ERGODICITY OF THE ACTION
OF THE POSITIVE RATIONALS
ON THE GROUP OF FINITE ADELES AND
THE BOST-CONNES PHASE TRANSITION THEOREM

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(Communicated by David R. Larson)

Abstract. We study relatively invariant measures with the multiplicators
$Q^+ \ni q \mapsto q^{-\beta}$ on the space $A_f$ of finite adeles. We prove that for
$\beta \in (0, 1]$ such measures are ergodic, and then deduce from this the uniqueness of KMS-$\beta$-
states for the Bost-Connes system. Combining this with a result of Blackadar
and Boca-Zaharescu, we also obtain ergodicity of the action of $Q^*$ on the full
adeles.

Bost and Connes [BC] constructed a remarkable C$^*$-dynamical system which
has a phase transition with spontaneous symmetry breaking involving an action
of the Galois group $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, and whose partition function is the Riemann $\zeta$
function. In their original definition the underlying algebra arises as the Hecke
algebra associated with an inclusion of certain $ax + b$ groups. Recently Laca and
Raeburn [LR, L2] have realized the Bost-Connes algebra as a full corner of the
crossed product algebra $C_0(A_f) \rtimes Q^+$. This new look at the system has allowed to
simplify significantly the proof of the existence of KMS-states for all temperatures,
and the classification of KMS-$\beta$-states for $\beta > 1$ [L1]. On the other hand, for $\beta \leq 1$
the uniqueness of KMS-$\beta$-states implies ergodicity of the action of $Q^+$ on $A_f$ for
certain measures (in particular, for the Haar measure). The aim of this note is
to give a direct proof of the ergodicity, and then to show that the uniqueness of
KMS-$\beta$-states easily follows from it.

So let $\mathcal{P}$ be the set of prime numbers, $A_f$ the restricted product of the fields
$\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$, $\mathcal{R} = \prod_p \mathbb{Z}_p$ its maximal compact subring, $W = \mathcal{R}^* = \prod_p \mathbb{Z}_p^*$. The group $Q^+_+$ of positive rationals is embedded diagonally into
$A_f$, and so acts by multiplication on the additive group of finite adeles. Then
the Bost-Connes algebra $C_0$ is the full corner of $C_0(A_f) \rtimes Q^+_+$ determined by the
characteristic function of $\mathcal{R}$ [L2]. The dynamics $\sigma_t$ is defined as follows [L1]: it is
trivial on $C_0(A_f)$, and $\sigma_t(u_q) = q^{it} u_q$, where $u_q$ is the multiplier of $C_0(A_f) \rtimes Q^+_+$
corresponding to $q \in Q^+_+$. Then [L1]) there is a one-to-one correspondence between

Received by the editors November 28, 2000 and, in revised form, May 11, 2001.
1991 Mathematics Subject Classification. Primary 46L55; Secondary 28D15.
This research was partially supported by Award No UMI-2092 of the Civilian Research &
Development Foundation.
(β, σ)-KMS-states on \(C_Q\) and measures \(μ\) on \(A_f\) such that

\[(1\beta) \quad μ(\mathcal{R}) = 1 \quad \text{and} \quad q_∗μ = q^βμ \quad \text{for all} \quad q \in \mathbb{Q}_+^* \quad (\text{i.e.,} \quad μ(q^{-1}X) = q^βμ(X)).\]

Namely, the KMS-state corresponding to \(μ\) is the restriction of the dual weight on \(C_0(A_f) \cong \mathbb{Q}_+^*\) to \(C_Q\).

Note (\[\square\]) that if \(β > 1\) and \(μ\) is a measure with property \((1\beta)\), then

\[μ(W) = \prod_{p \in \mathcal{P}} (1 - p^{-β}) = \frac{1}{ζ(β)} > 0,\]

since \(W = \mathcal{R} \setminus \bigcup_p p\mathcal{R}\). Moreover, the sets \(qW, \quad q \in \mathbb{Q}_+^*\), are disjoint, and their union is a set of full measure (since \(\sum_{n \in \mathbb{N}} μ(nW) = \frac{1}{ζ(β)} \sum_{n \in \mathbb{N}} n^{-β} = 1\)). Thus there exists a one-to-one correspondence between probability measures on \(W\) and measures on \(A_f\) satisfying \((1\beta)\) \([\square]\). On the other hand, if \(β ≤ 1\), then \(μ(W) = 0\).

**Proposition.** For \(β \in (0, 1]\) and any measure \(μ\) satisfying \((1\beta)\), the action of \(\mathbb{Q}_+^*\) on \(\mathcal{A}_f\) is ergodic.

**Proof.** Consider the space \(L^2(\mathcal{R}, dμ)\) and the subspace \(H\) of it consisting of the functions that are constant on \(\mathbb{N}\)-orbits. In other words,

\[H = \{f \in L^2(\mathcal{R}, dμ) \mid V_n f = f, \quad n \in \mathbb{N}\},\]

where \((V_n f)(x) = f(nx)\). Since any \(\mathbb{Q}_+^*\)-invariant subset of \(A_f\) is completely determined by its intersection with \(\mathcal{R}\), it suffices to prove that \(H\) consists of constant functions. For this we will compute the action of the projection \(P: L^2(\mathcal{R}, dμ) \to H\) on functions spanning a dense subspace of \(L^2(\mathcal{R}, dμ)\).

