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TREE-LIKE CONTINUA DO NOT ADMIT EXPANSIVE HOMEOMORPHISMS

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ABSTRACT. A homeomorphism $h : X \to X$ is called *expansive* provided that for some fixed $c > 0$ and every $x, y \in X$ there exists an integer n, dependent only on x and y, such that $d(h^n(x), h^n(y)) > c$. It is shown that if X is a tree-like continuum, then h cannot be expansive.

1. INTRODUCTION

A *continuum* is a nondegenerate compact connected metric space. A homeomorphism $h : X \to X$ is called *expansive* provided that for some fixed $c > 0$ and every $x, y \in X$ there exists an integer n, dependent only on x and y, such that $d(h^n(x), h^n(y)) > c$. Expansive homeomorphisms exhibit chaotic behavior in that no matter how close two points are, either their forward or reverse image will eventually be a certain distance apart. Plykin's attractors [4] and the dyadic solenoid [6] are examples of continua that admit expansive homeomorphisms.

If $\mathcal{U}$ is a collection of open sets, the *mesh* of $\mathcal{U}$ is defined as $\text{mesh}(\mathcal{U}) = \sup\{\text{diam}(U) : U \in \mathcal{U}\}$. If $\mathcal{U}$ is a finite open cover of continuum X, then the *nerve* of $\mathcal{U}$ is a geometric complex $\mathcal{N}(\mathcal{U})$ which has a vertex v_i that corresponds to each element U_i of $\mathcal{U}$ such that $\langle v_{i_1}, v_{i_2}, ..., v_{i_j}\rangle$ is a simplex of $\mathcal{N}(\mathcal{U})$ if and only if $U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_j} \neq \emptyset$. A continuum X is *arc-like* if for every $\epsilon > 0$, there exists a finite open cover $\mathcal{U}$ whose mesh is less than ϵ and whose nerve is an arc. Arc-like continua are also called *chainable* and *snake-like* continua. A continuum is *tree-like* if for every $\epsilon > 0$, there exists a finite open cover $\mathcal{U}$ whose mesh is less than ϵ and whose nerve contains no simple closed curves (i.e. a tree-graph). An open cover whose nerve is a tree is called a *tree-cover*. Equivalent definitions for a tree-like continuum, X, are the following:

1) For every $\epsilon > 0$, there is an onto map $g : X \to T$ such that $\text{diam}(g^{-1}(y)) < \epsilon$ for each $y \in T$ where T is a tree.

2) $X = \varprojlim\{T_i, f_i\}_{i=0}^{\infty}$. Where each T_i is a tree and each $f_i : T_{i+1} \longrightarrow T_i$ is a bonding map.

A continuum is 1-*dimensional* if for every $\epsilon > 0$ there exists a finite open cover $\mathcal{U}$ whose mesh is less than ϵ such that for every $y \in X$, y is in at most 2 elements

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of $\mathcal{U}$. A planar continuum X is a *non-separating plane* continuum provided that $\mathbb{R}^2 - X$ is connected. It is important to note that all 1-dimensional non-separating plane continua are tree-like. However, not all tree-like continua can be embedded in the plane.

In order for a homeomorphism h to be expansive, h must stretch subcontinua. Since compactness must be preserved, these subcontinua must either be stretched and wrapped or stretched and folded. If a continuum is tree-like, some folding must occur. In this paper, we will see that stretching and folding is not enough to produce an expansive homeomorphism since folding also pushes points closer together. However, stretching and folding is enough to produce *continuum-wise* expansive homeomorphisms, as there are several examples of arc-like and tree-like continua that admit continuum-wise expansive homeomorphisms [2].

In [3], Kato has shown that arc-like continua do not admit expansive homeomorphisms by first showing that the pseudo-arc does not admit an expansive homeomorphism and then by lifting a homeomorphism of an arc-like continuum to a homeomorphism of the pseudo-arc. Unfortunately, these techniques cannot be extended to tree-like continua. In F.W. Worth's Dissertation, it was shown that shift homeomorphisms from the *inverse limit of tree graphs* cannot be expansive homeomorphisms [7], and Kato has also shown that no *hereditarily decomposable* tree-like continuum can admit an expansive homeomorphism. In the sequel, these results are generalized and it is shown that no tree-like continuum can admit an expansive homeomorphism.

