BOUNDING THE FITTING HEIGHT OF A SOLVABLE GROUP
BY THE NUMBER OF ZEROS IN A CHARACTER TABLE

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ABSTRACT. In this paper, we bound the Fitting height of a solvable group by
the number of zeros in a character table.

1. Introduction

In this paper, all characters are complex characters and Irr(G) is the set of
irreducible characters of a finite group G. An old theorem of Burnside asserts that
any nonlinear \( \chi \in \text{Irr}(G) \) must vanish at some element of \( G \). In this paper, we
consider the following problem: given the number of zeros in a character table of
a finite group \( G \), what can be said about the structure of \( G \)? Although there are
many results about zeros of irreducible characters (see [1], [2], [3], [4]), our result is
that the Fitting height of a finite solvable group can be bounded by the number of
zeros in a character table.

For a finite solvable group \( G \), we define characteristic subgroups \( F_i(G) \) by let-
ting \( F_1(G) = F(G) \), the unique largest nilpotent normal subgroup of \( G \), and
\( F_{i+1}(G)/F_i(G) = F(G/F_i(G)) \). The Fitting height of group \( G \), denoted by \( nl(G) \),
is the smallest number \( l \) for which \( F_l(G) = G \). For a subset \( T \) of \( G \), let \( k_G(T) \)
be the number of \( G \)-conjugacy classes contained in \( T \). For \( \chi \in \text{Irr}(G) \), \( v_G(\chi) \) de-
notes the set of elements contained in \( G - F(G) \) on which \( \chi \) vanishes. It is clear
that \( v_G(\chi) \) is a union of some conjugacy classes of \( G \). Let \( \nu_G(\chi) = k_G(v_G(\chi)) \),
i.e. the number of \( G \)-conjugacy classes outside \( F(G) \) on which \( \chi \) vanishes, and let
\( \nu(G) = \max \{ \nu_G(\chi) | \chi \in \text{Irr}(G) \} \). Now our result can be stated as follows.

Theorem 1. Let \( G \) be a solvable group. Then \( nl(G) \leq (2\nu(G) + 5)/3 \).

Note that if we let \( m(G) \) be the maximal number of \( G \)-classes on which some
\( \chi \in \text{Irr}(G) \) vanishes, then we see that \( \nu(G) \leq m(G) \) and hence we also get the
bound \( \nu(G) \leq (2m(G) + 5)/3 \).

Throughout this paper, we use the following notations:
\( \text{cl}_G(g) := \text{the } G \text{-conjugacy class containing } g \in G \). Also if we say “class”, we
always mean “conjugacy class”.
\( \text{Irr}(G/N) := \text{Irr}(G) - \text{Irr}(G/N) \) where \( N \triangleleft G \). For a character \( \sigma \), \( \text{Irr}(\sigma) \) denotes
the set of irreducible constituents of \( \sigma \).

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class.
\[ \Phi(G) := \cap \{M| M \text{ is a maximal subgroup of } G\}; \]  
\( p \) always denotes a prime number.

2. Proof of Theorem 1

In this section, we assume that the letter \( G \) always denotes a finite solvable group, and we shall freely use the following known results:

1. Suppose \( G = MN \) is a Frobenius group with kernel \( N \) and complement \( M \). Then \( N \) is nilpotent and \( (|M|, |N|) = 1 \). Furthermore, if \( M \) is abelian, then \( M \) is cyclic; if \( 2||M| \), then \( N \) is abelian.

2. Let \( N \triangleleft G \) and \( \chi \in \text{Irr}(G) \). If \( e\chi = \lambda^G \) for some \( \lambda \in \text{Irr}(N) \) and some positive integer \( e \), then \( \chi \) vanishes on \( G - N \).

3. Let \( N < M \) be two normal subgroups of \( G \). If \( \psi \in \text{Irr}(M) \) vanishes on \( M - N \), then by Clifford’s theorem, each member of \( \text{Irr}(\psi^G) \) also vanishes on \( M - N \).

**Lemma 1** ([\( G \) Lemma 18.1]). If \( G/F(G) \) is abelian, then there exists \( \chi \in \text{Irr}(G) \) such that \( \chi(1) = |G : F(G)| \).

