A NOTE CONCERNING THE INDEX OF THE SHIFT

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Abstract. Let \( \mu \) be a finite, positive Borel measure with support in \( \{ z : |z| \leq 1 \} \) such that \( P^2(\mu) \) – the closure of the polynomials in \( L^2(\mu) \) – is irreducible and each point in \( D := \{ z : |z| < 1 \} \) is a bounded point evaluation for \( P^2(\mu) \). We show that if \( \mu(\partial D) > 0 \) and there is a nontrivial subarc \( \gamma \) of \( \partial D \) such that \( \int_{\gamma} \log(\frac{d\mu}{dm}) dm > -\infty \), then \( \dim(M \ominus zM) = 1 \) for each nontrivial closed invariant subspace \( M \) for the shift \( Mz \) on \( P^2(\mu) \).

1. Introduction

Given a finite, positive Borel measure \( \mu \) with compact support in the complex plane \( \mathbb{C} \) and \( 1 \leq t < \infty \), we let \( P^t(\mu) \) denote the closure of the polynomials in \( L^t(\mu) \). J. Thomson has established (see [T]) a direct sum decomposition of \( P^t(\mu) \) that involves the components of \( \text{abpe}(P^t(\mu)) \) – the set of analytic bounded point evaluations for \( P^t(\mu) \). In this brief paper we restrict our attention to the case \( t = 2 \) and assume that the support of \( \mu \) is contained in \( D := \{ z : |z| < 1 \} \), that \( \text{abpe}(P^2(\mu)) = D \) and that \( P^2(\mu) \) is irreducible (which means that \( P^2(\mu) \) contains no nontrivial characteristic functions). A consequence of these assumptions is that \( \mu(\partial D) \ll m \), where \( m \) denotes normalized Lebesgue measure on \( \partial D \). Another consequence is that multiplication by the independent variable \( z \) is a bounded operator on \( P^2(\mu) \), with closed range; we call this operator the shift and denote it by \( Mz \). If \( \mu(\partial D) = 0 \), then work of C. Apostol, H. Bercovici, C. Foias and C. Pearcy in [ABFP] shows that for any natural number \( n \), and for \( n = \infty \), there is a closed invariant subspace \( M \) for the shift on \( P^2(\mu) \) such that \( \dim(M \ominus zM) = n \). Information concerning the lattice of invariant subspaces for \( Mz \) on such \( P^2(\mu) \) spaces – in particular, the classical Bergman space \( L^2_{\alpha}(D) \) – contributes to a better understanding of bounded operators on separable Hilbert spaces in general (see [ABFP] or [HRS]). In contrast, if \( \mu(\partial D) > 0 \), then the lattice of invariant subspaces for \( Mz \) on \( P^2(\mu) \) appears to be somewhat limited and indeed the results to date have led the authors of [CM] to conjecture that (in this setting) the outcome follows that of the classical Hardy space \( H^2(D) \), and \( \dim(M \ominus zM) = 1 \) for each nontrivial closed invariant subspace \( M \) for the shift on \( P^2(\mu) \). Seminal work in support of

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this conjecture was done by R. Olin and J. Thomson in [OT] who established it in the special case that the support of \(\mu\) has an “outer hole”. The work of L. Miller, J. Thomson and L. Yang, who make use of results in [OT], all but dispatches with the conjecture in the case \(\mu|_D\) is area measure; see [M], [Y] and [TY]. Recently, the author of this paper has given an analytic condition that defines what it means for a measure to be so-called “strongly inscribed” and has shown that if a measure \(\mu\) is such, then there is a measure \(\mu_o\) whose support has an outer hole and for which the shifts on \(P^2(\mu)\) and \(P^2(\mu_o)\) are similar as operators (see [A1]). From this it follows that if \(\mu\) is strongly inscribed, then \(\dim(M \ominus z\mathcal{M}) = 1\) for each nontrivial closed invariant subspace \(M\) for the shift on \(P^2(\mu)\). Furthermore, the existence of a measure \(\mu_o\) (as described above) is basically determined by whether or not \(\mu\) is strongly inscribed. In this paper we use an idea or two from [A1] to show that the above conjecture holds whenever there is a nontrivial subarc \(\gamma\) of \(\partial D\) such that
\[
\int_\gamma \log \left(\frac{d\mu}{dm}\right)dm > -\infty;
\]
no special assumption is made concerning \(\mu|_D\) here.

2. AN INDEX THEOREM FOR THE SHIFT

Our first result of this section is well-known. Among the references that could be cited in its support is [OY1] (Lemma 2.6).

**Lemma 2.1.** Let \(\mu\) be a finite, positive Borel measure with compact support in \(C\) and let \(K\) be a compact subset of \(abpe(P^t(\mu))\). With \(\nu := \mu - \mu|_K\), there is a positive constant \(M\) such that
\[
||p||_{L^t(\mu)} \leq M||p||_{L^t(\nu)}
\]
for all polynomials \(p\).

