FINITE SUMS OF COMMUTATORS

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ABSTRACT. We show that elements of unital $C^*$-algebras without tracial states are finite sums of commutators. Moreover, the number of commutators involved is bounded, depending only on the given $C^*$-algebra.

1. Introduction

It was shown in [2] that in finite von Neumann algebras, elements with central trace zero are sums of at most 10 commutators. The $C^*$-algebra case was considered in [1]. The main result there states that if the unit of a $C^*$-algebra $A$ is properly infinite (i.e. there exist two orthogonal projections $p, q \in A$ equivalent to 1), then any hermitian element is a sum of at most five self-adjoint commutators. In this paper we consider the more general case of unital $C^*$-algebras $A$ without tracial states and improve the previous result of T. Fack. Note that if the unit of $A$ is properly infinite, then $A$ has no tracial states. The converse is known to be false, at least when $A$ is non-simple (see [4] for further details). $C^*$-algebras without tracial states have several nice characterizations, such as [3]. This paper also contains another simple proof of the latter result of [3, Lemma 1].

2. The result

Given $a, b \in A$, their commutator is $[a, b] = ab - ba$. A self-adjoint commutator is just a commutator of the form $[a^*, a] = a^*a - aa^*$.

Theorem 1. Let $A$ be a unital $C^*$-algebra. Then the following properties are equivalent:

1. $A$ has no tracial states.
2. There exist an integer $n \geq 2$ and elements $b_1, b_2, \ldots, b_n \in A$ such that

   $$\sum_{i=1}^{n} b_i^* b_i = 1 \quad \text{and} \quad \| \sum_{i=1}^{n} b_i b_i^* \| < 1.$$

3. There exists an integer $n \geq 2$ such that any element of $A$ can be expressed as a sum of $n$ commutators and any positive element can be expressed as a sum of at most $n$ self-adjoint commutators.

Remark 2. The equivalence of (1) and (2) is just [2, Lemma 1]. As mentioned, in this paper we give a new simple proof.
Remark 3. The integer \( n \) appearing in (2) above automatically satisfies property (3). If the unit of \( A \) is properly infinite, there exist two isometries \( v_1, v_2 \in A \) with orthogonal ranges. Let \( b_i = (1/\sqrt{2})v_i \) for \( i = 1, 2 \). Then \( b_1^*b_1 + b_2^*b_2 = 1 \) and \( b_1b_1^* + b_2b_2^* \leq 1/2 \); thus the property from (2) is achieved with \( n = 2 \). Thus in a properly infinite \( C^* \)-algebra, every element is the sum of two commutators, every positive element is the sum of two self-adjoint commutators, and every self-adjoint element is the sum of four self-adjoint commutators.

Proof. The implication (3) \( \Rightarrow \) (1) is trivial. (1) \( \Rightarrow \) (2). Consider

\[
R = \{ \sum_{i=1}^{s} (a_i^*a_i - a_i^*a_i) ; s \geq 1, a_i \in A \}
\]

the set of finite sums of self-adjoint commutators of \( A \). Note that \( R \subset A_{sa} \) is a real vector subspace of \( A_{sa} \). Put \( \delta = \text{dist}(1, R) \).

We show that \( \delta < 1 \). Suppose the contrary, i.e. \( \delta = 1 \). This is equivalent to

\[
||t + x|| \geq |t|, \quad \forall x \in R, \quad \forall t \in \mathbb{R}.
\]

It follows that \( \varphi(t + x) = t \) is a real bounded functional on \( \mathbb{R}^1 + R \) of norm 1. By the Hahn–Banach theorem it can be extended to a norm-1 functional on \( A_{sa} \) and furthermore to a bounded complex functional on \( A \), also denoted by \( \varphi \). Observe that \( \varphi \) is necessarily a tracial state on \( A \), which contradicts our hypothesis.

Because \( \delta < 1 \), there exist some elements \( a_1, a_2, \ldots, a_m \in A \) such that \( t_0 = \| 1 - \sum_{i=1}^{m} (a_i^*a_i - a_i^*a_i) \| < 1 \). In particular we have

\[
(1) \quad \sum_{i=1}^{m} a_i a_i^* \leq -1 + t_0 + \sum_{i=1}^{m} a_i^*a_i.
\]

Let \( k = \| \sum_{i=1}^{m} a_i^*a_i \| \) and \( a_{m+1} = (k - \sum_{i=1}^{m} a_i^*a_i)_{1/2} \). Then we have

\[
\sum_{i=1}^{m+1} a_i^*a_i = k;
\]

but on the other hand, by (1) we have also

\[
\sum_{i=1}^{m+1} a_i a_i^* \leq -1 + t_0 + k.
\]

The required properties are now fulfilled with \( n = m + 1 \) and \( b_i = (1/\sqrt{k})a_i \).

(2) \( \Rightarrow \) (3). Suppose that \( b_1, b_2, \ldots, b_n \) are as in (2). Define \( \Phi(a) = \sum b_i a b_i^* \). Then \( \Phi \) is a bounded positive map on \( A \) with norm \( \| \Phi \| = \| \sum b_i b_i^* \| < 1 \). It follows that \( Id_A - \Phi \) is invertible in the Banach algebra \( \mathcal{B}(A) \) of bounded maps on \( A \). Let

\[
\Psi = (Id_A - \Phi)^{-1}.
\]

Note that \( \Psi = \sum_{i=0}^{\infty} \Phi^i \), thus \( \Psi \) is positive too. By definition of \( \Psi \), for any \( a \in A \) we have

\[
a = (Id_A - \Phi)(\Psi(a)) = \Psi(a) - \sum_{i=1}^{n} b_i \Psi(a) b_i^*
\]

\[
= \sum_{i=1}^{n} b_i^* b_i \Psi(a) - \sum_{i=1}^{n} b_i \Psi(a) b_i^* = \sum_{i=1}^{n} [b_i^*, b_i \Psi(a)],
\]
so $a$ is a finite sum of at most $n$ commutators. If moreover $a$ is a positive element in $A$, then

$$a = (Id_A - \Phi)(\Psi(a)) = \Psi(a) - \sum_{i=1}^{n} b_i \Psi(a) b_i^*$$

$$= \sum_{i=1}^{n} \Psi(a)^{1/2} b_i^* b_i \Psi(a)^{1/2} = \sum_{i=1}^{n} [\Psi(a)^{1/2} b_i^*, b_i \Psi(a)^{1/2}],$$

so $a$ is a finite sum of at most $n$ self-adjoint commutators. \hfill \Box

3. Questions

For an infinite $C^*$-algebra $A$ (in the sense that it admits no tracial states) let $\nu(A)$ be the least positive integer such that any element of $A$ is a sum of at most $\nu(A)$ commutators. In all the examples that we know of, we have $\nu(A) = 2$. We believe that it is unlikely to always be the case.

In [3] it was shown that if $A$ is an unital exact $C^*$-algebra, then there exists an integer $m$ such that $\mathcal{M}_m(A)$ is properly infinite. It follows that $\nu(\mathcal{M}_m(A)) = 2$. Then a simple computation shows that $\nu(A) \leq 2m^2$. It would be interesting to answer the inverse problem, that is: assuming $\nu(A)$ is known, estimate the least positive integer $m$ such that $\nu(\mathcal{M}_m(A)) = 2$.

References


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