**Lemma 2.** Let \( M, N \) be two normal subgroups of \( G \) with \( 1 < N < M \). Then:

1. If \( k_G(M - N) = 1 \) and \( (|M : N|, |N|) = 1 \), then \( M \) is a Frobenius group with kernel \( N \) and prime order complement.

2. If \( G = M \) and \( k_G(G - N) = 1 \), then \( |G : N| = 2 \) and \( G \) is a Frobenius group with abelian kernel \( N \) of odd order.

3. If \( N = F(G) \) and \( M/N \) is abelian, then there exists \( \psi \in \text{Irr}(M|N) \) of degree \( |M : N| \) vanishing on \( M - N \), and every \( \chi \in \text{Irr}(\psi^G) \) also vanishes on \( M - N \).

**Proof.** (1) Write \( M = HN \), where \( H \cong M/N \). Since \( k_G(M - N) = 1 \), then \( M/N \) is a principal factor of \( G \), and hence is an elementary abelian group with prime power order \( p^r \). Also, we see that all elements in \( M - N \) have the same order \( p \) since \( (|M : N|, |N|) = 1 \). It implies that for any element \( h \in H - \{1\}, C_M(h) = H \). Therefore, \( M \) is a Frobenius group with abelian complement \( H \) so that \( H \) is a cyclic group of prime order \( p \).

(2) Since \( k_G(G - N) = 1 \), \( G/N \) is a principal factor of \( G \) and hence \( |G/N| = p \). Let \( x \in G - N \). Observe that the length of class \( cl_G(x) \) is less than or equal to the order of derived group \( G' \). It is easy to see that \( N = G' \), \( |G : N| = 2 \) and \( C_G(x) = \langle x \rangle \) with order 2. So, \( G = \langle x \rangle N \) is a Frobenius group with a complement \( \langle x \rangle \) of order 2, and hence the Frobenius kernel \( N \) is an abelian group of odd order.

(3) Since \( F(G/\Phi(G)) = F(G)/\Phi(G) \), we may assume that \( \Phi(G) = 1 \) and hence \( F(G) \) is abelian. Since \( F(M) = F(G) \), by Lemma 1, there exists \( \psi \in \text{Irr}(M|F(G)) \) such that \( \psi(1) = |M : F(G)| \). It is clear that \( \psi = \lambda^M \) for some non-principal linear \( \lambda \in \text{Irr}(F(G)) \) and hence \( \psi \) vanishes on \( M - F(G) \). Now for any \( \chi \in \text{Irr}(\psi^G) \), we see that \( \chi \in \text{Irr}(G|F(G)) \) and that \( \chi \) also vanishes on \( M - F(G) \) by Clifford’s theorem.

Lemma 3 is somewhat trivial, but is useful and is used over and over in the rest of this paper. Note that Lemma 3 is true for any finite group.

**Lemma 3.** Let \( N \triangleleft G \) and \( \psi \in \text{Irr}(G/N) \). Write \( \bar{G} = G/N \). Then:

1. \( cl_G(\bar{g}) \) (viewed as a subset of \( G \)) is again a \( G \)-class if and only if for every \( \chi \in \text{Irr}(G|N), \chi(\bar{g}) = 0 \); similarly, if \( A \) is a union of some cosets of \( N \), then \( k_G(A) = 1 \) if and only if \( k_G(\bar{A}) = 1 \) and each \( \chi \in \text{Irr}(G|N) \) vanishes on \( A \).
(2) If \( A = v_G(\psi) \) contains \( m \) classes of \( G/N \), then \( nv_G(\psi) \leq nv_G(\psi) \); with equality if and only if \( \chi \) vanishes on \( A \) for every \( \chi \in \text{Irr}(G/N) \); and if \( nv_G(\psi) + 1 = nv_G(\psi) \), then there exists a subset \( A_1 \) of \( A \) such that \( m - 1 = k_G(A_1) = K_G(A_1) \), and hence \( \chi \) vanishes on \( A_1 \) for every \( \chi \in \text{Irr}(G/N) \).

(3) \( nv(G) \leq nv(G) \).

**Proof.** (1) It follows that \( |C_G(g)| = |C_{G/N}(gN)| + \sum_{\chi \in \text{Irr}(G/N)} |\chi(g)|^2 \) and that \( cl_{G/N}(gN) = cl_G(g) \) if and only if \( |C_{G/N}(gN)| = |C_G(g)| \).

