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NEW IDENTITIES OF DIFFERENTIAL OPERATORS FROM ORBITAL INTEGRALS ON GL$(r, \mathbf{C})$

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ABSTRACT. We derive identities of differential operators on complex general linear groups which appear in the differential equations satisfied by weighted orbital integrals. These identities stem from and have applications to comparisons of metaplectic coverings.

1. INTRODUCTION

A basic object in the harmonic analysis of a real reductive group G is the orbital integral of a smooth compactly supported function f at a regular element $\gamma \in G$,

$$J_G(\gamma, f) = |D^G(\gamma)|^{1/2} \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx.$$

Harish-Chandra showed that orbital integrals satisfy a differential equation,

$$J_G(\gamma, zf) = \partial_G^G(\gamma, z) J_G(\gamma, f), \; z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}}),$$

when γ is restricted to the regular elements of a maximal subtorus of G. The truncation process in the Arthur-Selberg trace formula leads to the more general notion of a weighted orbital integral $J_M(\gamma, f)$, where M is now any Levi subgroup of G defined over $\mathbf{R}$. Arthur generalized the earlier differential equation to weighted orbital integrals. The more general equation has the form

$$J_M(\gamma, zf) = \sum_{L \in \mathcal{L}(M)} \partial_M^L(\gamma, z_L) J_L(\gamma, f), \; z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}}).$$

The Arthur-Selberg trace formula is useful for the comparison of automorphic representations of two groups. If $\mathbf{A}$ is the adele ring of a number field containing the nth roots of unity, then there exists an n-fold topological covering $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ of $\mathrm{GL}(r, \mathbf{A})$ called a metaplectic covering. In comparing the trace formulas of these two groups one is lead to comparisons of weighted orbital integrals at the local completions. In particular, one is lead to the comparison of the weighted orbital integrals of the complex group $\mathrm{GL}(r, \mathbf{C})$ and its n-fold metaplectic covering. Fortunately, this covering is trivial. As a consequence, many identities arising from

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this comparison may be expressed purely in terms of $\mathrm{GL}(r,\mathbf{C})$. The main result of this paper, Theorem 1, is such an identity. It is an identity,

$$\partial^G_M(\gamma^n, z) = \sum_{\eta \in \mu^M_n/\mu^G_n} \partial^G_M(\eta\gamma, z'),$$

in which $G = \mathrm{GL}(r,\mathbf{C})$ is regarded as a real Lie group. It relates the nth power map on G to the nth roots of unity. The comparison of global trace formulas ([Mez00]), which relies on Theorem 1, has implications in number theory.

Identities similar to Theorem 1 should also hold between the real general linear group and its two-fold metaplectic coverings. More generally, one should expect similar identities between differential operators on Chevalley groups and their metaplectic coverings over archimedean fields.

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2. Preliminaries

Let r be a positive integer. We write G for $\mathrm{GL}(r,\mathbf{C})$. Let M_0 be the diagonal subgroup of G. The set of Levi subgroups of G containing the minimal Levi subgroup M_0 is denoted by $\mathcal{L}$. We denote a generic Levi subgroup in $\mathcal{L}$ by M. The set of Levi subgroups of some element $L \in \mathcal{L}$, which also contain $M \subset L$, is written as $\mathcal{L}^L(M)$. The set of parabolic subgroups of G with Levi component M is written as $\mathcal{P}(M)$. Fix $P_0 \in \mathcal{P}(M_0)$ to be the upper-triangular parabolic subgroup.

Let $X(M)$ be the group of rational characters of M and set $\mathfrak{a}_M$=$\mathrm{Hom}_{\mathbf{Z}}(X(M),\mathbf{R})$. As usual, we define the homomorphism

$$H_M : M \to \mathfrak{a}_M$$

by the equation

$$e^{\langle H_M(\gamma),\xi\rangle} = |\xi(\gamma)|,\ \gamma = \gamma \in M,\ \xi \in X(M).$$

Given $P \in \mathcal{P}(M)$ with unipotent radical U_P, we may decompose any element $x \in P$ uniquely as $u_P m_P$, where $u_P \in U_P$ and $m_P \in M$. Define

$$H_P(x) = H_M(m_P),\ x \in P.$$

Let n be a positive integer and denote the group of nth roots of unity by μ_n. Let μ^M_n be the subgroup formed by the matrices in the center of M whose nonzero entries all lie in μ_n.

We shall only be concerned with the regular elements and conjugacy classes of G. We therefore define G_{reg} as the set of elements in G which are regular and semisimple, and

$$G_{\mathrm{oreg}} = \{\gamma \in G : \gamma^n \in G_{\mathrm{reg}}\}.$$

Let us describe the terms involved in the definition of an invariant orbital integral. Let G_γ be the centralizer in G of $\gamma \in G_{\mathrm{reg}}$. Set $\mathfrak{g}$ and $\mathfrak{g}_\gamma$ to be the complex Lie algebras of G and G_γ respectively. The Weyl discriminant is defined as

$$D^G(\gamma) = \det((1 - \mathrm{Ad}(\gamma))_{|\mathfrak{g}/\mathfrak{g}_\gamma}).$$

We define $C^\infty_c(G)$ to be the set of compactly supported functions on G which are smooth on the real Lie group G. The orbital integral of $f \in C^\infty_c(G)$ at $\gamma \in G_{\mathrm{reg}}$ is defined as

$$J_G(\gamma, f) = |D^G(\gamma)|_{\mathbf{C}}^{1/2} \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx.$$

Here, we are making an implicit choice of measure. It is well-known (§2 of [Art88]) that the integral converges.

