is a finitely generated left H^*-module. Let K be the kernel of an epimorphism of a finitely generated free left H^*-module F onto the left H^*-module A. Since H^* is a finitely generated module over R, F is a finitely generated module over R, and since A is a projective module over R, the R-module K is a direct summand of F. Therefore K is a finitely generated R-module, consequently K is a finitely generated left H^*-module, and A is a finitely presented left H^*-module. Let R_p denote the local ring at a prime ideal p of R, and let $M_p = R_p \otimes M$ for any module M over R. Then $\mathrm{Hom}_{H^*}(A, X)_p$ is naturally isomorphic to $\mathrm{Hom}_{H^*}(A, X_p) = \mathrm{Hom}_{H_p^*}(A_p, X_p)$ for any left H^*-module X by [1, Chapter I, §2, No. 9, Prop. 10]. A is a finitely generated, projective left module over H^* if, and only if, for every epimorphism of a left H^*-module X onto a left H^*-module Y, the induced map of $\mathrm{Hom}_{H^*}(A, X)$ into $\mathrm{Hom}_{H^*}(A, Y)$ is surjective. But an R-module homomorphism of $\mathrm{Hom}_{H^*}(A, X)$ into $\mathrm{Hom}_{H^*}(A, Y)$ is surjective if, and only if, the corresponding map of $\mathrm{Hom}_{H_p^*}(A_p, X_p)$ into $\mathrm{Hom}_{H_p^*}(A_p, Y_p)$ is surjective for every maximal ideal p of R by [1, Chapter II, §3, No. 3, Thm. 1]. Thus it is only necessary to prove Corollary 6 when R is a local ring. But then A and H are isomorphic left modules over H^* by Theorem 5, and H is a finitely generated, projective left module over H^* by Proposition 1. □

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