Statement (2) is immediate from statement (1), and (3) follows from (2).

**Lemma 4.** Suppose \( nl(G) \geq 2 \). Write \( F(G) = F \) and \( F_2(G) = F_2 \). Then:

1. There exists \( T_0 \triangleleft G \) such that \( T_0 \not< F \), \( nl(G/T_0) = nl(G) \) and \( F/T_0 \) is a principal factor of \( G \).
2. \( nl(G) \geq nl(G/F) + 1 \).
3. Let \( T \leq T_2 \) be a normal subgroup of \( G \) such that \( F \leq T \leq T_2 \) and \( T_2/F \) is a Frobenius group with cyclic complement \( M \). Then \( \chi \in \text{Irr}(G/F) \) vanishes on \( L = G/F \).

**Proof.** (1) Let \( F/\Phi(G) = D_1/\Phi(G) \times \cdots \times D_s/\Phi(G) \) be a direct product of minimal normal subgroups \( D_j/\Phi(G) \). Let \( T_j = \prod_{k \neq j} D_k \). It is clear that there exists some \( T_j \) such that \( nl(G/T_0) = nl(G) \) as desired.

Now we prove conclusions (2) and (3). Let \( L \triangleleft G \) maximal such that \( L/F \) is an abelian group. By Lemma 2(3), there exists \( \chi_0 \in \text{Irr}(G/F) \) such that \( \chi_0 \) vanishes on \( L - F \).

(2) It is easy to see that if \( nl(G) = 2 \), then \( nv(G) \geq 1 \). Suppose \( nl(G) \geq 3 \) and let \( \psi \in \text{Irr}(G/F) \) such that \( nv_{G/F}(\psi) = m = nv(G/F) \) and let \( A = v_{G/F}(\psi) \). If \( nv(G) = nv_{G/F}(\psi) = m \), then \( k_{G/F}(A) = m = k_G(A) \), and hence each \( \chi \in \text{Irr}(G/F) \) vanishes on \( A \) by Lemma 3. Thus \( \chi_0 \), as a member of \( \text{Irr}(G/F) \), vanishes not only on \( L - F \) but also on \( A \). Observe that \( A \subseteq G - F_2 \). It forces that \( nv_G(\chi_0) \geq m + 1 \), a contradiction.

(3) Let \( G \) be a counterexample. Since \( \chi_0 \) vanishes on \( L - F \) and \( \chi_0 \in \text{Irr}(G/F) \subseteq \text{Irr}(G/T) \), then \( k_G(L - F) = 1 \) and hence \( L/F \) is a \( p \)-principal factor of \( G \). In particular, \( F_2/F \) is a \( p \)-group because \( Z(F_2/F) \leq L/F \). Let \( N = O_p(G) \). Then by Lemma 3(1), \( k_{G/N}(L/N - F/N) = 1 \). Now Lemma 2(1) implies that \( L/N \) is a Frobenius group with cyclic complement \( L/F \) of prime order \( p \).

Since \( [F_2/F] \) is not a prime, we may choose \( M/L \) to be a \( p \)-principal factor of \( G \) with \( M \leq F_2 \). By the assumption of \( L \), it is easy to see that \( M/F \) is an extra-special \( p \)-group. Then there exists \( \psi \in \text{Irr}(M/F) \) such that \( \psi \) vanishes on \( M - L \). Now let \( \chi_1 \in \text{Irr}(\psi^G) \); then \( \chi_1 \) also vanishes on \( M - L \). Since \( ker(\chi_1) = F \) and \( F < T \) (note that we assume \( F_2/T \) is abelian), we have \( \chi_1 \in \text{Irr}(G/T) \). But since \( G \) is a counterexample, it forces that \( M - L \) is just a class of \( G \). Now applying Lemma 3(1), we see that \( \chi_0 \) (as a member of \( \text{Irr}(G/L) \)) vanishes not only on \( L - F \) but also on \( M - L \), so \( \chi_0 \) vanishes off two classes contained in \( F_2 - F \), a contradiction.

In particular, if \( nl(G) \geq 4 \) and \( F_2/T \) is a principal factor of \( G \), then \( |F_2 : F| \) is not prime because the alternative would force \( G/F_2 \leq Aut(F_2/F) \) to be abelian, and so that the same conclusion holds.

