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A NEW CONSTRUCTION OF THE KAC JORDAN SUPERALGEBRA

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To Irving Kaplansky

ABSTRACT. We present an elementary construction of the 10-dimensional simple Jordan superalgebra K_{10} of Kac using the 3-dimensional tiny Kaplansky superalgebra. This new realization of K_{10} enables us to derive many of its properties.

INTRODUCTION

The 10-dimensional Jordan superalgebra K_{10} of Kac [K] is the only exceptional finite-dimensional simple Jordan superalgebra over an algebraically closed field of characteristic 0. We give a new, elementary construction of it (over an arbitrary field of characteristic not 2) using the tiny Kaplansky superalgebra K. In our construction, K_{10} appears as a direct sum $F1 \oplus (K \otimes K)$ with unit element 1 and with a suitable multiplication (see (2.1)). In characteristic 3, K_{10} is not simple but possesses a simple ideal K_9 of dimension 9, which is just the tensor product $K \otimes K$ (of superalgebras).

Our realization of K_{10} makes it easy to deduce certain of its known properties: for example, that its Lie superalgebra of superderivations is isomorphic to a direct sum of two copies of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1,2)$, or that its Lie multiplication superalgebra is $\mathfrak{osp}(2,4)$ if -1 is a square in the field F.

The Kac superalgebra was originally constructed in [K] by Lie theoretical methods from a 3-grading of the simple Lie superalgebra $\mathrm{F}(4)$. No direct proof of the fact that it is a Jordan superalgebra is known. Indeed, in their work on Jordan superalgebras with semisimple even part, Racine and Zelmanov [RZ] remark "That K_{10}, and hence K_9, is a Jordan superalgebra can be obtained by Lie methods as in [K] but a direct proof would be desirable." Here we provide such a proof.

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1. The Kaplansky superalgebra

Suppose F is a field of characteristic not 2. The tiny ***Kaplansky superalgebra*** ([Kp, M]) is the 3-dimensional Jordan superalgebra $K = K_{\bar{0}} \oplus K_{\bar{1}}$, with $K_{\bar{0}} = Fe$ and $K_{\bar{1}} = Fx \oplus Fy$, and with multiplication given by

$$\begin{gathered} e^2 = e, \qquad ex = \frac{1}{2}x = xe, \qquad ey = \frac{1}{2}y = ye, \\ xy = e = -yx, \qquad x^2 = 0 = y^2. \end{gathered} \tag{1.1}$$

It can be seen using (1.1) that the even supersymmetric bilinear form $(\mid)$ defined on K by

$$(e \mid e) = \frac{1}{2}, \qquad (x \mid y) = 1, \qquad \text{and} \qquad (K_{\bar{0}} \mid K_{\bar{1}}) = 0$$

is associative ($(ab \mid c) = (a \mid bc)$ for all $a, b, c \in K$). Thus,

$$(ab \mid c) = (-1)^{\bar{a}(\bar{b}+\bar{c})}(bc \mid a) = (-1)^{\bar{c}(\bar{a}+\bar{b})}(ca \mid b) \tag{1.2}$$

for all $a, b, c \in K_{\bar{0}} \cup K_{\bar{1}}$. (Our convention is that $\bar{a} = i$ whenever $a \in K_i$.) Hence, since the supertrace, $\operatorname{str}(Ł_e) = \operatorname{tr}_{K_{\bar{0}}}(Ł_e) - \operatorname{tr}_{K_{\bar{1}}}(Ł_e)$, of the left multiplication operator $Ł_e$ of e on K is 0, it follows that

$$\begin{aligned} Ł_{K_i} = \{\phi \in \operatorname{End}_F(K)_i \mid (\phi(u) \mid v) \\ = (-1)^{i\bar{u}}(u \mid \phi(v)) \ \ \forall\, u, v \in K, \text{ and } \operatorname{str}(\phi) = 0\} \end{aligned}$$

for $i = \bar{0}, \bar{1}$. Moreover, $[Ł_K, Ł_K] = \operatorname{Der} K = \mathfrak{osp}(K) \cong \mathfrak{osp}(1, 2)$.