2. Main results

The structure of the proof is as follows:

1) For purposes of a contradiction, it is assumed that $h : X \longrightarrow X$ is an expansive homeomorphism of tree-like continuum X with expansive constant c and let $0 < \epsilon < c/3$.

2) A nondegenerate subcontinuum M is found such that $\mathrm{diam}(h^i(M)) \to 0$ as $i \to -\infty$.

3) For each k, finite sequences $\{a^k = x^k_i, ..., x^k_{i_k} = b^k\} \subset M$ are found such that $\mathrm{d}(x^k_i, x^k_{i+1}) < \delta_k$ and $\mathrm{d}(a^k, b^k) < \gamma_k$, where $\delta_k, \gamma_k \to 0$.

4) For each k, it is shown that there are elements $x^k_\alpha, x^k_\beta \in \{a^k = x^k_i, ..., x^k_{i_k} = b^k\}$ such that $\mathrm{d}(x^k_\alpha, x^k_\beta) < \gamma_k$, $\mathrm{d}(h^i(x^k_\alpha), h^i(x^k_\beta)) < \epsilon$ for all $i \leq 0$, and there exists an integer n, $0 \leq n \leq k$, such that $\epsilon/3 \leq \mathrm{d}(h^n(x^k_\alpha), h^n(x^k_\beta)) < \epsilon$. Let $y_k = h^n(x^k_\alpha)$ and $z_k = h^n(x^k_\beta)$.

5) For each k, it is shown that $\mathrm{d}(y_k, z_k) \geq \epsilon/3$ and $\mathrm{d}(h^i(y_k), h^i(z_k)) < \epsilon$ for all $i \leq k$.

6) Finally, it is shown that there exist limit points y, z of $\{z_k\}^\infty_{k=1}, \{y_k\}^\infty_{k=1}$, respectively, such that $y \neq z$ and $\mathrm{d}(h^i(y), h^i(z)) < 2\epsilon < c$ which is a contradiction.

The first proposition follows from the *Simple Chain Theorem* which can be found in most graduate texts such as [5].

Proposition 1. *Suppose X is connected and $a, b \in X$. For every $\epsilon > 0$ there exists a finite sequence $\{x_i\}^n_{i=1} \subset X$ such that $x_1 = a$, $x_n = b$, and $d(x_i, x_{i+1}) < \epsilon$.*

The previous sequence is called a *simple chain sequence* from a to b with *mesh* less than ϵ. The next theorem is due to Kato [2].

Theorem 2. *Let h be an expansive homeomorphism of a continuum X. There exists a nondegenerate subcontinuum M of X such that either* $\lim_{n\to\infty} \text{diam } h^n(M) = 0$ *or* $\lim_{n\to -\infty} \text{diam } h^n(M) = 0$.

Proposition 3. *Suppose $\mathcal{T}$ is a tree cover for continuum X. Let a and b be points of X in the same element T_1 of $\mathcal{T}$ and let $\{x_i\}_{i=1}^n$ be a simple chain sequence from a to b whose mesh is less than the Lebesgue number for $\mathcal{T}$. If x_2 and x_{n-1} are not in the same element of $\mathcal{T}$, then there exists a k, $2 \leq k \leq n-1$, such that $x_k \in T_1$.*

Proof. Suppose not. We may assume that x_2 and x_{n-1} are not in T_1. (Otherwise we are done.) Since each x_i and x_{i+1} are in a common element of $\mathcal{T}$, let T_{i+1} be an element of $\mathcal{T}$ that contains both x_i and x_{i+1}. Since x_2 and x_{n-1} are not in the same element of $\mathcal{T}$ we can conclude that T_1, T_2, and T_n are all different. Also, since T_i and T_{i+1} both contain x_i, T_i and T_{i+1} must intersect. Likewise, T_1 and T_n both contain x_n and hence must intersect. Therefore, the sequence $[T_1, T_2, ..., T_n, T_1]$ must contain a circle-chain. But that contradicts the fact that $\mathcal{T}$ is a tree cover. □

