PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 10, Pages 2821–2826
S 0002-9939(02)06542-5
Article electronically published on April 17, 2002

# APPLICATIONS OF A THEOREM OF H. CRAMÉR TO THE SELBERG CLASS

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(Communicated by Dennis A. Hejhal)

Abstract. We prove two results on the nature of the Dirichlet coefficients $a(n)$ of the $L$-functions in the extended Selberg class $\mathcal{S}^\sharp$. The first result asserts that if $a(n) = \phi(\log n)$ for some entire function $\phi(z)$ of order 1 and finite type, then $\phi(z)$ is constant. The second result states, roughly, that if $a(n)\phi(\log n)$ are still the coefficients of some $L$-function from $\mathcal{S}^\sharp$, then $\phi(z) = ce^{i\beta z}$ with $c \in \mathbb{C}$ and $\beta \in \mathbb{R}$. The proofs are based on an old result by Cramér and on the characterization of the functions of degree 1 of $\mathcal{S}^\sharp$.

## 1. Introduction

The nature of the Dirichlet coefficients of $L$-functions is often quite mysterious. In particular, one expects that, apart from the simplest cases, such coefficients have no "simple" expressions. A related problem is about the stability of such coefficients. A way of formulating such a problem is as follows: can one perturbate the coefficients of an $L$-function by multiplication by "simple" factors and still form an $L$-function? In this paper we analyze these problems for the $L$-functions in the Selberg class, in the case where "simple" means, roughly, "values of an entire function". We refer to Selberg [8], Conrey-Ghosh [2], Murty [7] and the survey paper [5] for the basic definitions and properties of the Selberg class $\mathcal{S}$ and of the extended Selberg class $\mathcal{S}^\sharp$. Here we only recall that $\mathcal{S}^\sharp$ consists, roughly, of the Dirichlet series with meromorphic continuation to $\mathbb{C}$ and satisfying a standard functional equation.

The starting point of our investigations is an old result by Cramér [3]; see also Ch. III of Bernstein [1]. Let $\alpha$ be any non-negative real number and let $E_\alpha$ denote the set of entire functions $\phi(z)$ of order 1 and type at most $\alpha$, *i.e.*, satisfying $\phi(z) \ll e^{(\alpha+\varepsilon)|z|}$ for every $\varepsilon > 0$. We recall that $\phi \in E_\alpha$ if and only if

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} z^n \qquad \text{with } \alpha_n \ll (\alpha+\varepsilon)^n; \tag{1.1}$$

see Ch.III of [1]. Given $\phi \in E_\alpha$ and a Dirichlet series $F(s)$, convergent in some right half-plane and with coefficients $a_n$, we write

$$F_\phi(s) = \sum_{n=1}^{\infty} a_n \phi(\log n) n^{-s}.$$

Received by the editors May 11, 2001.
2000 *Mathematics Subject Classification.* Primary 11M41.

Moreover, we denote by $\Gamma_F$ the set of singularities of $F(s)$ and by $|s-\Gamma_F|$ the distance of $s$ from the set $\Gamma_F$. With the above notation, Cramér's theorem states that $F_\phi(s)$ is holomorphic on any domain $\mathcal{D}$ containing a right half-plane and satisfying $|s-\Gamma_F|>\alpha$ for $s\in\mathcal{D}$.

For convenience, we report here a proof of Cramér's theorem. For $\sigma$ sufficiently large we have

$$F_\phi(s)=\sum_{m=0}^\infty \frac{\alpha_m}{m!}\sum_{n=1}^\infty a_n \log^m n n^{-s}=\sum_{m=0}^\infty \frac{\alpha_m}{m!}(-1)^m F^{(m)}(s).$$

Now let $s\in\mathcal{D}$ and $0<r<|s-\Gamma_F|$. Then

$$F^{(m)}(s)=\frac{m!}{2\pi i}\int_{|z-s|=r}\frac{F(z)}{(z-s)^{m+1}}dz \ll \frac{m!}{r^m},$$

and hence

$$\sum_{m=0}^\infty \frac{|\alpha_m|}{m!}|F^{(m)}(s)| \ll \sum_{m=0}^\infty \big(\frac{\alpha+\varepsilon}{r}\big)^m<\infty$$

since we may choose $r$ and $\varepsilon$ such that $\alpha+\varepsilon<r$. This proves Cramér's theorem; see Ch. III of [1] for other proofs.

Our results are as follows.

