

## A NOTE CONCERNING THE INDEX OF THE SHIFT

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(Communicated by Joseph A. Ball)

ABSTRACT. Let  $\mu$  be a finite, positive Borel measure with support in  $\{z : |z| \leq 1\}$  such that  $P^2(\mu)$  – the closure of the polynomials in  $L^2(\mu)$  – is irreducible and each point in  $\mathbb{D} := \{z : |z| < 1\}$  is a bounded point evaluation for  $P^2(\mu)$ . We show that if  $\mu(\partial\mathbb{D}) > 0$  and there is a nontrivial subarc  $\gamma$  of  $\partial\mathbb{D}$  such that

$$\int_{\gamma} \log\left(\frac{d\mu}{dm}\right) dm > -\infty,$$

then  $\dim(\mathcal{M} \ominus z\mathcal{M}) = 1$  for each nontrivial closed invariant subspace  $\mathcal{M}$  for the shift  $M_z$  on  $P^2(\mu)$ .

### 1. INTRODUCTION

Given a finite, positive Borel measure  $\mu$  with compact support in the complex plane  $\mathbb{C}$  and  $1 \leq t < \infty$ , we let  $P^t(\mu)$  denote the closure of the polynomials in  $L^t(\mu)$ . J. Thomson has established (see [T]) a direct sum decomposition of  $P^t(\mu)$  that involves the components of  $abpe(P^t(\mu))$  – the set of analytic bounded point evaluations for  $P^t(\mu)$ . In this brief paper we restrict our attention to the case  $t = 2$  and assume that the support of  $\mu$  is contained in  $\overline{\mathbb{D}}$  ( $\mathbb{D} := \{z : |z| < 1\}$ ), that  $abpe(P^2(\mu)) = \mathbb{D}$  and that  $P^2(\mu)$  is irreducible (which means that  $P^2(\mu)$  contains no nontrivial characteristic functions). A consequence of these assumptions is that  $\mu|_{\partial\mathbb{D}} \ll m$ , where  $m$  denotes normalized Lebesgue measure on  $\partial\mathbb{D}$ . Another consequence is that multiplication by the independent variable  $z$  is a bounded operator on  $P^2(\mu)$ , with closed range; we call this operator the shift and denote it by  $M_z$ . If  $\mu(\partial\mathbb{D}) = 0$ , then work of C. Apostol, H. Bercovici, C. Foias and C. Pearcy in [ABFP] shows that for any natural number  $n$ , and for  $n = \infty$ , there is a closed invariant subspace  $\mathcal{M}$  for the shift on  $P^2(\mu)$  such that  $\dim(\mathcal{M} \ominus z\mathcal{M}) = n$ . Information concerning the lattice of invariant subspaces for  $M_z$  on such  $P^2(\mu)$  spaces – in particular, the classical Bergman space  $L^2_a(\mathbb{D})$  – contributes to a better understanding of bounded operators on separable Hilbert spaces in general (see [ABFP] or [HRS]). In contrast, if  $\mu(\partial\mathbb{D}) > 0$ , then the lattice of invariant subspaces for  $M_z$  on  $P^2(\mu)$  appears to be somewhat limited and indeed the results to date have led the authors of [CY] to conjecture that (in this setting) the outcome follows that of the classical Hardy space  $H^2(\mathbb{D})$ , and  $\dim(\mathcal{M} \ominus z\mathcal{M}) = 1$  for each nontrivial closed invariant subspace  $\mathcal{M}$  for the shift on  $P^2(\mu)$ . Seminal work in support of

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Received by the editors April 17, 2001 and, in revised form, June 19, 2001.

2000 *Mathematics Subject Classification*. Primary 47A53, 47B20, 47B38; Secondary 30E10, 46E15.