Next, if h is a homeomorphism and n a positive integer, define $\mathcal{L}(h, n, \epsilon)$ to be a number greater than 0 such that

$$d(x, y) < \mathcal{L}(h, n, \epsilon) \text{ implies } d(h^i(x), h^i(y)) < \epsilon.$$

for all $-n \leq i \leq n$, and

$$Q_n(a, b) = \max_{-n \leq i \leq 0} \{d(h^i(a), h^i(b))\}.$$

Notice that for n fixed, $Q_n(a, b)$ is a metric and hence follows the triangle inequality.

Lemma 4. *Suppose $h : X \longrightarrow X$ is a homeomorphism of a continuum X and that $\{x_i\}_{i=1}^m$ is a simple chain sequence from a to b with mesh less than $\mathcal{L}(h, n, \epsilon/6)$. Also, suppose that $\{x_i\}_{i=1}^m$ is contained in some tree-cover $\mathcal{T}$ such that a and b are in the same element T_1 of $\mathcal{T}$ and that the mesh of $\{x_i\}_{i=1}^m$ is less than the Lebesgue number of $\mathcal{T}$. If $Q_n(a, b) \geq \epsilon$, then there exist $x_\alpha, x_\beta \in \{x_i\}_{i=1}^m$ such that x_α, x_β are in the same link of $\mathcal{T}$ and $\epsilon/3 \leq Q_n(x_\alpha, x_\beta) < \epsilon$.*

Proof. Given $T \in \mathcal{T}$ and subsequence $\{x_i\}_{i=m_1}^{m_2}$ of $\{x_i\}_{i=1}^m$ where $x_{m_1}, x_{m_2} \in T$, define $W(T, \{x_i\}_{i=m_1}^{m_2}) = \{x_\alpha, x_\beta\}$ where x_α, x_β are the elements of $T \cap \{x_i\}_{i=m_1}^{m_2}$ such that $Q_n(x_\alpha, x_\beta) \geq \epsilon$ and the difference between the indices, $|\alpha - \beta|$, is minimized. Notice that $W(T, \{x_i\}_{i=m_1}^{m_2})$ will often not exist. However, $W(T_1, \{x_i\}_{i=1}^m) = \{x_{\alpha_1}, x_{\beta_1}\}$ where $\alpha_1 < \beta_1$ exists since $Q_n(a, b) \geq \epsilon$. There are 3 cases to consider:

Notice that if $Q_n(x_{\alpha_1+1}, x_{\beta_1-1}) \geq \epsilon$, then x_{α_1+1} and x_{β_1-1} cannot both be in T_1 since $\beta_1 - 1 - (\alpha_1 + 1) < \beta_1 - \alpha_1$.

Case 1. $Q_n(x_{\alpha_1+1}, x_{\beta_1-1}) < \epsilon$ and $x_{\alpha_1+1}, x_{\beta_1-1}$ are in the same element of $\mathcal{T}$.

Since $\text{d}(h^i(x_{\alpha_1}), h^i(x_{\alpha_1+1})) < \epsilon/6$ and $\text{d}(h^i(x_{\beta_1}), h^i(x_{\beta_1-1})) < \epsilon/6$ for all $-n \leq i \leq 0$, we have $\epsilon/3 < Q_n(x_{\alpha_1}, x_{\beta_1}) - \epsilon/6 - \epsilon/6 < Q_n(x_{\alpha_1+1}, x_{\beta_1-1}) < \epsilon$ and we are done.

Case 2. x_{α_1+1} and x_{β_1-1} are not contained in the same element of $\mathcal{T}$.