For the sake of completeness, we now recall what it means for a measure to be strongly inscribed (cf. [A1], Definition 2.3).

**Definition 2.2.** Let \(\mu\) be a finite, positive Borel measure with support in \(\overline{D}\) such that \(P^2(\mu)\) is irreducible and \(abpe(P^2(\mu)) = D\). We say that \(\mu\) is strongly inscribed if there is a Jordan curve \(\Gamma\) in \(\overline{D}\) (\(\Omega := \text{inside}(\Gamma)\) and \(\omega_\Omega\) denotes harmonic measure on \(\Gamma\) for evaluation at some \(z_0\) in \(\Omega\)) having the properties:

1) \(\omega_\Omega(\partial D) > 0\), and
2) there exists \(\psi\), \(0 \leq \psi \in L^\infty(\omega_\Omega)\), such that \(log(\psi) \in L^1(\omega_\Omega)\) and 
\[
\int_\Gamma |p|^2\psi d\omega_\Omega \leq \int |p|^2 d\mu
\]
for all polynomials \(p\).

The next result is a straightforward consequence of the above definition; we state it without proof.

**Lemma 2.3.** Let \(\mu\) and \(\nu\) be finite, positive Borel measures with support in \(\overline{D}\) such that \(P^2(\mu)\) and \(P^2(\nu)\) are irreducible, and \(abpe(P^2(\mu)) = abpe(P^2(\nu)) = D\). Suppose \(0 \neq f \in H^\infty(D)\) and define \(\eta\) by \(d\eta = |f|d\mu\). Then \(P^2(\eta)\) is irreducible and \(abpe(P^2(\eta)) = D\). Furthermore, if \(\mu\) is strongly inscribed, then so is \(\eta\), and so also is \(\nu\), if \(\mu \leq \nu\).
Moreover, since the parameters discussed earlier, we may assume that $\nu = \beta$ and $\alpha$ remains to be shown in establishing Claim 1 is that $\Gamma$ is irreducible and $abpe(P^2(\nu)) = \mathbb{D}$. If $\mu(\partial \mathbb{D}) > 0$ and there is a nontrivial subarc $\gamma$ of $\partial \mathbb{D}$ such that

$$\int_{\gamma} \log \left( \frac{d\mu}{dm} \right) dm > -\infty,$$

then $\dim(\mathcal{M} \oplus z\mathcal{M}) = 1$ for each nontrivial closed invariant subspace $\mathcal{M}$ for the shift $M_z$ on $P^2(\mu)$.

Proof. Our objective is to show that $\mu$ is strongly inscribed; by [A1] (Corollary 2.5) this will establish the theorem. We begin by observing that we have some freedom to modify $\mu|_{\gamma}$. Indeed, define $\eta$ by

$$dn = d\mu - d\mu|_{\gamma} + wdm|_{\gamma},$$

where $0 \leq w \in L^1(m|_{\gamma})$ and $\int_{\gamma} \log(w)dm > -\infty$. Then $\eta$ is mutually absolutely continuous with respect to $\mu$ and there is a nonzero bounded outer function $f_1$ such that $|f_1|d\mu \leq d\eta$. So, by Lemma 2.3, $P^2(\eta)$ is irreducible and $abpe(P^2(\eta)) = \mathbb{D}$. Moreover, since $\int_{\gamma} \log \left( \frac{d\mu}{dm} \right) dm > -\infty$, there is another nonzero bounded outer function $f_2$ such that $|f_2|d\eta \leq d\mu$. Applying Lemma 2.3 once again, we now see that if $\eta$ is strongly inscribed, then so is $\mu$.

Our next step is to show that $abpe(P^2(\mu_o)) = \mathbb{D}$ for $\mu_o$ of the form

$$\mu_o := \mu - \mu|_S,$$

where $S = \mathbb{D} \cap \{ z : |z - e^{i\theta}| < r \}$, $e^{i\theta}$ is in the relative interior of $\gamma$ and $r > 0$ is chosen so that $(\partial \mathbb{D}) \cap \{ z : |z - e^{i\theta}| \leq r \}$ is contained in the relative interior of $\gamma$. Toward this objective, first observe that we may assume $\gamma$ is a proper (nontrivial) subarc of $\partial \mathbb{D}$. Let $\varphi$ be a conformal mapping from $\mathbb{D}$ one-to-one and onto $E := \mathbb{D} \setminus \{ z : |z - 1| \leq \frac{1}{3} \}$ such that $\Gamma := \varphi(\gamma) = \mathbb{D} \setminus \{ z : |z - 1| = \frac{1}{3} \}$, and let $\nu = \mu \circ \varphi^{-1}$. Let $W = \{ z : |z - 1| < \frac{1}{3} \}$ and let $\omega_W$ denote harmonic measure on $\partial W$ for evaluation at 1. Since we have the freedom to modify $\mu|_{\gamma}$ (within the parameters discussed earlier), we may assume that $\nu|_{\Gamma} = \omega_W|_{\Gamma}$. In what follows, let $\sigma = \nu + \omega_W$.

