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ERGODICITY OF THE ACTION OF THE POSITIVE RATIONALS ON THE GROUP OF FINITE ADELES AND THE BOST-CONNES PHASE TRANSITION THEOREM

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Abstract. We study relatively invariant measures with the multiplicators $\mathbb{Q}_+^* \ni q \mapsto q^{-\beta}$ on the space $\mathcal{A}_f$ of finite adeles. We prove that for $\beta \in (0,1]$ such measures are ergodic, and then deduce from this the uniqueness of KMS_β-states for the Bost-Connes system. Combining this with a result of Blackadar and Boca-Zaharescu, we also obtain ergodicity of the action of $\mathbb{Q}^*$ on the full adeles.

Bost and Connes [BC] constructed a remarkable C*-dynamical system which has a phase transition with spontaneous symmetry breaking involving an action of the Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, and whose partition function is the Riemann ζ function. In their original definition the underlying algebra arises as the Hecke algebra associated with an inclusion of certain $ax+b$ groups. Recently Laca and Raeburn [LR, L2] have realized the Bost-Connes algebra as a full corner of the crossed product algebra $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^*$. This new look at the system has allowed to simplify significantly the proof of the existence of KMS-states for all temperatures, and the classification of KMS_β-states for $\beta > 1$ [L1]. On the other hand, for $\beta \le 1$ the uniqueness of KMS_β-states implies ergodicity of the action of $\mathbb{Q}_+^*$ on $\mathcal{A}_f$ for certain measures (in particular, for the Haar measure). The aim of this note is to give a direct proof of the ergodicity, and then to show that the uniqueness of KMS_β-states easily follows from it.

So let $\mathcal{P}$ be the set of prime numbers, $\mathcal{A}_f$ the restricted product of the fields $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$, $p \in \mathcal{P}$, $\mathcal{R} = \prod_p \mathbb{Z}_p$ its maximal compact subring, $W = \mathcal{R}^* = \prod_p \mathbb{Z}_p^*$. The group $\mathbb{Q}_+^*$ of positive rationals is embedded diagonally into $\mathcal{A}_f$, and so acts by multiplication on the additive group of finite adeles. Then the Bost-Connes algebra $\mathcal{C}_\mathbb{Q}$ is the full corner of $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^*$ determined by the characteristic function of $\mathcal{R}$ [L2]. The dynamics σ_t is defined as follows [L1]: it is trivial on $C_0(\mathcal{A}_f)$, and $\sigma_t(u_q) = q^{it} u_q$, where u_q is the multiplier of $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^*$ corresponding to $q \in \mathbb{Q}_+^*$. Then ([L1]) there is a one-to-one correspondence between

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(β,σ_t)-KMS-states on $\mathcal{C}_{\mathbb{Q}}$ and measures μ on $\mathcal{A}_f$ such that

$$(1\beta)\qquad \mu(\mathcal{R})=1 \ \text{ and } \ q_*\mu=q^\beta\mu \ \text{ for all } q\in\mathbb{Q}^*_+ \ \ (\text{i.e., } \mu(q^{-1}X)=q^\beta\mu(X)).$$

Namely, the KMS-state corresponding to μ is the restriction of the dual weight on $C_0(\mathcal{A}_f)\rtimes\mathbb{Q}^*_+$ to $\mathcal{C}_{\mathbb{Q}}$.

Note ([L1]) that if $\beta>1$ and μ is a measure with property (1β), then

$$\mu(W)=\prod_{p\in\mathcal{P}}(1-p^{-\beta})=\frac{1}{\zeta(\beta)}>0,$$

since $W=\mathcal{R}\backslash\bigcup_p p\mathcal{R}$. Moreover, the sets qW, $q\in\mathbb{Q}^*_+$, are disjoint, and their union is a set of full measure (since $\sum_{n\in\mathbb{N}}\mu(nW)=\frac{1}{\zeta(\beta)}\sum_{n\in\mathbb{N}}n^{-\beta}=1$). Thus there exists a one-to-one correspondence between probability measures on W and measures on $\mathcal{A}_f$ satisfying (1β) [L1]. On the other hand, if $\beta\le 1$, then $\mu(W)=0$.