Let \(B\) be a finite subset of \(\mathcal{P}\). Consider the projection \(π_B: \mathcal{R} \to \prod_{p \in B} \mathbb{Z}_p\), and set \(μ_B = (π_B)_∗μ\). Then \(L^2(\prod_{p \in B} \mathbb{Z}_p, dμ_B)\) can be considered as a subspace of \(L^2(\mathcal{R}, dμ)\), and the union of these subspaces over all finite \(B\) is dense in \(L^2(\mathcal{R}, dμ)\). The characters of \(\prod_{p \in B} \mathbb{Z}_p\) span a dense subspace of \(L^2(\prod_{p \in B} \mathbb{Z}_p, dμ_B)\). Let \(N_B\) be the unitary multiplicative subgroup of \(\mathbb{N}\) generated by \(p \in B\). Note that the sets \(n \prod_{p \in B} \mathbb{Z}_p^n, \quad n \in \mathbb{N}_B\), are disjoint, their union is a subset of \(\prod_{p \in B} \mathbb{Z}_p\) of full measure (condition \((1\beta)\) implies that the set \(\{x \in \mathcal{R} \mid x_p = 0\}\) has zero measure), and the operator \(n^{-β/2}V_n^∗\) maps isometrically \(L^2(\prod_{p \in B} \mathbb{Z}_p^*, dμ_B)\) onto \(L^2(n \prod_{p \in B} \mathbb{Z}_p^*, dμ_B)\) for any \(n \in \mathbb{N}_B\). Hence the functions \(V_n^∗χ, \quad n \in \mathbb{N}_B, \quad χ \in (\prod_{p \in B} \mathbb{Z}_p^*)\), span a dense subspace of \(L^2(\prod_{p \in B} \mathbb{Z}_p, dμ_B)\). So we have to compute \(PV_n^∗χ\). But if \(g \in H\), then \((V_n^∗χ, g) = (χ, g)\), whence \(PV_n^∗χ = Pχ\). Thus we have only to compute \(Pχ\).

For a finite subset \(A\) of \(\mathcal{P}\), let \(H_A\) be the subspace consisting of the functions that are constant on \(\mathbb{N}_A\)-orbits, \(P_A\) the projection onto \(H_A\). Then \(P_A \setminus P\) as \(A \setminus \mathcal{P}\). Set

\[W_A = \prod_{p \in A} \mathbb{Z}_p^* \times \prod_{q \in \mathcal{P} \setminus A} \mathbb{Z}_q \subset \mathcal{R} .\]

Note, as above, that \(\bigcup_{n \in \mathbb{N}_A} nW_A\) is a subset of \(\mathcal{R}\) of full measure. We assert that

\[(2) \quad P_A f|_{\mathbb{N}_A x} \equiv \frac{1}{ζ_A(β)} \sum_{n \in \mathbb{N}_A} n^{-β} f(nx) \quad \text{for} \quad x \in W_A ,\]
Hence a factor, which is a reduction of the factor $L$ is the Lebesgue measure, and the flow $\lambda |_\beta$ tends to a finite value as $t/\beta \to t/\beta$.

Thus algebra consisting of $Q$ of Dirichlet series, $|_\beta$ function corresponding to the number character $\chi$\cite{[CT]}. Consider a standard measure space $A \times \nu$. Let $\nu'$ be a unique $\nu_\beta$-invariant ($\lambda \times \mu_{\beta}$)-measurable subsets of $R_+ \times A_f$, where $\lambda$ is the Lebesgue measure, and the flow $F_{t,\beta}^{\nu} = (t(x,a))$ of $R_+ \times A_f$. Then $F_{t,\beta}^{\nu}$ is ergodic by Proposition, and it is the flow of weights $A \times \nu_{\beta}$.

Theorem (\cite{[BC]}). For $\beta \in (0,1]$, $\mu_{\beta}$ is a unique measure satisfying $\lambda |_\beta$. The action of $Q^*_+ \times (A_f,\mu_{\beta})$ is ergodic, moreover, the action of $Q^*_+ \times (A,\nu_{\beta})$, where $A = R_+ \times A_f$ is the space of full adeles and $\nu_{\beta} = \lambda \times \mu_{\beta}$, is ergodic. Equivalently, $\phi_{\beta}$ is a unique $(\beta,\sigma_1)$-KMS state on $C_\beta$, and $\pi_{\phi_{\beta}}(C_\beta)$" is the hyperfinite factor of type $III_1$. 