Claim 1. $abpe(P^2(\sigma)) = \mathbb{D} \cup W$.

Now, by our hypothesis and a standard conformal mapping argument, $E = abpe(P^2(\nu)) (\subseteq abpe(P^2(\sigma)))$. Furthermore, since $|p|^2$ is subharmonic for any polynomial $p$, Harnack’s Inequality gives $W = abpe(P^2(\omega_W)) (\subseteq abpe(P^2(\sigma)))$. What remains to be shown in establishing Claim 1 is that $\Gamma \setminus \{ \alpha, \beta \} \subseteq abpe(P^2(\sigma))$, where $\alpha$ and $\beta$ are the endpoints of $\Gamma$; $Im(\alpha) > 0$ and $Im(\beta) < 0$ — see the figure. Since $abpe(P^2(\sigma))$ is an open subset of $\mathbb{C}$ and its components are simply connected, if
there exists $\xi$ in $\Gamma \setminus \{\alpha, \beta\}$ such that $\xi \not\in \text{abpe}(P^2(\sigma))$, then one of the two components of $\Gamma \setminus \{\xi\}$ has empty intersection with $\text{abpe}(P^2(\sigma))$; without loss, we may assume that the subarc of $\Gamma$ that has endpoints $\alpha$ and $\xi$—call this subarc $\Gamma_\alpha$—has empty intersection with $\text{abpe}(P^2(\sigma))$. Let $C$ be the chord of $W$ that has endpoints $\xi$ and $1 - \frac{1}{z}$ and let $V$ be the component of $W \setminus C$ that contains 1. Let $\omega_V$ denote harmonic measure on $\partial V$ for evaluation at 1 and let $\tau = \nu + \omega_V$. Since $|p|^2$ is subharmonic (for any polynomial $p$) and $V \subseteq W$,

$$||p||_{L^2(\tau)} \leq ||p||_{L^2(\sigma)}$$

for all polynomials $p$. Therefore, $\Gamma_\alpha \cap \text{abpe}(P^2(\sigma)) = \emptyset$ and so it follows that $\text{abpe}(P^2(\tau)) = E \cup V$. However, $P^2(\nu)$ and $P^2(\omega_V)$ are irreducible, and the measures $\nu$ and $\omega_V$ are nonzero and mutually absolutely continuous on their shared support (i.e., $\Gamma_\alpha$). By [1], Theorem 5.8, this outcome is not possible, and so we have a contradiction. Therefore, $\Gamma \setminus \{\alpha, \beta\} \subseteq \text{abpe}(P^2(\sigma))$, and so Claim 1 holds. As a footnote, we mention that there are other ways of establishing this claim, at least one of which uses results found in [OY2]. For convenience, we let $U = \mathbb{D} \cup \overline{W}$ (= $\text{abpe}(P^2(\sigma))$). Let $\Omega$ be a Jordan region such that $E \cap \Omega \neq \emptyset$ and $\overline{\Omega} \subseteq U$. Let $\nu^* = \nu - \nu|_{E \cap \Omega}$ and let $\sigma^* = \nu^* + \omega_W$. Since we have reduced our proof to the case that $\nu|_\Gamma = \omega_W|_\Gamma$, we have $\sigma^* = 2\omega_W$ on $\Gamma$. By Claim 1 and Lemma 2.1, $\text{abpe}(P^2(\sigma^*)) = U$. Let $\mathcal{P}$ denote the collection of polynomials and let $\mathcal{Q} = \{p(\frac{1}{z}) : p \text{ is a polynomial and } p(0) = 0\}$. Let $G$ denote the complement of $\overline{W}$ in the Riemann sphere and let $\Sigma$ denote the sweep of $\nu$ in $\overline{G}$ to $\partial G$ (= $\partial W$). Notice that $\Sigma \ll \omega_W$, and so we can find a nonzero function $h$ in $H^\infty(G)$, whose (conformal) pull-back to $\mathbb{D}$ is an outer function, such that

$$\int |q|^2 |h| \nu \leq \int |q|^2 d\omega_W$$

for all $q$ in $\mathcal{Q}$. Since $h|_E \circ \varphi$ is itself a nonzero bounded outer function, we can argue as we did at the beginning of this proof (via $\varphi^{-1}$, replacing $d\nu$ by $|h|d\nu$ if need be) and make one last reduction to the special case: there is a positive constant $c$ such that

$$\int |q|^2 d\sigma \leq c \cdot \int |q|^2 d\omega_W$$

for all $q$ in $\mathcal{Q}$. By our preliminary observation, we may still assume that $\nu|_\Gamma = \omega_W|_\Gamma$. Choose $\lambda$ in $E \cap \Omega$ and let $K = \overline{E} \cup (\partial W)$.