Proposition. *For $\beta\in(0,1]$ and any measure μ satisfying (1β), the action of $\mathbb{Q}^*_+$ on $(\mathcal{A}_f,\mu)$ is ergodic.*

Proof. Consider the space $L^2(\mathcal{R},d\mu)$ and the subspace H of it consisting of the functions that are constant on $\mathbb{N}$-orbits. In other words,

$$H=\{f\in L^2(\mathcal{R},d\mu)\,|\,V_nf=f,\ n\in\mathbb{N}\},$$

where $(V_nf)(x)=f(nx)$. Since any $\mathbb{Q}^*_+$-invariant subset of $\mathcal{A}_f$ is completely determined by its intersection with $\mathcal{R}$, it suffices to prove that H consists of constant functions. For this we will compute the action of the projection $P\colon L^2(\mathcal{R},d\mu)\to H$ on functions spanning a dense subspace of $L^2(\mathcal{R},d\mu)$.

Let B be a finite subset of $\mathcal{P}$. Consider the projection $\pi_B\colon\mathcal{R}\to\prod_{p\in B}\mathbb{Z}_p$, and set $\mu_B=(\pi_B)_*\mu$. Then $L^2(\prod_{p\in B}\mathbb{Z}_p,d\mu_B)$ can be considered as a subspace of $L^2(\mathcal{R},d\mu)$, and the union of these subspaces over all finite B is dense in $L^2(\mathcal{R},d\mu)$. The characters of $\prod_{p\in B}\mathbb{Z}^*_p$ span a dense subspace of $L^2(\prod_{p\in B}\mathbb{Z}^*_p,d\mu_B)$. Let $\mathbb{N}_B$ be the unital multiplicative subsemigroup of $\mathbb{N}$ generated by $p\in B$. Note that the sets $n\prod_{p\in B}\mathbb{Z}^*_p$, $n\in\mathbb{N}_B$, are disjoint, their union is a subset of $\prod_{p\in B}\mathbb{Z}_p$ of full measure (condition (1β) implies that the set $\{x\in\mathcal{R}\,|\,x_p=0\}$ has zero measure), and the operator $n^{-\beta/2}V^*_n$ maps isometrically $L^2(\prod_{p\in B}\mathbb{Z}^*_p,d\mu_B)$ onto $L^2(n\prod_{p\in B}\mathbb{Z}^*_p,d\mu_B)$ for any $n\in\mathbb{N}_B$. Hence the functions $V^*_n\chi$, $n\in\mathbb{N}_B$, $\chi\in(\prod_{p\in B}\mathbb{Z}^*_p)\hat{}$, span a dense subspace of $L^2(\prod_{p\in B}\mathbb{Z}_p,d\mu_B)$. So we have to compute $PV^*_n\chi$. But if $g\in H$, then $(V^*_n\chi,g)=(\chi,g)$, whence $PV^*_n\chi=P\chi$. Thus we have only to compute $P\chi$.

For a finite subset A of $\mathcal{P}$, let H_A be the subspace consisting of the functions that are constant on $\mathbb{N}_A$-orbits, P_A the projection onto H_A. Then $P_A\searrow P$ as $A\nearrow\mathcal{P}$. Set

$$W_A=\prod_{p\in A}\mathbb{Z}^*_p\times\prod_{q\in\mathcal{P}\backslash A}\mathbb{Z}_q\subset\mathcal{R}.$$