this conjecture was done by R. Olin and J. Thomson in [OT] who established it in the special case that the support of  $\mu$  has an “outer hole”. The work of L. Miller, J. Thomson and L. Yang, who make use of results in [OT], all but dispatches with the conjecture in the case  $\mu|_{\mathbb{D}}$  is area measure; see [M], [Y] and [TY]. Recently, the author of this paper has given an analytic condition that defines what it means for a measure to be so-called “strongly inscribed” and has shown that if a measure  $\mu$  is such, then there is a measure  $\mu_o$  whose support has an outer hole and for which the shifts on  $P^2(\mu)$  and  $P^2(\mu_o)$  are similar as operators (see [A1]). From this it follows that if  $\mu$  is strongly inscribed, then  $\dim(\mathcal{M} \ominus z\mathcal{M}) = 1$  for each nontrivial closed invariant subspace  $\mathcal{M}$  for the shift on  $P^2(\mu)$ . Furthermore, the existence of a measure  $\mu_o$  (as described above) is basically determined by whether or not  $\mu$  is strongly inscribed. In this paper we use an idea or two from [A1] to show that the above conjecture holds whenever there is a nontrivial subarc  $\gamma$  of  $\partial\mathbb{D}$  such that

$$\int_{\gamma} \log\left(\frac{d\mu}{dm}\right) dm > -\infty;$$

no special assumption is made concerning  $\mu|_{\mathbb{D}}$  here.

## 2. AN INDEX THEOREM FOR THE SHIFT

Our first result of this section is well-known. Among the references that could be cited in its support is [OY1] (Lemma 2.6).

**Lemma 2.1.** *Let  $\mu$  be a finite, positive Borel measure with compact support in  $\mathbb{C}$  and let  $K$  be a compact subset of  $\text{abpe}(P^t(\mu))$ . With  $\nu := \mu - \mu|_K$ , there is a positive constant  $M$  such that*

$$\|p\|_{L^t(\mu)} \leq M \|p\|_{L^t(\nu)}$$

for all polynomials  $p$ .

For the sake of completeness, we now recall what it means for a measure to be strongly inscribed (cf. [A1], Definition 2.3).

**Definition 2.2.** Let  $\mu$  be a finite, positive Borel measure with support in  $\overline{\mathbb{D}}$  such that  $P^2(\mu)$  is irreducible and  $\text{abpe}(P^2(\mu)) = \mathbb{D}$ . We say that  $\mu$  is *strongly inscribed* if there is a Jordan curve  $\Gamma$  in  $\overline{\mathbb{D}}$  ( $\Omega := \text{inside}(\Gamma)$ ) and  $\omega_{\Omega}$  denotes harmonic measure on  $\Gamma$  for evaluation at some  $z_o$  in  $\Omega$ ) having the properties:

- 1)  $\omega_{\Omega}(\partial\mathbb{D}) > 0$ , and
- 2) there exists  $\psi$ ,  $0 \leq \psi \in L^{\infty}(\omega_{\Omega})$ , such that  $\log(\psi) \in L^1(\omega_{\Omega})$  and  $\int_{\Gamma} |p|^2 \psi d\omega_{\Omega} \leq \int |p|^2 d\mu$  for all polynomials  $p$ .

The next result is a straightforward consequence of the above definition; we state it without proof.

**Lemma 2.3.** *Let  $\mu$  and  $\nu$  be finite, positive Borel measures with support in  $\overline{\mathbb{D}}$  such that  $P^2(\mu)$  and  $P^2(\nu)$  are irreducible, and  $\text{abpe}(P^2(\mu)) = \text{abpe}(P^2(\nu)) = \mathbb{D}$ . Suppose  $0 \neq f \in H^{\infty}(\mathbb{D})$  and define  $\eta$  by  $d\eta = |f|d\mu$ . Then  $P^2(\eta)$  is irreducible and  $\text{abpe}(P^2(\eta)) = \mathbb{D}$ . Furthermore, if  $\mu$  is strongly inscribed, then so is  $\eta$ , and so also is  $\nu$ , if  $\mu \leq \nu$ .*

**Theorem 2.4.** *Let  $\mu$  be a finite, positive Borel measure with support in  $\overline{\mathbb{D}}$  such that  $P^2(\mu)$  is irreducible and  $abpe(P^2(\mu)) = \mathbb{D}$ . If  $\mu(\partial\mathbb{D}) > 0$  and there is a nontrivial subarc  $\gamma$  of  $\partial\mathbb{D}$  such that*

$$\int_{\gamma} \log\left(\frac{d\mu}{dm}\right) dm > -\infty,$$

*then  $\dim(\mathcal{M} \ominus z\mathcal{M}) = 1$  for each nontrivial closed invariant subspace  $\mathcal{M}$  for the shift  $M_z$  on  $P^2(\mu)$ .*