Then, by Proposition 3, there exists a k_1, $\alpha_1 + 1 \leq k_1 \leq \beta_1 - 1$ such that $x_{k_1} \in T_1$. $Q_n(x_{\alpha_1}, x_{k_1}) < \epsilon$ and $Q_n(x_{\beta_1}, x_{k_1}) < \epsilon$ since $|k_1 - \alpha_1| < |\beta_1 - \alpha_1|$ and $|k_1 - \beta_1| < |\beta_1 - \alpha_1|$. Hence, by the triangle inequality, either $\epsilon/3 < Q_n(x_{\alpha_1}, x_{k_1}) < \epsilon$ or $\epsilon/3 < Q_n(x_{k_1}, x_{\beta_1}) < \epsilon$ and we are done. The next case creates an induction argument for this lemma.

Case 3. $\epsilon \leq Q_n(x_{\alpha_1+1}, x_{\beta_1-1})$ and x_{α_1+1} and x_{β_1-1} are in same element of $\mathcal{T}$, say T_2.

Then $W(T_2, \{x_i\}_{\alpha_1+1}^{\beta_1-1}) = \{x_{\alpha_2}, x_{\beta_2}\}$, where $\alpha_2 < \beta_2$.

Suppose $T_2, \ldots, T_j$ and $x_{\alpha_2}, x_{\beta_2}, \ldots, x_{\alpha_j}, x_{\beta_j}$ have been found, again we have 3 cases to consider:

Case 1-j. $Q_n(x_{\alpha_j+1}, x_{\beta_j-1}) < \epsilon$ and $x_{\alpha_j+1}, x_{\beta_j-1}$ are in the same element of $\mathcal{T}$.

As in Case 1, this implies that we are done.

Case 2-j. x_{α_j+1} and x_{β_j-1} are not contained in the same element of $\mathcal{T}$.

As in Case 2, this implies that we are done.

Case 3-j. $\epsilon \leq Q_n(x_{\alpha_j+1}, x_{\beta_j-1})$ and x_{α_j+1} and x_{β_j-1} are in the same element of $\mathcal{T}$, say T_{j+1}.

Then $W(T_{j+1}, \{x_i\}_{i=\alpha_j+1}^{\beta_j-1}) = \{x_{\alpha_{j+1}}, x_{\beta_{j+1}}\}$, where $\alpha_{j+1} < \beta_{j+1}$, and the induction continues.

Eventually, the induction must stop at some j_1. Otherwise, since $\alpha_{j+1} > \alpha_j$ and $\beta_{j+1} < \beta_j$, there would be a j_2 such that $|\beta_{j_2} - \alpha_{j_2}| \leq 1$, which would in turn imply $Q_n(x_{\alpha_{j_2}}, x_{\beta_{j_2}}) < \epsilon/6$ which is impossible. So if the induction stops at j_1, then Case 3-j_1 cannot be satisfied. Hence either Case 1-j_1 or Case 2-j_1 must be satisfied and the lemma is satisfied. □

Lemma 5. *Let $h : X \longrightarrow X$ be a homeomorphism of a compact space onto itself. Suppose that there exist sequences $\{y_i\}_{i=1}^{\infty}$, $\{z_i\}_{i=1}^{\infty}$ such that $d(h^k(y_n), h^k(z_n)) < \epsilon$ for all $k \leq n$. Then there exists a limit point y of $\{y_i\}_{i=1}^{\infty}$ and a limit point z of $\{z_i\}_{i=1}^{\infty}$ such that $d(h^k(y), h^k(z)) < 2\epsilon$ for all k.*

Proof. Let Y be the set of limit points of $\{y_i\}_{i=1}^{\infty}$. Pick y in Y and let $\{y_{\alpha_i}\}_{i=1}^{\infty}$ be a subseqence that converges to y. Let Z_α be the set of limit points of $\{z_{\alpha_i}\}_{i=1}^{\infty}$. Pick $z \in Z_\alpha$ and let $\{z_{\beta_i}\}_{i=1}^{\infty}$ be a subsequence of $\{z_{\alpha_i}\}_{i=1}^{\infty}$ that converges to z. Then $\{y_{\beta_i}\}_{i=1}^{\infty}$ is a subsequence of $\{y_{\alpha_i}\}_{i=1}^{\infty}$ and hence also converges to y.