where $\zeta_A(\beta) = \sum_{n \in N_A} n^{-\beta} = \prod_{p \in A} (1 - p^{-\beta})^{-1}$. Indeed, denoting the right hand part of (2) by $f_A$, for $g \in H_A$ we obtain

$$\langle f_A, g \rangle = \sum_{n \in N_A} \int_{nW_A} f_A(x) \overline{g(x)} \mu(x) = \sum_{n \in N_A} n^{-\beta} \int_{W_A} f_A(x) \overline{g(x)} \mu(x) = \zeta_A(\beta) \int_{W_A} f_A(x) \overline{g(x)} \mu(x) = \sum_{n \in N_A} \int_{nW_A} f(x) \overline{g(x)} \mu(x) = (f, g).$$

Returning to the computation of $P_{\chi}$, we see that

\begin{equation}
P_{A \chi |_{N_A x}} = \frac{\chi(n)}{\zeta_A(\beta)} \sum_{n \in N_A} n^{-\beta} \chi(n) = \chi(x) \prod_{p \in A} \frac{1 - p^{-\beta}}{1 - \chi(p)p^{-\beta}} \quad \text{for } x \in W_A.
\end{equation}

Hence

\begin{equation}
\|P_{\chi}\|_2 = \lim_{A,\beta \to A,\beta} \prod_{p \in A} \left| \frac{1 - p^{-\beta}}{1 - \chi(p)p^{-\beta}} \right|.
\end{equation}

If $\chi$ is trivial, then using (3) we see that $P_{A \chi} = \prod_{p \in B}(1 - p^{-\beta})$ for all $A \supset B$, hence $P_{\chi}$ is a constant. Suppose $\chi$ is non-trivial. The limit in (4) is an increasing function in $\beta$ on $(0, +\infty)$ (because each factor is increasing), which is equal to $|L(\beta, \chi)|\zeta(\beta)^{-1}$ for $\beta > 1$, where $L(\beta, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-\beta}$ is the Dirichlet $L$-function corresponding to the number character $\chi$\cite{[S]}. By elementary properties of Dirichlet series, $|L(\beta, \chi)|$ tends to a finite value as $\beta \to 1 + 0$, while $\zeta(\beta) \to \infty$. Thus $P_{\chi} = 0$. 

Since the set of measures satisfying (1) is convex and consists of ergodic measures, there exists at most one measure satisfying (1). Such a measure does exist. In fact, for each $\beta \in (0, +\infty)$ there is a unique $W$-invariant measure $\mu_{\beta}$ satisfying (1) $\beta$\cite{[BC],[L]}. Explicitly, $\mu_{\beta} = \prod_p \mu_{\beta,p}$, where $\mu_{\beta,p}$ is the measure on $Q_p$ such that $\mu_{1,p}$ is the Haar measure ($\mu_{1,p}(Z_{p}) = 1$), and

\[ \frac{d\mu_{\beta,p}}{d\mu_{1,p}}(a) = \frac{1 - p^{-\beta}}{1 - p^{-1}}|a|_{\beta}^{-1} \quad \text{for } a \in Q_p. \]

Let $\phi_{\beta}$ be the $(\beta, \sigma_1)$-KMS state on $C_\beta$ corresponding to $\mu_{\beta}$. Then $\pi_{\phi_{\beta}}(C_\beta)$" is a factor, which is a reduction of the factor $L^\infty(A_f, \mu_{\beta}) \times Q^*_+$. It is easy to describe its flow of weights \cite{[CT]}. Consider a standard measure space $X_{\beta}$ with the measure algebra consisting of $Q^*_+$-invariant ($\lambda \times \mu_{\beta}$)-measurable subsets of $R_+ \times A_f$, where $\lambda$ is the Lebesgue measure, and the flow $F_{t,\beta}$ on it coming from the action $t(x,a) = (t(x),a)$ of $R_+ \times A_f$. Then $F_{t,\beta}$ is ergodic by Proposition, and it is the flow of weights of the factors $L^\infty(A_f, \mu_{\beta}) \times Q^*_+$ and $\pi_{\phi_{\beta}}(C_\beta)$".
Proof. In view of the above description of the flow of weights, the factor $\pi_{\phi_\beta}(C_Q)^\prime$ is of type III$_1$ if and only if the action of $Q^\ast_+$ on $(\mathbb{R}_+ \times A_f, \lambda \times \mu_\beta)$ is ergodic, or equivalently, the action of $Q^\ast_+$ on $(A, \nu_\beta)$ is ergodic.

To prove the ergodicity, first note that the action of $W$ on $X_\beta$ is ergodic. Indeed, the induced flow on $X_\beta/W$ is the flow of weights of the factor $L_\infty(A_f/W, \mu_\beta) \rtimes Q_+^\ast$. It is easy to see (BC) that this factor is ITPFI with eigenvalue list
\[ \{p^{-n\beta}(1-p^{-\beta}) | n \geq 0\} \text{ for } p \in \mathbb{P}. \]
Hence it is of type III$_1$ by [B] (see also [BZ]). Thus its flow of weights is trivial, i.e. the action of $W$ on $X_\beta$ is ergodic. Since $W$ is compact, the action is transitive.

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The author is grateful to Marcelo Laca, Sergey Sinelshchikov and Yoshimichi Ueda for helpful discussions.

Acknowledgement

The author is grateful to Marcelo Laca, Sergey Sinelshchikov and Yoshimichi Ueda for helpful discussions.

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