**Theorem 1.** *Let $F\in\mathcal{S}^\sharp$. Suppose the coefficients $a(n)$ of $F(s)$ satisfy $a(n)=\phi(\log n)$ for some $\phi\in E_\alpha$. Then $F(s)=c\zeta(s)$ for some non-zero $c\in\mathbb{C}$.*

**Theorem 2.** *Let $\phi\in E_\alpha$ for some $\alpha\geq 0$. Then:*

i) *if for every $F\in\mathcal{S}^\sharp$ we have $F_\phi\in\mathcal{S}^\sharp$, then $\phi(z)$ is a non-zero complex constant;*

ii) *if for every entire $F\in\mathcal{S}^\sharp$ we have $F_\phi\in\mathcal{S}^\sharp$, then $\phi(z)=ce^{i\beta z}$ with some non-zero $c\in\mathbb{C}$ and some $\beta\in\mathbb{R}$.*

We remark that the hypothesis "for every (entire) $F\in\mathcal{S}^\sharp$" in both assertions of Theorem 2 can be replaced by the weaker hypothesis "for every (entire) $F\in\mathcal{S}^\sharp$ with degree $d=1$", or even by a weaker hypothesis. This will be clear from the proof. We also remark that Theorem 2 is related to the problem of the rigidity of the Selberg class; see [6] and Vorhauer-Wirsing [9]. In Section 4 we give a conjecture in this direction.

We finally remark that in the proofs we will repeatedly use the characterization of functions in $\mathcal{S}^\sharp$ with degree $0\leq d\leq 1$; see Theorems 1 and 2 of [4]. In particular, we will use the following facts: the functions of degree $d=0$ are Dirichlet polynomials, there are no functions with degree $0<d<1$, and the functions of degree $d=1$ are linear combinations of Dirichlet $L$-functions over Dirichlet polynomials.

## 2. Proof of Theorem 1

Let $d_F$ denote the degree of $F\in\mathcal{S}^\sharp$ and $m_F$ be the order of pole of $F(s)$ at $s=1$. We first deal with the easy case of functions with $d_F=0$. By Theorem 1 of [4], in such a case $F(s)$ is a Dirichlet polynomial, and hence $\phi(\log n)=0$ for $n$ sufficiently large. Therefore, for $r$ sufficiently large the function $\phi(z)$ has $\gg e^r$ zeros for $|z|\leq r$, which implies $\phi(z)=0$ identically.

Suppose now $d_F>0$, and hence $d_F\geq 1$ by Theorem 1 of [4]. Moreover, assume first that $F(s)$ is entire, thus $\overline{F}(s)=\overline{F(\overline{s})}$ is entire as well and has coefficients $\overline{a(n)}$.

Hence by Cramér's theorem we have that

$$\overline{F}_\phi(s) = \sum_{n=1}^\infty |a(n)|^2 n^{-s}$$

is entire. Therefore, by Landau's theorem $\overline{F}_\phi(s)$ is everywhere absolutely convergent and hence the coefficients $a(n)$ satisfy $a(n) \ll n^{-A}$ for any $A > 0$. Thus $F(s)$ is everywhere absolutely convergent, which contradicts the properties of the Lindelöf $\mu$-function of $F(s)$; see Section 2 of [5].

In order to treat the remaining case $d_F \geq 1$ and $m_F \geq 1$ we need the following

**Lemma.** *Let $F \in \mathcal{S}^\sharp$ and $\phi \in E_\alpha$. If $F_\phi \in \mathcal{S}^\sharp$, then $d_{F_\phi} \leq d_F$.*

*Proof.* By means of the coefficients $\alpha_n$ of $\phi(z)$ in (1.1) we construct the function

$$\gamma(z) = \sum_{n=0}^\infty \alpha_n z^{-n-1}.$$

Clearly, $\gamma(z)$ is holomorphic for $|z| > \alpha$. Moreover, $\gamma(z)$ is related to $\phi(z)$ by

$$\phi(z) = \frac{1}{2\pi i}\int_{|w|=\alpha+\varepsilon} \gamma(w) e^{zw} dw,$$

and hence

$$F_\phi(s) = \frac{1}{2\pi i}\int_{|w|=\alpha+\varepsilon} F(s-w)\gamma(w) dw;$$

see Ch. III of [1]. Therefore, writing $\alpha' = \alpha + \varepsilon$ for $\sigma < 0$ the Lindelöf $\mu$-functions of $F_\phi(s)$ and $F(s)$ satisfy

$$(\tfrac{1}{2} - \sigma) d_{F_\phi} = \mu_{F_\phi}(\sigma) \leq \mu_F(\sigma - \alpha') = (\tfrac{1}{2} - \sigma + \alpha') d_F.$$