*Proof.* Our objective is to show that  $\mu$  is strongly inscribed; by [A1] (Corollary 2.5) this will establish the theorem. We begin by observing that we have some freedom to modify  $\mu|_{\gamma}$ . Indeed, define  $\eta$  by

$$d\eta = d\mu - d\mu|_{\gamma} + wdm|_{\gamma},$$

where  $0 \leq w \in L^1(m|_{\gamma})$  and  $\int_{\gamma} \log(w) dm > -\infty$ . Then  $\eta$  is mutually absolutely continuous with respect to  $\mu$  and there is a nonzero bounded outer function  $f_1$  such that  $|f_1|d\mu \leq d\eta$ . So, by Lemma 2.3,  $P^2(\eta)$  is irreducible and  $abpe(P^2(\eta)) = \mathbb{D}$ . Moreover, since  $\int_{\gamma} \log\left(\frac{d\mu}{dm}\right) dm > -\infty$ , there is another nonzero bounded outer function  $f_2$  such that  $|f_2|d\eta \leq d\mu$ . Applying Lemma 2.3 once again, we now see that if  $\eta$  is strongly inscribed, then so is  $\mu$ .

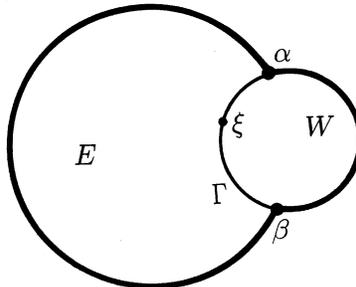
Our next step is to show that  $abpe(P^2(\mu_o)) = \mathbb{D}$  for  $\mu_o$  of the form

$$\mu_o := \mu - \mu|_S,$$

where  $S = \mathbb{D} \cap \{z : |z - e^{i\theta}| < r\}$ ,  $e^{i\theta}$  is in the relative interior of  $\gamma$  and  $r > 0$  is chosen so that  $(\partial\mathbb{D}) \cap \{z : |z - e^{i\theta}| \leq r\}$  is contained in the relative interior of  $\gamma$ . Toward this objective, first observe that we may assume  $\gamma$  is a proper (nontrivial) subarc of  $\partial\mathbb{D}$ . Let  $\varphi$  be a conformal mapping from  $\mathbb{D}$  one-to-one and onto  $E := \mathbb{D} \setminus \{z : |z - 1| \leq \frac{1}{2}\}$  such that  $\Gamma := \varphi(\gamma) = \overline{\mathbb{D}} \cap \{z : |z - 1| = \frac{1}{2}\}$ , and let  $\nu = \mu \circ \varphi^{-1}$ . Let  $W = \{z : |z - 1| < \frac{1}{2}\}$  and let  $\omega_W$  denote harmonic measure on  $\partial W$  for evaluation at 1. Since we have the freedom to modify  $\mu|_{\gamma}$  (within the parameters discussed earlier), we may assume that  $\nu|_{\Gamma} = \omega_W|_{\Gamma}$ . In what follows, let  $\sigma = \nu + \omega_W$ .

Claim 1.  $abpe(P^2(\sigma)) = \mathbb{D} \cup W$ .

Now, by our hypothesis and a standard conformal mapping argument,  $E = abpe(P^2(\nu)) (\subseteq abpe(P^2(\sigma)))$ . Furthermore, since  $|p|^2$  is subharmonic for any polynomial  $p$ , Harnack's Inequality gives  $W = abpe(P^2(\omega_W)) (\subseteq abpe(P^2(\sigma)))$ . What remains to be shown in establishing Claim 1 is that  $\Gamma \setminus \{\alpha, \beta\} \subseteq abpe(P^2(\sigma))$ , where  $\alpha$  and  $\beta$  are the endpoints of  $\Gamma$ ;  $Im(\alpha) > 0$  and  $Im(\beta) < 0$  – see the figure. Since  $abpe(P^2(\sigma))$  is an open subset of  $\mathbb{C}$  and its components are simply connected, if