Proposition 1.3. *For all $u, v, w \in K_{\bar{0}} \cup K_{\bar{1}}$,*

$$(uv)w = (u \mid v)w + \frac{1}{2}\Big(u(v \mid w) + (-1)^{\bar{u}\bar{v}}v(u \mid w)\Big). \tag{1.4}$$

Proof. Observe first that the expression in (1.4) is supersymmetric in u and v. If $u = v = e$, then (1.4) reduces to $ew = \frac{1}{2}w + (e \mid w)e$, which is valid for all homogeneous elements w. If $u = e$ and $v \in K_{\bar{1}}$, then (1.4) becomes

$$\frac{1}{2}vw = \frac{1}{2}\Big(e(v \mid w) + v(e \mid w)\Big),$$

which is true. Finally, if $u, v \in K_{\bar{1}}$, then $uv = (u \mid v)e$, and in this case, (1.4) says

$$(u \mid v)ew = (u \mid v)w + \frac{1}{2}\Big(u(v \mid w) + (-1)^{\bar{u}\bar{v}}v(u \mid w)\Big).$$

For $w = e$ this reduces to $(u \mid v)e = (u \mid v)e$. If instead $w \in K_{\bar{1}}$, then (1.4) holds because $(u \mid v)w + (v \mid w)u + (w \mid u)v = 0$, as the expression on the left is a skew-symmetric trilinear map on a 2-dimensional vector space (namely $K_{\bar{1}}$). $\square$

Corollary 1.5. *For all $u, v \in K_{\bar{0}} \cup K_{\bar{1}}$,*

$$[Ł_u, Ł_v] = \frac{1}{2}\Big(u(v \mid -\,) - (-1)^{\bar{u}\bar{v}}v(u \mid -\,)\Big) \quad \textit{and} \tag{1.6}$$

$$Ł_u \circ Ł_v = (u \mid v)\ \mathrm{I} + \frac{3}{2}\Big(u(v \mid -\,) + (-1)^{\bar{u}\bar{v}}v(u \mid -\,)\Big), \tag{1.7}$$

where $\phi \circ \psi = \phi\psi + (-1)^{\bar{\phi}\bar{\psi}}\psi\phi$ for all homogeneous $\phi, \psi \in \operatorname{End}_F(K)$.

Proof. From (1.4) we have

$$
\begin{aligned}
u(vw) - (-1)^{\bar{u}\bar{v}} v(uw) &= u(v \mid w) + \frac{1}{2}\Big((u \mid v)w + (-1)^{\bar{u}\bar{v}} v(u \mid w)\Big) \\
&\qquad - (-1)^{\bar{u}\bar{v}} \left(v(u \mid w) + \frac{1}{2}\Big((v \mid u)w + (-1)^{\bar{u}\bar{w}}(v \mid w)u\Big)\right) \\
&= \frac{1}{2}\Big(u(v \mid w) - (-1)^{\bar{u}\bar{v}} v(u \mid w)\Big),
\end{aligned}
$$

which demonstrates (1.6). The verification of (1.7) is similar. □

2. The Jordan superalgebra J

In this section we consider the superalgebra

$$J = F1 \oplus (K \otimes K)$$

with unit element 1 and with product defined by

$$(a \otimes b)(c \otimes d) = (-1)^{\bar{b}\bar{c}}\Big(ac \otimes bd - \frac{3}{4}(a \mid c)(b \mid d)1\Big) \tag{2.1}$$

for homogeneous elements $a, b, c, d \in K$. (All tensor products are over F unless specified otherwise.) By its definition J is a (super-)commutative superalgebra. Our aim is to show that J is a Jordan superalgebra. First we establish some properties of the multiplication.

Proposition 2.2. *For homogeneous elements $a, b, c, d \in K$,*

$$
\begin{aligned}
&[L_{a\otimes b}, L_{c\otimes d}](1) = 0, \\
&[L_{a\otimes b}, L_{c\otimes d}]|_{K\otimes K} = (-1)^{\bar{b}\bar{c}}\frac{1}{2}\Big([L_a, L_c] \otimes (b \mid d)\,\mathrm{I} + (a \mid c)\,\mathrm{I} \otimes [L_b, L_d]\Big).
\end{aligned}
\tag{2.3}
$$

Proof. First let us remark that

$$(\phi \otimes \psi)(u \otimes v) = (-1)^{\bar{\psi}\bar{u}}\phi(u) \otimes \psi(v)$$

for homogeneous $\phi, \psi \in \mathrm{End}_F(K)$, $u, v \in K$. The first equation in (2.3) is clear by supercommutativity. As for the second, note that