Following §§1-2 of [Art88], we define the weighted orbital integral at $\gamma \in M \cap G_{\mathrm{reg}}$ by

$$J_M(\gamma, f) = |D^G(\gamma)|_{\mathbf{C}}^{1/2} \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) v_M(x) dx, \ f \in C_c^\infty(G).$$

The weight factor v_M is defined at the end of §1 of [Art88].

Orbital integrals, weighted or not, clearly depend only on the conjugacy class of γ. Jordan canonical form therefore allows us to assume that γ belongs to $M_0 \cap G_{\mathrm{oreg}}$ without loss of generality.

3. Differential equations

Let $\mathfrak{g}_{\mathbf{R}}$ and $\mathfrak{h}$ be the real Lie algebras of G and M_0, respectively, regarded as Lie groups. Put $\mathfrak{g}_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$ and $\mathfrak{h}_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{h}$. We write $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ and $\mathcal{U}(\mathfrak{h}_{\mathbf{C}})$ for the respective universal enveloping algebras of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{h}_{\mathbf{C}}$. The respective centers of $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ and $\mathcal{U}(\mathfrak{h}_{\mathbf{C}})$ are denoted by $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ and $\mathcal{Z}(\mathfrak{h}_{\mathbf{C}})$. The foregoing notation applies to M in the same way it applies to G. We therefore denote the real Lie algebra of M by $\mathfrak{m}_{\mathbf{R}}$ and write $\mathcal{Z}(\mathfrak{m}_{\mathbf{C}})$ without any further explanation.

Let W be the complex Weyl group of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. The Harish-Chandra map (Theorem 8.18 of [Kna86]) provides a monomorphism $z \mapsto z_M$ from $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ into $\mathcal{Z}(\mathfrak{m}_{\mathbf{C}})$. It also provides an algebra isomorphism $z \mapsto z_{M_0}$ from $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ into the W-invariant elements of $\mathcal{Z}(\mathfrak{h}_{\mathbf{C}})$. The element $z_{M_0} \in \mathcal{Z}(\mathfrak{h}_{\mathbf{C}})$ is associated to a unique W-invariant differential operator γ_z on $\mathfrak{h}$ (not to be confused with an element in G). This differential operator is supported at the origin, when regarded as a distribution. It therefore has a Fourier transform $\hat{\gamma_z}$, which is a W-invariant polynomial on $\mathfrak{h}_{\mathbf{C}}^*$. All W-invariant polynomials on $\mathfrak{h}_{\mathbf{C}}^*$ are obtained uniquely in such a fashion. We may therefore define z' to be the unique element in $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ such that

$$\hat{\gamma_{z'}}(\nu) = \hat{\gamma_z}(n^{-1}\nu), \ \nu \in \mathfrak{h}_{\mathbf{C}}^*.$$

It is simple to show that this map is an algebra automorphism of $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$.

We may regard an element z of $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ as a differential operator on G in the obvious way ((A. 99) of [Kna86]) and denote its action on $f \in C_c^\infty(G)$ by zf. According to Proposition 11.1 of [Art88], there are differential operators,

$$\partial_M^L(\gamma, z), \ \gamma \in M_0 \cap G_{\mathrm{reg}}, \ L \in \mathcal{L},$$

on $M_0 \cap G_{\mathrm{reg}}$, such that

$$J_M(\gamma, zf) = \sum_{L \in \mathcal{L}(M)} \partial_M^L(\gamma, z_L) J_L(\gamma, f), \ \gamma \in M_0 \cap G_{\mathrm{reg}}.$$

These differential operators are described in some detail in §12 of [Art88]. We shall review some of this description later on. For the present, we remark that the dependence of $\partial_M^G(\gamma, z)$ on $\gamma \in M_0 \cap G_{\mathrm{reg}}$ is given exclusively through the adjoint action

$$\mathrm{Ad} : M_0 \to \mathrm{GL}(\mathfrak{g}_{\mathbf{C}}).$$

As μ_n^G is contained in the kernel of this map, it follows that

$$\partial_M^G(\eta\gamma, z) = \partial_M^G(\gamma, z),\ \gamma \in M_0 \cap G_{\mathrm{reg}},\ z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}}),\ \eta \in \mu_n^G.$$

Our main result is

Theorem 1. *Suppose* $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$. *Then*

$$\partial_M^G(\gamma^n, z) = \sum_{\eta \in \mu_n^M/\mu_n^G} \partial_M^G(\eta\gamma, z'),\ \gamma \in M_0 \cap G_{\mathrm{oreg}}.$$

The proof of this theorem will consume the rest of this section.