there exists  $\xi$  in  $\Gamma \setminus \{\alpha, \beta\}$  such that  $\xi \notin abpe(P^2(\sigma))$ , then one of the two components of  $\Gamma \setminus \{\xi\}$  has empty intersection with  $abpe(P^2(\sigma))$ ; without loss, we may assume that the subarc of  $\Gamma$  that has endpoints  $\alpha$  and  $\xi$  – call this subarc  $\Gamma_\alpha$  – has empty intersection with  $abpe(P^2(\sigma))$ . Let  $\mathcal{C}$  be the chord of  $W$  that has endpoints  $\xi$  and  $1 - \frac{i}{2}$  and let  $V$  be the component of  $W \setminus \mathcal{C}$  that contains 1. Let  $\omega_V$  denote harmonic measure on  $\partial V$  for evaluation at 1 and let  $\tau = \nu + \omega_V$ . Since  $|p|^2$  is subharmonic (for any polynomial  $p$ ) and  $V \subseteq W$ ,

$$\|p\|_{L^2(\tau)} \leq \|p\|_{L^2(\sigma)}$$

for all polynomials  $p$ . Therefore,  $\Gamma_\alpha \cap abpe(P^2(\tau)) = \emptyset$  and so it follows that  $abpe(P^2(\tau)) = E \cup V$ . However,  $P^2(\nu)$  and  $P^2(\omega_V)$  are irreducible, and the measures  $\nu$  and  $\omega_V$  are nonzero and mutually absolutely continuous on their shared support (i.e.,  $\Gamma_\alpha$ ). By [T], Theorem 5.8, this outcome is not possible, and so we have a contradiction. Therefore,  $\Gamma \setminus \{\alpha, \beta\} \subseteq abpe(P^2(\sigma))$ , and so Claim 1 holds. As a footnote, we mention that there are other ways of establishing this claim, at least one of which uses results found in [OY2]. For convenience, we let  $U = \mathbb{D} \cup W (= abpe(P^2(\sigma)))$ . Let  $\Omega$  be a Jordan region such that  $E \cap \Omega \neq \emptyset$  and  $\overline{\Omega} \subseteq U$ . Let  $\nu^* = \nu - \nu|_{(E \cap \Omega)}$  and let  $\sigma^* = \nu^* + \omega_W$ . Since we have reduced our proof to the case that  $\nu|_\Gamma = \omega_W|_\Gamma$ , we have  $\sigma^* = 2\omega_W$  on  $\Gamma$ . By Claim 1 and Lemma 2.1,  $abpe(P^2(\sigma^*)) = U$ . Let  $\mathcal{P}$  denote the collection of polynomials and let  $\mathcal{Q} = \{p(\frac{1}{z-1}) : p \text{ is a polynomial and } p(0) = 0\}$ . Let  $G$  denote the complement of  $\overline{W}$  in the Riemann sphere and let  $\Sigma$  denote the sweep of  $\nu$  in  $\overline{G}$  to  $\partial G (= \partial W)$ . Notice that  $\Sigma \ll \omega_W$ , and so we can find a nonzero function  $h$  in  $H^\infty(G)$ , whose (conformal) pull-back to  $\mathbb{D}$  is an outer function, such that

$$\int |q|^2 |h| d\nu \leq \int |q|^2 d\omega_W$$

for all  $q$  in  $\mathcal{Q}$ . Since  $h|_{E \circ \varphi}$  is itself a nonzero bounded outer function, we can argue as we did at the beginning of this proof (via  $\varphi^{-1}$ , replacing  $d\nu$  by  $|h|d\nu$  if need be) and make one last reduction to the special case: there is a positive constant  $c$  such that

$$\int |q|^2 d\sigma \leq c \cdot \int |q|^2 d\omega_W$$

for all  $q$  in  $\mathcal{Q}$ . By our preliminary observation, we may still assume that  $\nu|_\Gamma = \omega_W|_\Gamma$ . Choose  $\lambda$  in  $E \cap \Omega$  and let  $K = \overline{E} \cup (\partial W)$ .

Claim 2.  $z \rightarrow \frac{1}{z-\lambda} \notin R^2(K, \sigma^*)$  – the closure in  $L^2(\sigma^*)$  of the rational functions with poles off  $K$ .