(2.4)

$$
\begin{aligned}
&[L_{a\otimes b}, L_{c\otimes d}](u \otimes v) \\
&\quad = (-1)^{\bar{d}\bar{u}}(a \otimes b)\Big(cu \otimes dv - \frac{3}{4}(c \mid u)(d \mid v)1\Big) \\
&\qquad - (-1)^{(\bar{a}+\bar{b})(\bar{c}+\bar{d})}(-1)^{\bar{b}\bar{u}}(c \otimes d)\Big(au \otimes bv - \frac{3}{4}(a \mid u)(b \mid v)1\Big) \\
&\quad = (-1)^{\bar{d}\bar{u}}(-1)^{\bar{b}(\bar{c}+\bar{u})}\left(a(cu) \otimes b(dv) - \frac{3}{4}(a \mid cu)(b \mid dv)1 - \frac{3}{4}a(c \mid u) \otimes b(d \mid v)\right. \\
&\qquad \left. - (-1)^{\bar{a}\bar{c}+\bar{b}\bar{d}}\Big(c(au) \otimes d(bv) - \frac{3}{4}(c \mid au)(d \mid bv)1 - \frac{3}{4}c(a \mid u) \otimes d(b \mid v)\Big)\right).
\end{aligned}
$$

By associativity $(a \mid cu) = (-1)^{\bar{a}\bar{c}}(c \mid au)$ and $(b \mid dv) = (-1)^{\bar{b}\bar{d}}(d \mid bv)$, so that the $F1$-component of (2.4) is 0. Moreover,

$$\begin{aligned} a(cu) \otimes b(dv) - &(-1)^{\bar{a}\bar{c}+\bar{b}\bar{d}} c(au) \otimes d(bv) \\ &= \frac{1}{2}\Big([\text{Ł}_a, \text{Ł}_c](u) \otimes \big(\text{Ł}_b \circ \text{Ł}_d\big)(v) + \big(\text{Ł}_a \circ \text{Ł}_c\big)(u) \otimes [\text{Ł}_b, \text{Ł}_d](v)\Big), \end{aligned}$$

and by (1.6),

$$\begin{aligned} a(c \mid u) \otimes b(d \mid v) - &(-1)^{\bar{a}\bar{c}+\bar{b}\bar{d}} c(a \mid u) \otimes d(b \mid v) \\ &= \frac{1}{2}\Big(a(c \mid u) - (-1)^{\bar{a}\bar{c}} c(a \mid u)\Big) \otimes \Big(b(d \mid v) + (-1)^{\bar{b}\bar{d}} d(b \mid v)\Big) \\ &\qquad + \Big(a(c \mid u) + (-1)^{\bar{a}\bar{c}} c(a \mid u)\Big) \otimes \Big(b(d \mid v) - (-1)^{\bar{b}\bar{d}} d(b \mid v)\Big) \\ &= [\text{Ł}_a, \text{Ł}_c](u) \otimes \Big(b(d \mid v) + (-1)^{\bar{b}\bar{d}} d(b \mid v)\Big) \\ &\qquad + \Big(a(c \mid u) + (-1)^{\bar{a}\bar{c}} c(a \mid v)\Big) \otimes [\text{Ł}_b, \text{Ł}_d](v). \end{aligned}$$

Therefore, it follows from (1.7) that

$$\begin{aligned} &[\text{Ł}_{a\otimes b}, \text{Ł}_{c\otimes d}]|_{K\otimes K} \\ &\quad = (-1)^{\bar{b}\bar{c}}\Bigg([\text{Ł}_a, \text{Ł}_c] \otimes \Big(\frac{1}{2}\text{Ł}_b \circ \text{Ł}_d - \frac{3}{4}\Big(b(d \mid -) + (-1)^{\bar{b}\bar{d}} d(b \mid -)\Big)\Big) \\ &\qquad + \Big(\frac{1}{2}\text{Ł}_a \circ \text{Ł}_c - \frac{3}{4}\Big(a(c \mid -) + (-1)^{\bar{a}\bar{c}} c(a \mid -)\Big)\Big) \otimes [\text{Ł}_b, \text{Ł}_d]\Bigg) \\ &\quad = (-1)^{\bar{b}\bar{c}}\frac{1}{2}\Big([\text{Ł}_a, \text{Ł}_c] \otimes (b \mid d)\, \mathrm{I} + (a \mid c)\, \mathrm{I} \otimes [\text{Ł}_b, \text{Ł}_d]\Big), \end{aligned}$$

as desired. □

Corollary 2.5. *For $a, b, c, d \in K_{\bar{0}} \cup K_{\bar{1}}$, $[\text{Ł}_{a\otimes b}, \text{Ł}_{c\otimes d}]$ is a superderivation of J.*