Before embarking upon it we must fix some more notation. As a real vector space, $\mathfrak{g}_{\mathbf{R}}$ is isomorphic to

$$\mathfrak{gl}(r, \mathbf{R}) \oplus i\mathfrak{gl}(r, \mathbf{R}).$$

In order to avoid confusion with the complex structure which stems from the complexification of $\mathfrak{g}_{\mathbf{R}}$, let us agree to replace the imaginary number i above with I. Thus,

$$\mathfrak{g}_{\mathbf{R}} \cong \mathfrak{gl}(r, \mathbf{R}) \oplus I\mathfrak{gl}(r, \mathbf{R}).$$

Let $H_k \in \mathfrak{gl}(r, \mathbf{R})$, $1 \leq k \leq r$, be given by the matrix with one in the kth diagonal entry and zeros elsewhere. The elements $H_1, \ldots, H_r$ generate a Cartan subalgebra in $\mathfrak{gl}(r, \mathbf{C}) = \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{gl}(r, \mathbf{R})$. Let Σ be the set of roots of $\mathfrak{gl}(r, \mathbf{C})$ and $\{X_\alpha : \alpha \in \Sigma\} \subset \mathfrak{gl}(r, \mathbf{R})$ be the usual basis of the nonzero eigenspaces of $\mathfrak{gl}(r, \mathbf{C})$ with respect to this Cartan subalgebra. We extend this basis by

$$\{IX_\alpha : \alpha \in \Sigma\} \cup \{H_k,\ IH_k : 1 \leq k \leq r\}$$

to obtain a basis of $\mathfrak{g}_{\mathbf{R}}$ which corresponds to its root space decomposition.

Using the linear map defined on page 1255 of [Art82], one can show that $\mathfrak{a}_{M_0}$ embeds into $\mathfrak{g}_{\mathbf{R}}$ such that

$$\mathfrak{h} = \mathfrak{a}_{M_0} \oplus I\mathfrak{a}_{M_0}$$

is a Cartan subalgebra of the split real form of $\mathfrak{g}_{\mathbf{C}}$. Under this identification of $\mathfrak{a}_{M_0}$ with a subspace of $\mathfrak{h}$, $\mathfrak{a}_{M_0}$ is the real linear span of the elements $H_1, \ldots, H_r$.

Set

$$X_\alpha^{\pm} = (X_\alpha \pm iIX_\alpha)/2,\ \alpha \in \Sigma,$$

and

$$H_k^{\pm} = (H_k \pm iIH_k)/2,\ 1 \leq k \leq r.$$

Then $\mathfrak{g}^+$, the complex Lie algebra generated by

$$\{X_\alpha^+ : \alpha \in \Sigma\} \cup \{H_k^+ :\ 1 \leq k \leq r\},$$

is isomorphic to $\mathfrak{gl}(r, \mathbf{C})$ and is an ideal of $\mathfrak{g}_{\mathbf{C}}$. The same is true for $\mathfrak{g}^- \subset \mathfrak{g}_{\mathbf{C}}$, the complex Lie algebra generated by

$$\{X_\alpha^- : \alpha \in \Sigma\} \cup \{H_k^- :\ 1 \leq k \leq r\}.$$

As a result, $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ is isomorphic to $\mathfrak{gl}(r, \mathbf{C}) \times \mathfrak{gl}(r, \mathbf{C})$ as a complex Lie algebra. We may and shall identify Σ with the roots of $\mathfrak{g}^+$ or $\mathfrak{g}^-$. With these identifications, X_α^+ and X_α^- span the respective root spaces of $\mathfrak{g}^+$ and $\mathfrak{g}^-$ corresponding to $\alpha \in \Sigma$. Defining $\Sigma^+ \subset \Sigma$ to be the subset of positive roots with respect to the upper-triangular nilpotent subalgebra, we find that the complexified Lie algebra of U_{P_0} is spanned by $\{X_\alpha^+, X_\alpha^- : \alpha \in \Sigma^+\}$.

Given $P \in \mathcal{P}(M)$, let $\mathfrak{p}^1$ denote the real Lie algebra of

$$\{x \in P : H_P(x) = 0\}.$$

Let $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})_{M_0}$ be the subalgebra of elements in $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ which are invariant under the adjoint action of M_0. On page 223 of [Art76], it is shown that there is a linear map μ_P (not to be confused with the group μ_n^M) from $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})_{M_0}$ to $S(\mathfrak{a}_{M_0,\mathbf{C}})$, the symmetric algebra of $\mathfrak{a}_{M_0,\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{a}_{M_0}$. This map sends $X \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})_{M_0}$ to the unique element in $S(\mathfrak{a}_{M_0,\mathbf{C}})$ such that $X - \mu_P(X)$ lies in $\mathfrak{p}^1\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$. This map is fundamental to the understanding of our differential operators. The following lemma determines an upper bound for the degree of $\mu_P(X)$, for X in the subalgebra $\mathcal{U}(\mathfrak{g}^{\pm})_{M_0} = \mathcal{U}(\mathfrak{g}_{\mathbf{C}})_{M_0} \cap \mathcal{U}(\mathfrak{g}^{\pm})$. We write $\lfloor t \rfloor$ for the greatest integer less than or equal to $t \in \mathbf{R}$.