The lemma now follows dividing both sides by $-\sigma$ and letting $\sigma \to -\infty$. □

Since $a(n) = \phi(\log n)$ we have $F(s) = \zeta_\phi(s)$, and hence by the Lemma we get $d_F \leq d_\zeta = 1$. Therefore $d_F = 1$; thus by Theorem 2 of [4] we have $m_F = 1$. Hence we write

$$F(s) = c\zeta(s) + G(s) \tag{2.1}$$

with a non-zero $c \in \mathbb{C}$ and an entire Dirichlet series

$$G(s) = \sum_{n=1}^\infty c(n) n^{-s}$$

with coefficients given by $c(n) = \psi(\log n)$, where $\psi(z) = \phi(z) - c$ belongs to $E_\alpha$ as well. Arguing as before, by an application of Cramér's theorem to $G(s)$ we deduce that $c(n) \ll n^{-A}$ for any $A > 0$, and hence $G(\sigma + it)$ is almost periodic in $t$ for every $\sigma \in \mathbb{R}$.

Since $d_F = 1$ and $m_F = 1$, by Theorem 2 of [4] the function $F(s)$ satisfies a functional equation of type

$$Q^s \Gamma(\frac{s}{2}) F(s) = \omega Q^{1-s} \Gamma(\frac{1-s}{2}) \overline{F}(1-s) \tag{2.2}$$

with some $Q > 0$ and $|\omega| = 1$. Hence from (2.1) and (2.2) we have

$$G(s) = \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\big(\omega Q^{1-2s}\overline{F}(1-s) - c\pi^{s-\frac{1}{2}}\zeta(1-s)\big).$$

Therefore, unless $G(s)=0$ identically, the function $\Gamma(\frac{1-s}{2})/\Gamma(\frac{s}{2})$ is almost periodic in $t$ for any given $\sigma<0$, which contradicts Stirling's formula. Hence $G(s)=0$ identically, and Theorem 1 follows from (2.1).

## 3. Proof of Theorem 2

The first assertion of Theorem 2 is an immediate consequence of Theorem 1. Choose in fact $F(s)=\zeta(s)$, so that the coefficients of $F_\phi(s)$ are $a(n)=\phi(\log n)$. Hence by Theorem 1 we have $\phi(z)=c$ identically.

In order to prove the second assertion, we choose $F(s)=L(s,\chi_1)$ with $\chi_1$ primitive Dirichlet character modulo $q_1>1$. By the Lemma we have $d_{F_\phi}\le 1$, and by the same argument at the beginning of the proof of Theorem 1 we also have $d_{F_\phi}>0$. Thus by Theorem 1 of [4] we obtain that $d_{F_\phi}=1$, and hence by Theorem 2 of [4] we have

$$F_\phi(s)=\sum_\chi P_\chi(s+i\theta)L(s+i\theta,\chi^*), \tag{3.1}$$

where $\chi$ runs over the characters modulo a certain integer $q$, the $P_\chi(s)$ are Dirichlet polynomials, $\theta$ is a certain real number and $\chi^*$ is the primitive character inducing $\chi$.

Let $p$ be a sufficiently large prime. Comparing $p$-th coefficients in (3.1) we get

$$\chi_1(p)\phi(\log p)p^{i\theta}=\sum_\chi c_\chi\chi^*(p), \quad c_\chi\in\mathbb{C}. \tag{3.2}$$

Since the right-hand side of (3.2) is periodic of period $q$, for sufficiently large primes $p_1,p_2$ with $p_2\equiv p_1 \pmod q$ and coprime to $q_1$ we have

$$\chi_1(p_1)\psi(\log p_1)=\chi_1(p_2)\psi(\log p_2) \qquad \text{with } \psi(z)=\phi(z)e^{i\theta z},$$

and hence $|\psi(\log p_1)|=|\psi(\log p_2)|=c_0$, say. Writing

$$\Psi(z)=\psi(z)\overline{\psi}(z)-c_0^2,$$

we have $\Psi\in E_{2(\alpha+|\theta|)}$ and $\Psi(\log p)=0$ for sufficiently large primes $p$ in a suitable arithmetic progression (mod $q$) and coprime to $q_1$. Therefore, by the argument at the beginning of the proof of Theorem 1 we have $\Psi(z)=0$ identically, and hence