To see this, let  $\{p_n\}_{n=1}^\infty$  and  $\{q_n\}_{n=1}^\infty$  be sequences in  $\mathcal{P}$  and  $\mathcal{Q}$  respectively such that  $\|p_n + q_n\|_{L^2(\sigma^*)} \rightarrow 0$ , as  $n \rightarrow \infty$ . Then  $\|p_n + q_n\|_{L^2(\omega_W)} \rightarrow 0$  (as  $n \rightarrow \infty$ ) and so it follows from a theorem of M. Riesz (see [H], page 151) that  $\|q_n\|_{L^2(\omega_W)} \rightarrow 0$ . By this and our reduction, we have

- (a)  $\{q_n\}_{n=1}^\infty$  converges to 0 uniformly on compact subsets of  $G$ , and
- (b)  $\|q_n\|_{L^2(\sigma^*)} \rightarrow 0$ , as  $n \rightarrow \infty$ .

Since  $\|p_n + q_n\|_{L^2(\sigma^*)} \rightarrow 0$ , (b) implies that  $\|p_n\|_{L^2(\sigma^*)} \rightarrow 0$ , as  $n \rightarrow \infty$ . So, by (a) and since  $\lambda \in abpe(P^2(\sigma^*))$ , we can now find  $r > 0$  such that  $\{p_n + q_n\}_{n=1}^\infty$  converges to 0 uniformly on  $\{z : |z - \lambda| \leq r\}$ . From Runge’s Theorem it now follows that  $\lambda$  is an analytic bounded point evaluation for  $R^2(K, \sigma^*)$ , and therefore Claim 2 holds. Now by Claim 2, there exists  $g$  in  $L^2(\sigma^*)$  such that  $\int g f d\sigma^* = 0$  for

all  $f$  in  $R^2(K, \sigma^*)$  (i.e.,  $\bar{g} \perp R^2(K, \sigma^*)$ ) and yet  $\int \frac{g(z)}{z-\lambda} d\sigma^*(z) \neq 0$ . Recall that the Cauchy transform

$$\hat{g}(\zeta) := \int \frac{g(z)}{z-\zeta} d\sigma^*(z)$$

is defined and analytic off the support of  $\sigma^*$  and, since  $\bar{g} \perp R^2(K, \sigma^*)$ ,  $\hat{g} \equiv 0$  on both  $W$  and  $\mathbb{C} \setminus \bar{U}$ . Applying a well-known technique (see the proof of Lemma 6 in [OT] or the proof of Lemma 7 in [A2]), we find  $g = 0$  on  $(\partial W) \setminus \Gamma$ . Evidently, therefore,  $z \rightarrow \frac{1}{z-\lambda} \notin P^2(\nu^*)$ . From this it follows that  $z \rightarrow \frac{1}{z-\kappa} \notin P^2(\mu^*)$ , where  $\mu^* := \nu^* \circ \varphi$  ( $= \mu - \mu|_{\varphi^{-1}(E \cap \Omega)}$ ) and  $\kappa$  is any point in  $\varphi^{-1}(E \cap \Omega)$ . Since this holds for all  $\Omega$  as described above, if we select a particular  $\mu_o$  as defined in the early stages of this proof, then we necessarily have  $abpe(P^2(\mu_o)) = \mathbb{D}$ . Now by [T], Theorem 5.8, there is a Borel partition  $\{\Delta_0, \Delta_1\}$  of the support of  $\mu_o$  such that

$$P^2(\mu_o) = L^2(\mu_o|_{\Delta_0}) \oplus P^2(\mu_o|_{\Delta_1}),$$

where  $P^2(\mu_o|_{\Delta_1})$  is irreducible and  $abpe(P^2(\mu_o|_{\Delta_1})) = abpe(P^2(\mu_o))$  ( $= \mathbb{D}$ ). We can proceed with  $\mu_o|_{\Delta_1}$ , or bypass this direct sum decomposition and argue as above to show that  $\chi_B \notin P^2(\mu_o)$  for any Borel subset  $B$  of  $\partial\mathbb{D}$  such that  $\mu(B) > 0$ . Consequently,  $P^2(\mu_o)$  is irreducible, and so  $L^2(\mu_o|_{\Delta_0})$  is trivial. Notice that the support of  $\mu_o$  has an outer hole and the boundary of this outer hole contains a nontrivial subarc of  $\partial\mathbb{D}$ . Therefore, by Remark 3.5 of [A1],  $\mu_o$  is strongly inscribed. Since  $\mu_o \leq \mu$ , Lemma 2.3 now tells us that  $\mu$  is strongly inscribed, and the proof is complete.  $\square$

#### ACKNOWLEDGEMENT

The author is indebted to the referee for a careful reading of this paper and thoughtful suggestions.

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