**Lemma 5.** Suppose \( nl(G) \geq 3 \). Then \( nv(G) \geq 2 \) and with equality if and only if \( G \cong S_4 \).
Proof. Write $F(G) = F, F_2(G) = F_2$. By Lemma 4(2), we have $nv(G) \geq 1 + nv(G/F) \geq 2$. It is clear that $nv(S_4) = 2$. We now prove $G \cong S_4$ provided that $nv(G) = 2$ and $nl(G) \geq 3$.

By Lemma 4(2), we see that $nl(G) = 3$ and $nv(G/F) = 1$.

Step 1. If $F$ is a minimal normal subgroup of $G$, then $G \cong S_4$.

We claim that $G/F_2$ is abelian. Assume this is not the case. Let $N \triangleleft G$ maximal such that $F_2 \leq N$ and $G/N$ is not abelian, and let $Z > N$ with $Z/N = Z(G/N)$.

By Lemma 12.3, there exists $\phi \in \text{Irr}(G/N)$ such that $\phi$ vanishes on $G - Z$. Since $1 = nv(G/F) \geq nv_{G/F}(\phi)$, we have $k_{G/F}(G/F - Z/F) = 1$. Now applying Lemma 2(2), we have $|G : Z| = 2$. But it is impossible because $|G/Z| = \phi(1)^2 \neq 2$.

Now $G/F_2$ is abelian; by Lemma 2(3), there exists $\chi_1 \in \text{Irr}(G/F_2/F_2/F)$ such that $\chi_1$ vanishes on $G - F_2$. Since $nv(G/F) = 1$ and then $k_{G/F}(G/F - F_2/F) = 1$, Lemma 2(2) yields that $G/F$ is a Frobenius group with complement of order 2 and abelian kernel $F_2/F$ of odd order. If $k_{G}(G - F_2) = 1$, then Lemma 2(2) yields that $G$ is a Frobenius group with kernel $F_2$ and thus $F_2$ is nilpotent, a contradiction. Thus $k_{G}(G - F_2) = 2$ and then we easily deduce that $|C_G(g)| = 4$ for every $g \in G - F_2$.

It follows that $F$ is an elementary abelian 2-group.

Let $P \in \text{syl}_2(G)$ and let $g \in P - F$. Observe that $|P/P'| = |C_{P/P'}(gP')| = \sum_{\chi \in \text{Irr}(P/P')}|\chi(g)|^2 = \sum_{\chi \in \text{Irr}(P)}|\chi(g)|^2 = |\chi_2(g)| = |\chi_2(g)| = |\chi_2(g)| = 4$. It follows that $|P/P'| = 4$, and hence by O. Taussky’s Theorem [3, III, Theorem 11.9], $P$ has a cyclic subgroup with index 2. Since $F$ is elementary abelian, it forces that $|F| \leq 4$.

Now observe that $2'|F_2/F$ acts faithfully on 2-group $F$. It is easy to check that $|F_2/F| = 3$ and $|F| = 4$, so that $G \cong S_4$, as desired.

Step 2. $G \cong S_4$.

By Lemma 4(1), we may choose $T \triangleleft G, T < F$ such that $nl(G/T) = nl(G) = 3$ and $F/T$ is a principal factor of $G$. By Lemma 4(2) and Lemma 3(3), we have $nv(G/T) = 2$, and then $G/T \cong S_4$ by induction and step 1. Now it suffices to show $T = 1$. Suppose $T \neq 1$. To see a contradiction, we may assume that $T$ is a minimal normal subgroup of $G$. Write $G = G/T$. It is clear that $F(G) = F_2, F_2(G) = F_2$. Applying Lemma 2(3), we may choose $\chi_1, \chi_2 \in \text{Irr}(G)$ such that $\chi_1$ vanishes on $G - F_2$ and $\chi_2$ vanishes on $F_2 - F$. Since $2 = k_{G}(G - F_2) = nv_G(\chi_1) \leq nv_G(\chi_1) \leq 2$, it follows that $k_{G}(G - F_2) = k_{G}(G - F_2) = 2$. By Lemma 3, each member $\chi$ of $\text{Irr}(G/T)$ vanishes on two $G$-classes: $G - F_2$, and hence $v_G(\chi) = G - F_2$ because $nv_G(\chi) \leq 2$.