$$\psi(z)\overline{\psi}(z)=c_0^2. \tag{3.3}$$

By (3.3) we have two cases: either $\psi(z)=0$ identically, or $\psi(z)=ce^{bz}$. In the first case we also have $\phi(z)=0$ identically, which contradicts our hypothesis that $F_\phi\in\mathcal{S}^\sharp$. In the second case we have

$$\phi(z)=ce^{(b-i\theta)z}, \tag{3.4}$$

and hence $\phi(\log n)=cn^{b-i\theta}$. Since $F_\phi\in\mathcal{S}^\sharp$ and has degree $d_{F_\phi}=1$, we may inductively repeat the operation of convolution by $\phi(\log n)$, thus getting

$$F_{\phi^k}\in\mathcal{S}^\sharp \text{ and } d_{F_{\phi^k}}=1, \qquad k=1,2,\dots.$$

From Theorem 2 of [4] we have that every $F\in\mathcal{S}^\sharp$ with $d_F=1$ satisfies the Ramanujan conjecture, *i.e.*, the coefficients $a(n)$ satisfy $a(n)\ll n^\varepsilon$ for every $\varepsilon>0$. Therefore, for every $k\ge 1$

$$|\chi_1(n)\phi(\log n)^k|\le |c|^k n^{k\Re b}\ll n^\varepsilon,$$

and hence $\Re b \leq 0$. Moreover, if $\Re b < 0$, then for $k$ sufficiently large the coefficients $a_k(n)$ of $F_{\phi^k}(s)$ satisfy

$$a_k(n) \ll n^{-2},$$

say, a contradiction with the properties of the Lindelöf $\mu$-function. Hence $\Re b = 0$, and Theorem 2 follows from (3.4).

## 4. A conjecture

Given two sequences $\boldsymbol{a} = a(n)$ and $\boldsymbol{b} = b(n)$ we define their distance as

$$d(\boldsymbol{a}, \boldsymbol{b}) = \inf\{\alpha > 0 : \exists\, \phi \in E_\alpha \text{ such that } a(n) = b(n)\phi(\log n) \text{ or } b(n) = a(n)\phi(\log n)\}.$$

Clearly, $d(\boldsymbol{a}, \boldsymbol{b}) = +\infty$ if there exists no such function $\phi(z)$. Note that the distance $d$ satisfies the following three properties: $d(\boldsymbol{a}, \boldsymbol{b}) = d(\boldsymbol{b}, \boldsymbol{a})$, $d(\boldsymbol{a}, \boldsymbol{b}) \leq d(\boldsymbol{a}, \boldsymbol{c}) + d(\boldsymbol{c}, \boldsymbol{b})$ and $d(\boldsymbol{a}, \boldsymbol{a}) = 0$. Analogously, we define $d(F, G) = d(\boldsymbol{a}, \boldsymbol{b})$ for two Dirichlet series $F(s)$ and $G(s)$ with coefficients $a(n)$ and $b(n)$, respectively. With this notation, Theorem 2 suggests that if $F, G \in \mathcal{S}^\sharp$ have $d(F, G) < +\infty$, then $F(s) = cG_\beta(s)$ for some non-zero $c \in \mathbb{C}$ and $\beta \in \mathbb{R}$, and $d(F, G) = |\beta|$. Here $G_\beta(s)$ denotes the shift $G(s + i\beta)$. Note that this agrees with Sarnak's rigidity conjecture; see [6] and [9].

We also write $F \sim G$ if $d(F, G) < +\infty$, and consider the quotient $\mathcal{S}^\sharp/\!\sim\, = \bigcup [F]$, where $[F]$ denotes the class of $F \in \mathcal{S}^\sharp$ and the union is over a set of representatives. Moreover, we identify each class $[F]$ with the set of all distances between functions in $[F]$. With such a notation we conjecture that

$$[F] = \begin{cases} \{0\} & \text{if } m_F \geq 1, \\ \mathbb{R} & \text{if } m_F = 0. \end{cases}$$

More precisely, we expect that $G \in [F]$ if and only if $G(s) = cF_\theta(s)$ with some non-zero $c \in \mathbb{C}$ and $\theta \in \mathbb{R}$, and hence $d(F, G) = |\theta|$.

## Acknowledgments

This research was partially supported by the Istituto Nazionale di Alta Matematica, by a MURST grant and the KBN grant 2 PO3A 024 17.

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