Lemma 1. *Suppose $P \in \mathcal{P}(M)$ and $\alpha_1, \dots, \alpha_k \in \Sigma$, such that*

$$X^+_{\alpha_1} \cdots X^+_{\alpha_k} \in \mathcal{U}(\mathfrak{g}^+)_{M_0}.$$

Then $\mu_P(X^+_{\alpha_1} \cdots X^+_{\alpha_k})$ has degree no more than $\lfloor k/2 \rfloor$. This statement is also true if the "+" in the superscript is replaced by "−".

Proof. It suffices to consider the case

$$X^+_{\alpha_1} \cdots X^+_{\alpha_k} \in \mathcal{U}(\mathfrak{g}^+)_{M_0}.$$

Clearly, this is the case if and only if $\sum_{j=1}^k \alpha_j = 0$. The Weyl group W_0^G has a canonical action on Σ as its permutation group. Since

$$\sum_{j=1}^k \alpha_j^w = 0, \ w \in W_0^G,$$

we may assume that $P \supset P_0$. In this case, $\mathfrak{p}^1\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ contains the complex linear span of

$$\{X^+_\alpha : \alpha \in \Sigma^+\} \cup \{IH_j : 1 \le j \le r\}.$$

Since $\sum_{j=1}^k \alpha_j = 0$, at least one of $X^+_{\alpha_1}, \dots, X^+_{\alpha_k}$ belongs to $\mathfrak{p}^1\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$. We prove the lemma by induction on $k \ge 2$. If $k = 2$, then $\alpha_1 = -\alpha_2$. If $X^+_{\alpha_1}$ belongs to $\mathfrak{p}^1\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$, then $\mu_P(X^+_{\alpha_1}X^+_{-\alpha_1})$ is zero. Otherwise, $X^+_{-\alpha_1}$ belongs to $\mathfrak{p}^1\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ and

$$X^+_{\alpha_1}X^+_{-\alpha_1} = X^+_{-\alpha_1}X^+_{\alpha_1} + H$$

for some element H in $\mathfrak{h}_{\mathbf{C}}$. It is apparent that $X^+_{-\alpha_1}X^+_{\alpha_1}$ lies in $\mathfrak{p}^1\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$. Since $\mathfrak{h}_{\mathbf{C}}$ is identified with

$$\mathbf{C} \otimes_{\mathbf{R}} (\mathfrak{a}_{M_0} \oplus I\mathfrak{a}_{M_0}) = \mathfrak{a}_{M_0,\mathbf{C}} \oplus I\mathfrak{a}_{M_0,\mathbf{C}},$$

there exists a unique element H' in $\mathfrak{a}_{M_0,\mathbf{C}}$ such that $H - H'$ belongs to $I\mathfrak{a}_{M_0,\mathbf{C}} \subset \mathfrak{p}^1\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$. By definition then, $\mu_P(X^+_{\alpha_1}X^+_{-\alpha_1}) = H'$ is of degree one. Now suppose that $k > 2$. Fix $1 \le j \le k$ such that $X^+_{\alpha_j}$ belongs to $\mathfrak{p}^1\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$. Permuting $X^+_{\alpha_j}$ in $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ $j-1$ times, we obtain

$$\begin{aligned} X^+_{\alpha_1} \cdots X^+_{\alpha_k} &= X^+_{\alpha_j}X^+_{\alpha_1} \cdots X^+_{\alpha_{j-1}}X^+_{\alpha_{j+1}} \cdots X^+_{\alpha_k} \\ &+ \sum_{\ell=1}^{j-1} X^+_{\alpha_1} \cdots X^+_{\alpha_{j-\ell-1}}[X^+_{\alpha_{j-\ell}}, X^+_{\alpha_j}]X^+_{\alpha_{j-\ell+1}} \cdots X^+_{\alpha_{j-1}}X^+_{\alpha_{j+1}} \cdots X^+_{\alpha_k}. \end{aligned}$$

The first term in this expansion belongs to $\mathfrak{p}^1\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$. We therefore focus on the remaining terms. The Lie bracket $[X^+_{\alpha_{j-\ell}}, X^+_{\alpha_j}]$ vanishes if $\alpha_{j-\ell}+\alpha_j$ does not belong to Σ. Suppose therefore that $\alpha_{j-\ell}+\alpha_j \in \Sigma$. If $\alpha_{j-\ell} = -\alpha_j$, then $[X^+_{\alpha_{j-\ell}}, X^+_{\alpha_j}] = H$ for some $H \in \mathfrak{h}_{\mathbf{C}}$, and

$$X^+_{\alpha_1} \cdots X^+_{\alpha_{j-\ell-1}} [X^+_{\alpha_{j-\ell}}, X^+_{\alpha_j}] X^+_{\alpha_{j-\ell+1}} \cdots X^+_{\alpha_{j-1}} X^+_{\alpha_{j+1}} \cdots X^+_{\alpha_k}$$

is equal to

$$X^+_{\alpha_1} \cdots X^+_{\alpha_{j-\ell-1}} H X^+_{\alpha_{j-\ell+1}} \cdots X^+_{\alpha_{j-1}} X^+_{\alpha_{j+1}} \cdots X^+_{\alpha_k}.$$