Now we conclude that $|C_G(g)| = |C_G(\bar{g})| = 4$ for $g \in G - F_2$. Note that each member $\chi$ of $\text{Irr}(G/T)$ does not vanish on $F_2 - F$; we have $k_{G}(F_2 - F) < k_{G}(F_2 - F)$ by Lemma 3. It follows that $2 = k_{G}(F_2 - F)$ because $2 = 1 + k_{G}(F_2 - F) \leq k_{G}(F_2 - F) \leq nv_G(\chi_2) \leq 2$.

Since $k_{G}(F_2 - F) = 2$, it is easy to check that for every $x \in F_2 - F, |C_G(x)| = 6$. Observe that $|C_G(\bar{x})| = 3$. It forces that $T$ is a 2-group. Now let $P \in \text{syl}_2(G)$.

Arguing as in Step 1, we see that $|P/P'| = 4$ and $P$ has an element of order $|P|/2$. But $P$ has no such element in $P - F$ (note that for every $g \in P - F, |C_G(g)| = 4$ or in $F$, a contradiction.

Theorem 1. Let $G$ be a solvable group. Then $nl(G) \leq (2nv(G) + 5)/3$.

Proof. By Lemma 4 and Lemma 5, we may assume $nl(G) \geq 4$. Let $G$ be a minimal counterexample with $nl(G) = m + 2$. By Lemma 3(3) and Lemma 4(1), we may assume that $\Phi(G) = 1$ and $F(G)$ is the unique minimal normal subgroup of $G$ and
hence \(|F_2(G) : F(G)|, |F(G)|\) = 1. For the sake of simplicity, write \(F_3 = F_3(G), F_2 = F_2(G)\) and \(F = F(G)\).

First we claim that \(nl(G) \geq 5\). Suppose \(nl(G) = 4\) with

\[
4 = nl(G) > (2nv(G) + 5)/3.
\]

By Lemma 4(2) and Lemma 5, we see that \(nv(G) = 3\), \(nv(G/F) = 2\) and \(G/F \cong S_4\).

Also, there exists \(\chi \in \text{Irr}(G/F_2)\) such that \(\chi\) vanishes on \(G/F_3\). Since \(nv(G) = 3\) and \(2 = k_{G/F}(G/F - F_3/F) \leq k_G(G/F_3) \leq 3\), by Lemma 3, there exists \(g_0 \in G/F_3\) such that \(c_{G/F}(g_0F)\) is again a class of \(G\) and then \(\chi(g_0) = 0\) for all \(\chi \in \text{Irr}(G/F)\).

Since \(F_2/F\) is an abelian group of order 4, by Lemma 2(3), there exists \(\chi \in \text{Irr}(\psi(G))\) such that \(\chi\) vanishes on \(F_2 - F\), where \(\psi \in \text{Irr}(F_2)\) with \(\psi(1) = 4\). Now we see that \(\chi\) vanishes on \(g_0\) and \(F_2 - F\). If \(k_G(F_2 - F) = 1\), then Lemma 2(1) implies that \(F_2\) is a Frobenius group (note that \((|F_2/F|, |F|)\) = 1 as assumed in above paragraph), and so its abelian complement \(F_2/F\) is cyclic, a contradiction. Now \(k_G(F_2 - F) > 1\). Since \(nv_G(\chi) \leq 3\), it follows that \(v_G(\chi) = c_G(g_0) \cup (F_2 - F)\) and that \(\chi\) is an extension of \(\psi\) (otherwise, it is easy to check that \(\chi\) vanishes on \(G/F_3\) or \(G/F_2\). Now \(x_0 \in \text{Irr}(G)\) by \([H]\) Corollary 6.17, and \(x_0\chi\) vanishes on \(\{G/F_3\} \cup \{F_2 - F\}\) and then \(nv(G) \geq nv_G(x_0\chi) \geq 4\), a contradiction.