Note, as above, that $\bigcup_{n\in\mathbb{N}_A}nW_A$ is a subset of $\mathcal{R}$ of full measure. We assert that

$$(2)\qquad P_Af|_{\mathbb{N}_Ax}\equiv\frac{1}{\zeta_A(\beta)}\sum_{n\in\mathbb{N}_A}n^{-\beta}f(nx)\ \text{ for }\ x\in W_A,$$

where $\zeta_A(\beta)=\sum_{n\in\mathbb{N}_A} n^{-\beta}=\prod_{p\in A}(1-p^{-\beta})^{-1}$. Indeed, denoting the right hand part of (2) by f_A, for $g\in H_A$ we obtain

$$\begin{aligned}(f_A,g) &= \sum_{n\in\mathbb{N}_A}\int_{nW_A} f_A(x)\overline{g(x)}d\mu(x)=\sum_{n\in\mathbb{N}_A} n^{-\beta}\int_{W_A} f_A(x)\overline{g(x)}d\mu(x)\\ &= \zeta_A(\beta)\int_{W_A} f_A(x)\overline{g(x)}d\mu(x)=\sum_{n\in\mathbb{N}_A} n^{-\beta}\int_{W_A} f(nx)\overline{g(x)}d\mu(x)\\ &= \sum_{n\in\mathbb{N}_A}\int_{nW_A} f(x)\overline{g(x)}d\mu(x)=(f,g).\end{aligned}$$

Returning to the computation of $P\chi$, we see that

$$(3)\qquad P_A\chi|_{\mathbb{N}_A x}\equiv\frac{\chi(x)}{\zeta_A(\beta)}\sum_{n\in\mathbb{N}_A} n^{-\beta}\chi(n)=\chi(x)\prod_{p\in A}\frac{1-p^{-\beta}}{1-\chi(p)p^{-\beta}}\quad\text{for}\quad x\in W_A.$$

Hence

$$(4)\qquad \|P\chi\|_2=\lim_{A\nearrow\mathcal{P}}\prod_{p\in A}\left|\frac{1-p^{-\beta}}{1-\chi(p)p^{-\beta}}\right|.$$

If χ is trivial, then using (3) we see that $P_A\chi\equiv\prod_{p\in B}(1-p^{-\beta})$ for all $A\supset B$, hence $P\chi$ is a constant. Suppose χ is non-trivial. The limit in (4) is an increasing function in β on $(0,+\infty)$ (because each factor is increasing), which is equal to $|L(\beta,\chi)|\zeta(\beta)^{-1}$ for $\beta>1$, where $L(\beta,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-\beta}$ is the Dirichlet L-function corresponding to the number character χ [S]. By elementary properties of Dirichlet series, $|L(\beta,\chi)|$ tends to a finite value as $\beta\to 1+0$, while $\zeta(\beta)\to\infty$. Thus $P\chi=0$. □

Since the set of measures satisfying (1β) is convex and consists of ergodic measures, there exists at most one measure satisfying (1β). Such a measure does exist. In fact, for each $\beta\in(0,+\infty)$ there is a unique W-invariant measure μ_β satisfying (1β) [BC, L1]. Explicitly, $\mu_\beta=\prod_p\mu_{\beta,p}$, where $\mu_{\beta,p}$ is the measure on $\mathbb{Q}_p$ such that $\mu_{1,p}$ is the Haar measure ($\mu_{1,p}(\mathbb{Z}_p)=1$), and

$$\frac{d\mu_{\beta,p}}{d\mu_{1,p}}(a)=\frac{1-p^{-\beta}}{1-p^{-1}}|a|_p^{\beta-1}\quad\text{for}\quad a\in\mathbb{Q}_p.$$

Let ϕ_β be the (β,σ_t)-KMS state on $\mathcal{C}_{\mathbb{Q}}$ corresponding to μ_β. Then $\pi_{\phi_\beta}(\mathcal{C}_{\mathbb{Q}})''$ is a factor, which is a reduction of the factor $L^\infty(\mathcal{A}_f,\mu_\beta)\rtimes\mathbb{Q}_+^*$. It is easy to describe its flow of weights [CT]. Consider a standard measure space X_β with the measure algebra consisting of $\mathbb{Q}_+^*$-invariant $(\lambda\times\mu_\beta)$-measurable subsets of $\mathbb{R}_+\times\mathcal{A}_f$, where λ is the Lebesgue measure, and the flow F_t^β on it coming from the action $t(x,a)=(e^{-t/\beta}x,a)$ of $\mathbb{R}$ on $\mathbb{R}_+\times\mathcal{A}_f$. Then F_t^β is ergodic by Proposition, and it is the flow of weights of the factors $L^\infty(\mathcal{A}_f,\mu_\beta)\rtimes\mathbb{Q}_+^*$ and $\pi_{\phi_\beta}(\mathcal{C}_{\mathbb{Q}})''$.