Permuting H in this expression $j-\ell-1$ times, we obtain

$$\begin{aligned} &H X^+_{\alpha_1} \cdots X^+_{\alpha_{j-\ell-1}} X^+_{\alpha_{j-\ell+1}} \cdots X^+_{\alpha_k} \\ &\quad - \sum_{m=1}^{j-\ell-1} \alpha_m(H) X^+_{\alpha_1} \cdots X^+_{\alpha_{j-\ell-1}} X^+_{\alpha_{j-\ell+1}} \cdots X^+_{\alpha_k}. \end{aligned}$$

By induction, we may conclude that the degree of

$$\mu_P(\alpha_m(H) X^+_{\alpha_1} \cdots X^+_{\alpha_{j-\ell-1}} X^+_{\alpha_{j-\ell+1}} \cdots X^+_{\alpha_k})$$

is no more than $\lfloor (k-2)/2 \rfloor$. If we choose $H' \in \mathfrak{a}_{M_0,\mathbf{C}}$ as in the case $k=2$, then

$$\begin{aligned} &\mu_P(H X^+_{\alpha_1} \cdots X^+_{\alpha_{j-\ell-1}} X^+_{\alpha_{j-\ell+1}} \cdots X^+_{\alpha_k}) \\ &\quad = \mu_P(H' X^+_{\alpha_1} \cdots X^+_{\alpha_{j-\ell-1}} X^+_{\alpha_{j-\ell+1}} \cdots X^+_{\alpha_k}) \\ &\quad = H' \mu_P(X^+_{\alpha_1} \cdots X^+_{\alpha_{j-\ell-1}} X^+_{\alpha_{j-\ell+1}} \cdots X^+_{\alpha_k}). \end{aligned}$$

By induction, the degree of the final term in this equation is no more than $1 + \lfloor (k-2)/2 \rfloor$. This takes care of the case $\alpha_{j-\ell} = -\alpha_j$. If $\alpha_{j-\ell} \neq -\alpha_j$, then

$$[X^+_{\alpha_{j-\ell}}, X^+_{\alpha_j}] = \pm X^+_{\alpha_{j-\ell}+\alpha_j},$$

and we may apply the induction assumption to conclude that the degree of

$$\mu_P(X^+_{\alpha_1} \cdots X^+_{\alpha_{j-\ell-1}} X^+_{\alpha_{j-\ell}+\alpha_j} X^+_{\alpha_{j-\ell+1}} \cdots X^+_{\alpha_k})$$

is no more than $\lfloor (k-1)/2 \rfloor$. This completes the lemma. □

We shall now say more about the structure of $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ and our differential operators. Since the Lie bracket of any element in $\mathfrak{g}^+$ with any element in $\mathfrak{g}^-$ is zero, the elements of $\mathfrak{g}^+$ commute with the elements of $\mathfrak{g}^-$ in $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$. It follows that $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ is isomorphic to $\mathcal{U}(\mathfrak{g}^+) \otimes \mathcal{U}(\mathfrak{g}^-)$ as an algebra. The complex structure of $\mathfrak{g}_{\mathbf{R}}$ is inherited by $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$. One can use this complex structure to show that $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ is isomorphic to $\mathcal{Z}(\mathfrak{g}^+) \otimes \mathcal{Z}(\mathfrak{g}^-)$. It is well-known that $\mathcal{Z}(\mathfrak{g}^\pm) \cong \mathcal{Z}(\mathfrak{gl}(r, \mathbf{C}))$ is isomorphic to a polynomial algebra in r generators (Theorem 8.19 of [Kna86]). Furthermore, each generator, regarded as an element of the filtered algebra $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$, has degree no more than r (Proposition 5.32 of [Kna96]).