By Lemma 4(1), we can choose \(T_1 \triangleleft G, F \leq T_1 < F_2\) such that \(nl(G/T_1) = m + 1\) and \(F_2/T_1\) is a principal factor of \(G\). Similarly, let \(K, K_1 \triangleleft G\) such that \(K/T_1 = F/(G/T_1), K_1/T_1 = F_2(G/T_1)\), and let \(T_2 \triangleleft G\) with \(K \leq T_2 < K_1\) such that \(K_1/T_2\) is a principal factor and \(nl(G/T_2) = m\). Write \(s = nv(G/T_2)\). By induction, we have \(m = nl(G/T_2) \leq (2s + 5)/3\). Then the hypothesis that \(G\) is a counterexample implies \((2nv(G) + 5)/3 < nl(G) = m + 2 \leq (2s + 11)/3\), so that \(nv(G) \leq s + 2\). Now applying Lemma 3(3) and Lemma 4(2), we conclude \(s = nv(G/T_2) \leq nv(G/F_2) < nv(G/F) < nv(G) \leq s + 2\), which forces that \(nv(G) = s + 2\), \(nv(G/F) = s + 1\) and \(nv(G/F_2) = s\). Because \(K/T_1\) is the Fitting subgroup of \(G/T_1\), by Lemma 3(3) and Lemma 4(2), we have \(s = nv(G/K) < nv(G/T_1) = nv(G/F) = s + 1\), thus \(nv(G/T_2) = s + 1\).

Choose \(\chi_2 \in \text{Irr}(G/T_2)\) such that \(nv_G(T_2)(\chi_2) = nv(G/T_2) = s\) and let \(A = v_G(T_2)(\chi_2)\). Clearly, \(A\) (viewed as a subset of \(G/T_1\)) is the union of \(s + 1\) \(G/T_1\)-classes because \(s + 1 = nv(G/T_1)\).

Case 1. Suppose \(k_{G/T_1}(A) = s\).

By Lemma 3, for any \(\chi \in \text{Irr}(G/T_1|T_2/T_1), \chi\) vanishes on \(A\). Since \(nl(G/T_1) = m + 1 \geq 4\), by Lemma 4(3), there exists \(x_1 \in \text{Irr}(G/T_1|T_2/T_1)\) such that \(x_1\) vanishes on at least two \(G/T_1\)-classes contained in \(K_1/T_1 - K/T_1\). Observe that \(A \cap K_1 = \emptyset\).

Since \(\chi_1\) also vanishes on \(A\), we see that \(nv(G/T_1) \geq nv_{G/T_1}(\chi_1) \geq s + 2\), a contradiction.

Case 2. Suppose \(k_{G/T_1}(A) = s + 1\).

Then by Lemma 3(2), there exists a subset \(A_1\) of \(A\) such that \(k_{G/T_1}(A_1) = k_{G/T_2}(A_1) = s - 1\), and hence for every \(\chi \in \text{Irr}(G/T_1|T_2/T_1), \chi\) vanishes on \(A_1\). By Lemma 4(3), there exists \(\chi_1 \in \text{Irr}(G/T_1|T_2/T_1)\) which vanishes on \(C\): two \(G/T_1\)-classes contained in \(K_1/T_1 - K/T_1\). Note that \(C \cap A = \emptyset\).

Now \(\chi_1\) vanishes on \(s + 1\) \(G/T_1\)-classes: \(B = A_1 \cup C; \chi_2\) vanishes on \(s + 1\) \(G/T_1\)-classes: \(A\). Since \(nv_G(\chi_1) \leq nv(G) = s + 2\), by Lemma 3(2), there exists a subset \(B_1\) of \(B\) such that \(k_{G/B_1} = s = k_{G/T_1}(B_1)\), and for every \(\chi \in \text{Irr}(G/T_1), \chi\) vanishes on \(B_1\). Similarly, there exists \(A_2 \subseteq A\) which contains \(s\) classes of \(G\), and for every \(\chi \in \text{Irr}(G/T_1), \chi\) vanishes on \(A_2\). Thus for every \(\chi \in \text{Irr}(G/T_1), \chi\) vanishes on \(A_2 \cup B_1\), which contains at least \(s + 1\) classes of \(G\). By Lemma 4(3), there exists
\( \chi_3 \in \text{Irr}(G|T_1) \) which vanishes on at least two \( G \)-classes \( D \) contained in \( F_2 - F \). Note that \((A_2 \cup B_1) \cap (F_2 - F) = \emptyset \). Now \( \chi_3 \) vanishes on at least \( s + 3 \) \( G \)-classes: \( A_2 \cup B_1 \cup D \) and then \( nv(G) \geq nv_G(\chi_3) \geq s + 3 \), a contradiction. The proof is complete.

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