Claim 2. $z \rightarrow \frac{1}{z-\lambda} \not\in R^2(K, \sigma^*)$—the closure in $L^2(\sigma^*)$ of the rational functions with poles off $K$.

To see this, let $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ be sequences in $\mathcal{P}$ and $\mathcal{Q}$ respectively such that $||p_n + q_n||_{L_2(\sigma^*)} \rightarrow 0$, as $n \rightarrow \infty$. Then $||p_n + q_n||_{L_2(\omega_W)} \rightarrow 0$ (as $n \rightarrow \infty$) and so it follows from a theorem of M. Riesz (see [1], page 151) that $||q_n||_{L_2(\omega_W)} \rightarrow 0$. By this and our reduction, we have

(a) $\{q_n\}_{n=1}^\infty$ converges to 0 uniformly on compact subsets of $G$, and

(b) $||q_n||_{L_2(\sigma^*)} \rightarrow 0$, as $n \rightarrow \infty$.

Since $||p_n + q_n||_{L_2(\sigma^*)} \rightarrow 0$, (b) implies that $||p_n||_{L_2(\sigma^*)} \rightarrow 0$, as $n \rightarrow \infty$. So, by (a) and since $\lambda \in \text{abpe}(P^2(\sigma^*))$, we can now find $r > 0$ such that $\{p_n + q_n\}_{n=1}^\infty$ converges to 0 uniformly on $\{z : |z - \lambda| \leq r\}$. From Runge’s Theorem it now follows that $\lambda$ is an analytic bounded point evaluation for $R^2(K, \sigma^*)$, and therefore Claim 2 holds. Now by Claim 2, there exists $g$ in $L^2(\sigma^*)$ such that $\int gf d\sigma^* = 0$ for
all $f$ in $R^2(K, \sigma^*)$ (i.e., $\overline{g} \perp R^2(K, \sigma^*)$) and yet $\int \frac{g(z)}{z-\zeta}d\sigma^*(z) \neq 0$. Recall that the Cauchy transform

$$\hat{g}(\zeta) := \int \frac{g(z)}{z-\zeta}d\sigma^*(z)$$

is defined and analytic off the support of $\sigma^*$ and, since $\overline{g} \perp R^2(K, \sigma^*)$, $\hat{g} \equiv 0$ on both $W$ and $C \setminus \Gamma$. Applying a well-known technique (see the proof of Lemma 6 in [OT] or the proof of Lemma 7 in [A2]), we find $g = 0$ on $(\partial W) \setminus \Gamma$. Evidently, therefore, $z \rightarrow \frac{1}{z-\zeta} \notin P^2(\nu^*)$. From this it follows that $z \rightarrow \frac{1}{z-\zeta} \notin P^2(\mu^*)$, where $\mu^*:=\nu^* \circ \varphi (= \mu - \mu|_{\varphi^{-1}(E \cap \Omega)})$ and $\kappa$ is any point in $\varphi^{-1}(E \cap \Omega)$. Since this holds for all $\Omega$ as described above, if we select a particular $\mu_0$ as defined in the early stages of this proof, then we necessarily have $abpe(P^2(\mu_0)) = \mathbb{D}$. Now by [1], Theorem 5.8, there is a Borel partition $\{\Delta_0, \Delta_1\}$ of the support of $\mu_0$ such that

$$P^2(\mu_0) = L^2(\mu_0|_{\Delta_0}) \oplus P^2(\mu_0|_{\Delta_1}),$$

where $P^2(\mu_0|_{\Delta_1})$ is irreducible and $abpe(P^2(\mu_0|_{\Delta_1})) = abpe(P^2(\mu_0)) (= \mathbb{D})$. We can proceed with $\mu_0|_{\Delta_1}$, or bypass this direct sum decomposition and argue as above to show that $\chi_B \notin P^2(\mu_0)$ for any Borel subset $B$ of $\partial \mathbb{D}$ such that $\mu(B) > 0$. Consequently, $P^2(\mu_0)$ is irreducible, and so $L^2(\mu_0|_{\Delta_0})$ is trivial. Notice that the support of $\mu_0$ has an outer hole and the boundary of this outer hole contains a nontrivial subarc of $\partial \mathbb{D}$. Therefore, by Remark 3.5 of [A1], $\mu_0$ is strongly inscribed. Since $\mu_0 \leq \mu$, Lemma 2.3 now tells us that $\mu$ is strongly inscribed, and the proof is complete.

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**References**


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