In order to describe our differential operators in more detail, we specialize some of the ideas of §12 of [Art88] to our context. Let $\mathfrak{q}_{\mathbf{C}}$ be the complex linear span of

$$\{X^+_\alpha, X^-_\alpha : \alpha \in \Sigma\},$$

and let $\mathcal{Q}$ be the image of $S(\mathfrak{q}_{\mathbf{C}})$ in $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ under the symmetrization map. Given $\gamma \in M_0 \cap G_{\text{reg}}$, there is a linear isomorphism

$$\Gamma_\gamma : \mathcal{Q} \otimes S(\mathfrak{h}_{\mathbf{C}}) \to \mathcal{U}(\mathfrak{g}_{\mathbf{C}}).$$

Let $\mathcal{Q}'$ be the subspace of codimension one in $\mathcal{Q}$ consisting of elements with zero constant term. Then for any $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$, there are unique elements $X_1, \dots, X_k$ in $\mathcal{Q}' \cap \mathcal{U}(\mathfrak{g}_{\mathbf{C}})_{M_0}$, elements $u_1, \dots, u_m$ in $S(\mathfrak{h}_{\mathbf{C}})$ and analytic functions

$$\{a_{j\ell} : 1 \le j \le k,\ 1 \le \ell \le m\}$$

on $M_0 \cap G_{\mathrm{reg}}$ such that

$$z - \sum_{j=1}^{k} \sum_{\ell=1}^{m} a_{j\ell}(\gamma)\Gamma_\gamma(X_j \otimes u_\ell) \in S(\mathfrak{h}_{\mathbf{C}}). \tag{1}$$

Lemma 12.1 of [Art88] asserts that if $G \neq M$, then there exist constants c_j, $1 \le j \le k$, depending only on the homogeneous component of $\mu_P(X_j)$ of degree $\dim(A_M/A_G)$, such that $\partial_M^G(\gamma, z)$ is equal to

$$\sum_{j=1}^{k} c_j \sum_{\ell=1}^{m} |D^G(\gamma)|_{\mathbf{C}}^{1/2} a_{j\ell}(\gamma)\partial(u_\ell) \circ |D^G(\gamma)|_{\mathbf{C}}^{-1/2},\ \gamma \in M_0 \cap G_{\mathrm{reg}}. \tag{2}$$

On the other hand, for $G = M$, Lemma 12.1 and Lemma 12.4 of [Art88] imply that

$$\partial_G^G(\gamma, z) = \partial_{M_0}^{M_0}(\gamma, z_{M_0}) = z_{M_0},\ \gamma \in M_0 \cap G_{\mathrm{reg}},\ z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}}).$$

We first use these expansions to prove Theorem 1 in some special cases. Suppose $G = M$. Let $\varphi \in C^\infty(M_0)$ and set

$$\varphi^n(\gamma) = \varphi(\gamma^n),\ \gamma \in M_0.$$

It is easily verified that

$$(\partial_G^G(\gamma^n, z)\varphi^n)(\gamma) = (z_{M_0}\varphi)(\gamma^n) = (z'_{M_0}\varphi^n)(\gamma) = (\partial_G^G(\gamma, z')\varphi^n)(\gamma),$$

for $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ and $\gamma \in M_0 \cap G_{\mathrm{oreg}}$. This establishes Theorem 1 in the case $M = G$, so we may make the induction assumption that the theorem holds if G is replaced by $L \in \mathcal{L}$ with $L \subsetneq G$. This assumption will remain in force for the remainder of this section.

We now examine the case $r = 2$, that is $G = \mathrm{GL}(2, \mathbf{C})$. In this case Σ^+ consists of a single root α and $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ is generated by $H_1^\pm + H_2^\pm$ and the Casimir elements,

$$z^\pm = (H_1^\pm - H_2^\pm)^2/2 + X_\alpha^\pm X_{-\alpha}^\pm + X_{-\alpha}^\pm X_\alpha^\pm \in \mathcal{Z}(\mathfrak{g}^\pm).$$

Since the elements $H_1^\pm + H_2^\pm$ lie in $\mathfrak{h}_{\mathbf{C}}$, they are easily dealt with. We therefore ignore their treatment and consider only the Casimir elements. It is straightforward to show that

$$n^{-2}(z^\pm)' = z^\pm.$$

Define the quasicharacter ξ_α on M_0 by

$$\xi_\alpha(\exp(H)) = e^{\alpha(H)},\ H \in \mathfrak{h}.$$

Then, following a computation on pages 357-358 of [Kna86], there exist constants $c^\pm$, depending on $X_{-\alpha}^\pm X_\alpha^\pm$ respectively (cf. equation (2)), such that

$$\partial_{M_0}^G(\gamma, z^\pm) = c^\pm \frac{-2\xi_\alpha(\gamma)}{(1 - \xi_\alpha(\gamma))^2},\ \gamma \in M_0 \cap G_{\mathrm{reg}}.$$

Consequently, for $\gamma \in M_0 \cap G_{\text{oreg}}$, we have

$$\begin{aligned}\sum_{\eta \in \mu_n^{M_0}/\mu_n^G} \partial_{M_0}^G(\eta\gamma, (z^\pm)') &= \sum_{\eta \in \mu_n^{M_0}/\mu_n^G} c^\pm n^{-2} \frac{-2\xi_\alpha(\eta\gamma)}{(1-\xi_\alpha(\eta\gamma))^2} \\ &= \sum_{\eta \in \mu_n} c^\pm n^{-2} \frac{-2\eta\xi_\alpha(\gamma)}{(1-\eta\xi_\alpha(\gamma))^2}.\end{aligned}$$

One can use the Taylor series expansion of the rational expression $1/(1-Y)$ to show that

$$\sum_{\eta \in \mu_n} \frac{\eta Y}{(1-\eta Y)^2} = \frac{n^2 Y^n}{(1-Y^n)^2}.$$

Thus,

$$-2c^\pm n^{-2} \sum_{\eta \in \mu_n} \frac{\eta\xi_\alpha(\gamma)}{(1-\eta\xi_\alpha(\gamma))^2} = c^\pm \frac{-2\xi_\alpha(\gamma^n)}{(1-\xi_\alpha(\gamma^n))^2} = \partial_{M_0}^G(\gamma^n, z^\pm).$$

This takes care of Theorem 1 in the case that $r = 2$ and $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ is a generator.