For each positive integer n, there exists $m_n \geq n$ such that $\mathrm{d}(y_{\beta_{m_n}}, y) < \mathcal{L}(h, n, \epsilon/2)$ and $\mathrm{d}(z_{\beta_{m_n}}, z) < \mathcal{L}(h, n, \epsilon/2)$. Thus,

$$\begin{aligned} d(h^k(y), h^k(z)) < d(h^k(y), h^k(y_{\beta_{m_n}})) + d(h^k(y_{\beta_{m_n}}), h^k(z_{\beta_{m_n}})) \\ + d(h^k(z_{\beta_{m_n}}), h^k(z)) < \epsilon/2 + \epsilon + \epsilon/2 \end{aligned}$$

for all $-n \leq k \leq n$. Since n is arbitrary, the lemma holds. □

Theorem 6. *Tree-like continua do not admit expansive homeomorphisms.*

Proof. Suppose that $h : X \longrightarrow X$ is an expansive homeomorphism of tree-like continuum X with expansive constant c. Let ϵ be chosen such that $0 < \epsilon < c/3$. By Theorem 2, there exists a nondegenerate subcontinuum M such that either $\lim_{n\to\infty}$ diam $h^n(M) = 0$ or $\lim_{n\to-\infty}$ diam $h^n(M) = 0$. Without loss of generality, we may assume that $\mathrm{diam}(h^i(M)) < \epsilon$ for all $i \leq 0$. Let $\{\delta_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that each $\delta_k < \mathcal{L}(h, k, \epsilon/6)$. Let $\mathcal{T}_k$ be a tree-cover of X with mesh $< \delta_k$. Let A_k be any $|\mathcal{T}_k| + 1$ elements of M. By the pigeon-hole principle, for each N, there must be at least 2 elements $a_N^k, b_N^k \in A_k$ such that $h^N(a_N^k), h^N(b_N^k)$ are in a common element of $\mathcal{T}_k$. Since A_k is finite, we may conclude that there are two elements $a_k, b_k \in A_k$ and a sequence of increasing integers $\{N_j\}_{j=1}^{\infty}$ such that $h^{N_j}(a_k)$ and $h^{N_j}(b_k)$ are in a common element of $\mathcal{T}_k$ for each j. Also, since h is expansive, there exists an integer n_k such that $\mathrm{d}(h^{n_k}(a_k), h^{n_k}(b_k)) \geq c > \epsilon$.

Pick $N_{j_k} \geq n_k$. By Lemma 4, there exists $x^k_\alpha, x^k_\beta \in h^{N_{j_k}}(M)$ such that $\epsilon/3 \leq Q_{N_{j_k}}(x^k_\alpha, x^k_\beta) < \epsilon$ and $\mathrm{d}(x^k_\alpha, x^k_\beta) < \delta_k$. Hence, $\mathrm{d}(h^i(x^k_\alpha), h^i(x^k_\beta)) < \epsilon$ for all $i \leq k$.

Now, let m_k be the positive integer such that $\mathrm{d}(h^{-m_k}(x^k_\alpha), h^{-m_k}(x^k_\beta)) \geq \epsilon/3$. Let $y_k = h^{-m_k}(x^k_\alpha)$ and $z_k = h^{-m_k}(x^k_\beta)$. Then $\mathrm{d}(h^i(y_k), h^i(z_k)) < \epsilon$ for all $i < k + m_k$. By Lemma 5, there exist limit points y of $\{y_k\}_{k=1}^\infty$ and z of $\{z_k\}_{k=1}^\infty$ such that $\mathrm{d}(h^i(y), h^i(z)) \leq 2\epsilon < c$ for all i. However, since $\mathrm{d}(y_k, z_k) \geq \epsilon/3$, y and z must be distinct. Therefore, h is not expansive. □

A continuum is *decomposable* if it is the union of two of its proper subcontinuum and *indecomposable* otherwise. A continuum is *hereditarily indecomposable* if every subcontinuum is indecomposable.

Question 1 ([3]). *Does there exist a hereditarily indecomposable continuum that admits an expansive homeomorphism?*

Question 2. *Does there exist a non-separating plane continuum that admits an expansive homeomorphism?*

If so, then it cannot be 1-dimensional.

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