Proof. Because K is a Jordan superalgebra, for $a, b \in K_{\bar{0}} \cup K_{\bar{1}}$, the mapping $[\text{Ł}_a, \text{Ł}_b]$ is a superderivation of K. By the associativity of $(\,|\,)$, it is super-skew-symmetric relative to $(\,|\,)$. Consequently, from (2.3) we can conclude that $[\text{Ł}_{a\otimes b}, \text{Ł}_{c\otimes d}]$ is a superderivation of J. □

Theorem 2.6. *$J = F1 \oplus (K \otimes K)$ is a Jordan superalgebra if* char $F \neq 2$.

Proof. A Jordan algebra over a field F of characteristic not 2 or 3 is characterized by commutativity and the identity $\sum_{\text{cyclic}}[\text{Ł}_{uv}, \text{Ł}_w] = 0$. Hence a Jordan superalgebra over F is characterized by supercommutativity and

$$\sum_{\text{supercyclic}} [\text{Ł}_{uv}, \text{Ł}_w] = [\text{Ł}_{uv}, \text{Ł}_w] + (-1)^{\bar{u}(\bar{v}+\bar{w})}[\text{Ł}_{vw}, \text{Ł}_u] + (-1)^{(\bar{u}+\bar{v})\bar{w}}[\text{Ł}_{wu}, \text{Ł}_v] = 0.$$

In the supercommutative algebra J, we have from (1.2) and (2.3) that for all homogeneous $a, b, c, d, u, v \in K$,

$$\begin{aligned}
&\sum_{\text{supercyclic}} [\mathrm{L}_{(a\otimes b)(c\otimes d)}, \mathrm{L}_{u\otimes v}] \\
&\qquad = \sum_{\text{supercyclic}} (-1)^{\bar b\bar c}[\mathrm{L}_{ac\otimes bd}, \mathrm{L}_{u\otimes v}] \\
&\qquad = \sum_{\text{supercyclic}} (-1)^{\bar b(\bar c+\bar u)+\bar d\bar v}\frac{1}{2}\Big([\mathrm{L}_{ac}, \mathrm{L}_u]\otimes (bd \mid v)\ \mathrm{I} + (ac \mid u)\ \mathrm{I}\otimes[\mathrm{L}_{bd}, \mathrm{L}_v]\Big) \\
&\qquad = (-1)^{\bar b(\bar c+\bar u)+\bar d\bar u}\frac{1}{2}\Bigg\{\Bigg(\sum_{\text{supercyclic}}[\mathrm{L}_{ac}, \mathrm{L}_u]\Bigg)\otimes (bd \mid v)\ \mathrm{I} \\
&\qquad\qquad\qquad + (ac \mid u)\ \mathrm{I}\otimes\Bigg(\sum_{\text{supercyclic}}[\mathrm{L}_{bd}, \mathrm{L}_v]\Bigg)\Bigg\} \\
&\qquad = 0.
\end{aligned}$$

Thus, J is a Jordan superalgebra whenever the field has characteristic different from 2 or 3.

If R is the ring of fractions $R = S^{-1}\mathbb{Z}$ relative to the multiplicative set of integers $S = \{2^n \mid n = 0, 1, \dots\}$, we could have defined K and J over R as above. (Denote the result by J_R.) Then $J_{\mathbb{Q}} = \mathbb{Q}\otimes_R J_R$ is a Jordan superalgebra over the rationals $\mathbb{Q}$, and so is J_R over R. But if F is a field having characteristic not 2, then $J = F\otimes_R J_R$ is a Jordan superalgebra over F. □

From the computations and results above, it is easy to draw the following conclusions about the Jordan superalgebra J. Here we use the associator $(u, v, w) = (uv)w - u(vw)$ in J in the statements.