The following splitting properties will show that if Theorem 1 is true for the generators of $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$, then it is in fact true for all elements of $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$. In Lemma 2 and Corollary 1, r is again an arbitrary positive integer.

Lemma 2. *Suppose z_1 and z_2 belong to $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$. Then*

$$\partial_M^G(\gamma, z_1 z_2) = \sum_{L \in \mathcal{L}(M)} \partial_M^L(\gamma, z_{1,L}) \partial_L^G(\gamma, z_2), \ \gamma \in M_0 \cap G_{\text{reg}}.$$

Proof. We assume inductively that the lemma holds if G is replaced $L \in \mathcal{L}$ with $L \subsetneq G$. Then, on the one hand, for $f \in C_c^\infty(G)$,

$$J_M(\gamma, z_1 z_2 f) = \sum_{L_1 \in \mathcal{L}(M)} \partial_M^{L_1}(\gamma, (z_1 z_2)_{L_1}) J_{L_1}(\gamma, f), \ \gamma \in M_0 \cap G_{\text{reg}}.$$

On the other hand,

$$\begin{aligned} J_M(\gamma, z_1 z_2 f) &= \sum_{L_1 \in \mathcal{L}(M)} \partial_M^{L_1}(\gamma, z_{1,L_1}) J_{L_1}(\gamma, z_2 f) \\ &= \sum_{L_1 \in \mathcal{L}(M)} \partial_M^{L_1}(\gamma, z_{1,L_1}) \sum_{L_2 \in \mathcal{L}(L_1)} \partial_{L_1}^{L_2}(\gamma, z_{2,L_2}) J_{L_2}(\gamma, f) \\ &= \sum_{L_1 \in \mathcal{L}(M)} \sum_{L \in \mathcal{L}^{L_1}(M)} \partial_M^L(\gamma, z_{1,L}) \partial_L^{L_1}(\gamma, z_{2,L_1}) J_{L_1}(\gamma, f). \end{aligned}$$

Taking the difference of these two expansions and applying the latest induction assumption, we obtain

$$\left(\partial_M^G(\gamma, z_1 z_2) - \sum_{L \in \mathcal{L}(M)} \partial_M^L(\gamma, z_{1,L}) \partial_L^G(\gamma, z_2) \right) J_G(\gamma, f) = 0, \ \gamma \in M_0 \cap G_{\text{reg}}.$$

Since any smooth function on $M_0 \cap G_{\text{reg}}$ is locally of the form

$$\gamma \mapsto J_G(\gamma, f), \ \gamma \in M_0 \cap G_{\text{reg}},$$

we obtain the lemma. $\square$

Corollary 1. *Suppose z_1 and z_2 belong to $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ and $\gamma \in M_0 \cap G_{\text{oreg}}$. Then*

$$\sum_{\eta\in\mu_n^M/\mu_n^G} \partial_M^G(\eta\gamma, z_1z_2) = \sum_{L\in\mathcal{L}(M)} \sum_{\eta_1\in\mu_n^M/\mu_n^L} \partial_M^L(\eta_1\gamma, z_{1,L}) \sum_{\eta_2\in\mu_n^L/\mu_n^G} \partial_L^G(\eta_1\eta_2\gamma, z_2).$$

Proof. The corollary follows immediately from Lemma 2 and the fact that

$$\partial_M^L(\eta_1\gamma, z_{1,L}) = \partial_M^L(\gamma, z_{1,L}),\ \eta_1 \in \mu_n^L,$$

as mentioned at the beginning of this section. □

Returning to the case $r = 2$, we point out that $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ is a polynomial in $z^\pm$ and $H_1^\pm + H_2^\pm$. Since

$$z \mapsto \partial_{M_0}^G(\gamma, z),\ z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}}),$$

is a linear map for any $\gamma \in M_0 \cap G_{\text{reg}}$, it suffices to show the identity of Theorem 1 in the case that $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ is a monomial in $z^\pm$ and $H_1^\pm + H_2^\pm$. To this end, suppose that $z = z^\pm z_1$ for some monomial $z_1 \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$. (We leave the simpler case that $z = (H_1^\pm + H_2^\pm)z_1$ to the reader.) Then, by Corollary 1, if $\gamma \in M_0 \cap G_{\text{oreg}}$,

$$\begin{aligned}\sum_{\eta\in\mu_n^{M_0}/\mu_n^G} \partial_{M_0}^G(\eta\gamma, z') &= \sum_{\eta\in\mu_n^{M_0}/\mu_n^G} \partial_{M_0}^G(\gamma, (z^\pm)' z_1') \\ &= \sum_{L\in\mathcal{L}} \sum_{\eta_1\in\mu_n^{M_0}/\mu_n^L} \partial_{M_0}^L(\eta_1\gamma, (z_L^\pm)') \sum_{\eta_2\in\mu_n^L/\mu_n^G} \partial_L^G(\eta_1\eta_2\gamma, z_1').\end{aligned}$$