Theorem ([BC]). *For $\beta\in(0,1]$, μ_β is a unique measure satisfying (1β). The action of $\mathbb{Q}_+^*$ on $(\mathcal{A}_f,\mu_\beta)$ is ergodic, moreover, the action of $\mathbb{Q}^*$ on $(\mathcal{A},\nu_\beta)$, where $\mathcal{A}=\mathbb{R}\times\mathcal{A}_f$ is the space of full adeles and $\nu_\beta=\lambda\times\mu_\beta$, is ergodic. Equivalently, ϕ_β is a unique (β,σ_t)-KMS state on $\mathcal{C}_{\mathbb{Q}}$, and $\pi_{\phi_\beta}(\mathcal{C}_{\mathbb{Q}})''$ is the hyperfinite factor of type III_1.*

Proof. In view of the above description of the flow of weights, the factor $\pi_{\phi_\beta}(\mathcal{C}_\mathbb{Q})''$ is of type III_1 if and only if the action of $\mathbb{Q}^*_+$ on $(\mathbb{R}_+\times\mathcal{A}_f, \lambda\times\mu_\beta)$ is ergodic, or equivalently, the action of $\mathbb{Q}^*$ on $(\mathcal{A},\nu_\beta)$ is ergodic.

To prove the ergodicity, first note that the action of W on X_β is ergodic. Indeed, the induced flow on X_β/W is the flow of weights of the factor

$$L^\infty(\mathcal{A}_f/W,\mu_\beta)\rtimes\mathbb{Q}^*_+ = (L^\infty(\mathcal{A}_f,\mu_\beta)\rtimes\mathbb{Q}^*_+)^W.$$

It is easy to see ([BC]) that this factor is ITPFI with eigenvalue list

$$\{p^{-n\beta}(1-p^{-\beta})\,|\,n\ge 0\}_{p\in\mathcal{P}}.$$

Hence it is of type III_1 by [B] (see also [BZ]). Thus its flow of weights is trivial, i.e. the action of W on X_β is ergodic. Since W is compact, the action is transitive. So we may identify X_β with W/W_β for some closed subgroup W_β of W. Then the flow F^β_t is given by a continuous one-parametric subgroup of W/W_β. Since W/W_β is totally disconnected, this one-parametric subgroup is trivial, and since the flow is ergodic, $W_\beta = W$. Thus X_β is singlepoint, and the action of $\mathbb{Q}^*_+$ on $(\mathbb{R}_+\times\mathcal{A}_f,\lambda\times\mu_\beta)$ is ergodic. ◻

Remarks. (i) In order to prove that ϕ_β is a unique KMS_β-state, it is enough to know that μ_β is ergodic. Indeed, this means that ϕ_β is an extremal KMS_β-state. Since ϕ_β is a unique W-invariant KMS_β-state, we can argue as in the proof of [BC, Theorem 25]: if ψ is an extremal KMS_β-state, then

$$\int_W w_*\psi\,dw = \phi_\beta.$$

Since KMS_β-states form a simplex, we conclude that $\psi=\phi_\beta$.

Thus, the uniqueness and the assertion about the type for KMS_β-states follow easily from ergodicity of the action of $\mathbb{Q}^*$ on $(\mathcal{A},\nu_\beta)$.

(ii) A slight modification of the argument in the proof of the Theorem gives the following general result, apparently well-known to specialists: if M is a factor and G a compact totally disconnected group acting on M such that M^G is a factor of type III_1, then M is also of type III_1.

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