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# FINITE SUMS OF COMMUTATORS

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ABSTRACT. We show that elements of unital $C^*$-algebras without tracial states are finite sums of commutators. Moreover, the number of commutators involved is bounded, depending only on the given $C^*$-algebra.

## 1. Introduction

It was shown in [2] that in finite von Neumann algebras, elements with central trace zero are sums of at most 10 commutators. The $C^*$-algebra case was considered in [1]. The main result there states that if the unit of a $C^*$-algebra $A$ is properly infinite (i.e. there exist two orthogonal projections $p, q \in A$ equivalent to 1), then any hermitian element is a sum of at most five self-adjoint commutators. In this paper we consider the more general case of unital $C^*$-algebras $A$ without tracial states and improve the previous result of T. Fack. Note that if the unit of $A$ is properly infinite, then $A$ has no tracial states. The converse is known to be false, at least when $A$ is non-simple (see [4] for further details). $C^*$-algebras without tracial states have several nice characterizations, such as [3]. This paper also contains another simple proof of the latter result of [3, Lemma 1].

## 2. The result

Given $a, b \in A$, their commutator is $[a, b] = ab - ba$. A self-adjoint commutator is just a commutator of the form $[a^*, a] = a^*a - aa^*$.

**Theorem 1.** *Let $A$ be a unital $C^*$-algebra. Then the following properties are equivalent:*

(1) *$A$ has no tracial states.*
(2) *There exist an integer $n \geq 2$ and elements $b_1, b_2, \ldots, b_n \in A$ such that*

$$\sum_{i=1}^{n} b_i^* b_i = 1 \quad \text{and} \quad \| \sum_{i=1}^{n} b_i b_i^* \| < 1.$$

(3) *There exists an integer $n \geq 2$ such that any element of $A$ can be expressed as a sum of $n$ commutators and any positive element can be expressed as a sum of at most $n$ self-adjoint commutators.*

*Remark* 2. The equivalence of (1) and (2) is just [3, Lemma 1]. As mentioned, in this paper we give a new simple proof.

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*Remark* 3. The integer $n$ appearing in (2) above automatically satisfies property (3). If the unit of $A$ is properly infinite, there exist two isometries $v_1, v_2 \in A$ with orthogonal ranges. Let $b_i = (1/\sqrt{2})v_i$ for $i = 1, 2$. Then $b_1^*b_1 + b_2^*b_2 = 1$ and $b_1b_1^* + b_2b_2^* \leq 1/2$; thus the property from (2) is achieved with $n = 2$. Thus in a properly infinite $C^*$-algebra, every element is the sum of two commutators, every positive element is the sum of two self-adjoint commutators, and every self-adjoint element is the sum of four self-adjoint commutators.

*Proof.* The implication (3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2). Consider

$$R = \{\sum_{i=1}^{s}(a_i^*a_i - a_ia_i^*)\ ;\ s \geq 1, a_i \in A\}$$

the set of **finite** sums of self-adjoint commutators of $A$. Note that $R \subset A_{sa}$ is a **real vector subspace** of $A_{sa}$. Put $\delta = \mathrm{dist}(1, R)$.

We show that $\delta < 1$. Suppose the contrary, i.e. $\delta = 1$. This is equivalent to

$$\|t + x\| \geq |t|, \quad \forall x \in R, \quad \forall t \in \mathbb{R}.$$

It follows that $\varphi(t + x) = t$ is a real bounded functional on $\mathbb{R}1 + R$ of norm 1. By the Hahn–Banach theorem it can be extended to a norm-1 functional on $A_{sa}$ and furthermore to a bounded **complex** functional on $A$, also denoted by $\varphi$. Observe that $\varphi$ is necessarily a tracial state on $A$, which contradicts our hypothesis.

Because $\delta < 1$, there exist some elements $a_1, a_2, \dots, a_m \in A$ such that $t_0 = \|1 - \sum_{i=1}^{m}(a_i^*a_i - a_ia_i^*)\| < 1$. In particular we have

$$\sum_{i=1}^{m} a_ia_i^* \leq -1 + t_0 + \sum_{i=1}^{m} a_i^*a_i. \tag{1}$$

Let $k = \|\sum_{i=1}^{m} a_i^*a_i\|$ and $a_{m+1} = (k - \sum_{i=1}^{m} a_i^*a_i)^{1/2}$. Then we have

$$\sum_{i=1}^{m+1} a_i^*a_i = k;$$

but on the other hand, by (1) we have also

$$\sum_{i=1}^{m+1} a_ia_i^* \leq -1 + t_0 + k.$$

The required properties are now fulfilled with $n = m + 1$ and $b_i = (1/\sqrt{k})a_i$.

(2) $\Rightarrow$ (3). Suppose that $b_1, b_2, \dots, b_n$ are as in (2). Define $\Phi(a) = \sum b_iab_i^*$. Then $\Phi$ is a bounded positive map on $A$ with norm $\|\Phi\| = \|\sum b_ib_i^*\| < 1$. It follows that $Id_A - \Phi$ is invertible in the Banach algebra $\mathcal{B}(A)$ of bounded maps on $A$. Let

$$\Psi = (Id_A - \Phi)^{-1}.$$

Note that $\Psi = \sum_{i=0}^{\infty} \Phi^i$, thus $\Psi$ is positive too. By definition of $\Psi$, for any $a \in A$ we have

$$\begin{aligned} a &= (Id_A - \Phi)(\Psi(a)) = \Psi(a) - \sum_{i=1}^{n} b_i\Psi(a)b_i^* \\ &= \sum_{i=1}^{n} b_i^*b_i\Psi(a) - \sum_{i=1}^{n} b_i\Psi(a)b_i^* = \sum_{i=1}^{n}[b_i^*, b_i\Psi(a)], \end{aligned}$$

so $a$ is a finite sum of at most $n$ commutators. If moreover $a$ is a positive element in $A$, then

$$\begin{aligned} a &= (Id_A - \Phi)(\Psi(a)) = \Psi(a) - \sum_{i=1}^{n} b_i \Psi(a) b_i^* \\ &= \sum_{i=1}^{n} \Psi(a)^{1/2} b_i^* b_i \Psi(a)^{1/2} - \sum_{i=1}^{n} b_i \Psi(a) b_i^* = \sum_{i=1}^{n} [\Psi(a)^{1/2} b_i^*, b_i \Psi(a)^{1/2}], \end{aligned}$$

so $a$ is a finite sum of at most $n$ self-adjoint commutators. □

## 3. Questions

For an infinite $C^*$-algebra $A$ (in the sense that it admits no tracial states) let $\nu(A)$ be the least positive integer such that any element of $A$ is a sum of at most $\nu(A)$ commutators. In all the examples that we know of, we have $\nu(A) = 2$. We believe that it is unlikely to always be the case.

In [3] it was shown that if $A$ is an unital exact $C^*$-algebra, then there exists an integer $m$ such that $\mathbb{M}_m(A)$ is properly infinite. It follows that $\nu(\mathbb{M}_m(A)) = 2$. Then a simple computation shows that $\nu(A) \leq 2m^2$. It would be interesting to answer the inverse problem, that is: assuming $\nu(A)$ is known, estimate the least positive integer $m$ such that $\nu(\mathbb{M}_m(A)) = 2$.

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