We may assume, by induction on the degree of the monomials in $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$, that

$$\sum_{\eta_2\in\mu_n^L/\mu_n^G} \partial_L^G(\eta_1\eta_2\gamma, z_1') = \partial_L^G((\eta_1\gamma)^n, z_1) = \partial_L^G(\gamma^n, z_1).$$

Combining this equation with Lemma 2, the right-hand side of the earlier expansion is equal to

$$\sum_{L\in\mathcal{L}} \partial_{M_0}^L(\gamma^n, z_L^\pm)\partial_L^G(\gamma^n, z_1) = \partial_{M_0}^G(\gamma^n, z^\pm z_1) = \partial_{M_0}^G(\gamma^n, z).$$

This completes the proof of Theorem 1 in the case $r = 2$.

We are almost ready to prove the theorem in general. The following lemma uses a descent property to allow us to restrict the proof to the case $M = M_0$.

Lemma 3. *Suppose $M \neq M_0$ and $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$. Then*

$$\sum_{\eta\in\mu_n^M/\mu_n^G} \partial_M^G(\eta\gamma, z') = \partial_M^G(\gamma^n, z),\ \gamma \in M_0 \cap G_{\text{oreg}}.$$

Proof. According to (3.5) of [Art], there exist constants $d_{M_0}^G(M, L)$, such that $d_{M_0}^G(M, G)$ vanish for $M \supsetneq M_0$, and

$$\partial_M^G(\gamma, z) = \sum_{L\in\mathcal{L}} d_{M_0}^G(M, L)\partial_{M_0}^L(\gamma, z_L),\ \gamma \in M_0 \cap G_{\text{reg}}.$$

If $\gamma \in M_0 \cap G_{\text{oreg}}$ and $M \neq M_0$, then by induction

$$\partial_M^G(\gamma^n, z) = \sum_{\{L\in\mathcal{L}: L\neq G\}} d_{M_0}^G(M, L) \sum_{\eta\in\mu_n^{M_0}/\mu_n^L} \partial_{M_0}^L(\eta\gamma, z_L').$$

By Lemma 3 of [Mez], the canonical map

$$\mu_n^M/\mu_n^G \to \mu_n^{M_0}/\mu_n^L$$

is bijective if $d^G_{M_0}(M,L) \neq 0$. Thus the right-hand side is equal to

$$\sum_{\eta\in\mu_n^M/\mu_n^G}\ \sum_{\{L\in\mathcal{L}:L\neq G\}} d^G_{M_0}(M,L)\partial^L_M(\eta\gamma, z'_L) = \sum_{\eta\in\mu_n^M/\mu_n^G}\partial^G_M(\eta\gamma,z'),$$

and the lemma is complete. □

Proof of Theorem 1. Taking the preceding arguments into consideration, we need only show that

$$\sum_{\eta\in\mu_n^{M_0}/\mu_n^G}\partial^G_{M_0}(\eta\gamma,z') = \partial^G_{M_0}(\gamma^n,z),\ \gamma\in M_0\cap G_{\mathrm{oreg}},$$

for $z\in\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ and $r>2$. In fact, the application of Lemma 2 and Corollary 1 in the case $r=2$ generalizes trivially to the case $r>2$, so we need only consider the case that $z\in\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ is a generator. As mentioned earlier, this implies that z has degree no more than r. It follows from the definition of the linear map Γ_γ ((10.47) of [Kna86]) that the elements, $X_1,\dots,X_k$, of equation (1) have degree no more than r. Lemma 1 then tells us that $\mu_P(X_1),\dots,\mu_P(X_k)$ each have degree no more than $\lfloor r/2\rfloor$. For $r>2$, we have

$$\lfloor r/2\rfloor < r-1 = \dim(A_{M_0}/A_G).$$

Consequently the homogeneous component of $\mu_P(X_j)$ of degree $\dim(A_{M_0}/A_G)$ is zero for $1\le j\le k$. By (12.5) of [Art88], this implies that the constants, $c_1,\dots,c_k$ of equation (2) are all zero as well. Therefore,

$$\sum_{\eta\in\mu_n^{M_0}/\mu_n^G}\partial^G_{M_0}(\eta\gamma,z') = 0 = \partial^G_{M_0}(\gamma^n,z),\gamma\in M_0\cap G_{\mathrm{oreg}},$$

and the theorem is proved. □

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