Proposition 2.7. *For the Jordan superalgebra $J = F1 \oplus (K \otimes K)$:*

(a) $(J, J, J) = K \otimes K$.
(b) $J = F1 \oplus (J, J, J)$.
(c) *If* char $F = 3$, *then* $K \otimes K = (J, J, J)$ *is an ideal of* J.
(d) $e_1 := -\frac{1}{2}1 + 2e\otimes e$ *and* $e_2 := \frac{3}{2}1 - 2e\otimes e$ *are orthogonal idempotents in* J. *Relative to* e_1 *the Peirce spaces* $J^{(\lambda)} := \{u \in J \mid e_1u = \lambda u\}$ *are given by*

$$J^{(1)} = Fe_1, \quad J^{(0)} = Fe_2 \oplus (K_{\bar 1}\otimes K_{\bar 1}), \quad J^{(\frac{1}{2})} = (e\otimes K_{\bar 1}) \oplus (K_{\bar 1}\otimes e).$$

Thus, $J_{\bar 0} = J^{(1)} \oplus J^{(0)}$ *and* $J_{\bar 1} = J^{(\frac{1}{2})}$. *The space* $J^{(0)}$ *is the Jordan algebra of a nondegenerate symmetric bilinear form, and* e_2 *is its unit element.*

Theorem 2.8. (a) *If* char $F \neq 2, 3$, $J = F1 \oplus (K \otimes K)$ *is a simple Jordan superalgebra.*
(b) *If* char $F = 3$, $(J, J, J) = K \otimes K$ *is a simple Jordan superalgebra.*
(c) Der J = Inder J *and it is isomorphic to* $\mathfrak{osp}(1, 2) \oplus \mathfrak{osp}(1, 2)$.

Proof. We begin with (c). Any superderivation of J annihilates $F1$ and leaves invariant $(J, J, J) = K \otimes K$. Therefore, using the natural identifications and the fact that $\operatorname{Der} K = \operatorname{Inder} K \cong \mathfrak{osp}(1, 2)$, we have by (2.3) that

$$\operatorname{Der} K \oplus \operatorname{Der} K \cong (\operatorname{Der} K \otimes \mathrm{I}) \oplus (\mathrm{I}\otimes \operatorname{Der} K) = \operatorname{Inder} J. \tag{2.9}$$

Now one may check that $\operatorname{Inder} J = \operatorname{Der} J$ as follows: First for any $D \in (\operatorname{Der} J)_{\bar{0}}$, $D|_{J_{\bar{0}}} \in \operatorname{Der}(J_{\bar{0}})$. Since $J_{\bar{0}}$ is a direct sum of a 1-dimensional ideal $J^{(1)} = Fe_1$ and the Jordan algebra $J^{(0)} = Fe_2 \oplus (K_{\bar{1}} \otimes K_{\bar{1}})$ of a bilinear form, $\operatorname{Der}(J_{\bar{0}}) = \operatorname{Inder} J_{\bar{0}}$. Thus, there is a $\tilde{D} \in (\operatorname{Inder} J)_{\bar{0}}$ with $(D - \tilde{D})(J_{\bar{0}}) = 0$. It follows easily that $D = \tilde{D}$. Now suppose $E \in (\operatorname{Der} J)_{\bar{1}}$. Then there is a $\tilde{E} \in (\operatorname{Inder} J)_{\bar{1}}$ with $E(e_1) = \tilde{E}(e_1)$ because $(\operatorname{Inder} J)_{\bar{1}} e_1 = J_{\bar{1}}$. For any $z \in J_{\bar{1}}$, $e_1 z = \frac{1}{2} z$, so $(E - \tilde{E})(z) = 2(E - \tilde{E})(e_1 z) = 2e_1(E - \tilde{E})(z)$. Thus, $(E - \tilde{E})(z) \in J_{\bar{0}} \cap J^{(\frac{1}{2})} = 0$, so that $(E - \tilde{E})(J_{\bar{1}}) = 0$. As a consequence, $E = \tilde{E}$ and $(\operatorname{Der} J)_{\bar{1}} = (\operatorname{Inder} J)_{\bar{1}}$.

Since $\operatorname{Der} K$ acts irreducibly on K, so does $\operatorname{Der} J = \operatorname{Inder} J$ on $K \otimes K = (J, J, J)$, which is an ideal of J if and only if the characteristic of F is 3. From this (a) and (b) follow. $\square$

The Lie multiplication superalgebra of J. To compute the Lie multiplication superalgebra of J we need some preliminaries. Let H be a unital Jordan superalgebra with a normalized trace, that is, a linear map $\mathfrak{t} : H \to F$ such that $\mathfrak{t}(1) = 1$ and $\mathfrak{t}\Big((H, H, H)\Big) = 0$. Assume that $(H, H, H) \neq 0$. Let $H_0 = \ker \mathfrak{t}$. Then for any $x, y \in H_0$, $xy = \mathfrak{t}(xy)1 + x * y$, where $x * y = xy - \mathfrak{t}(xy)1$. The *Lie multiplication superalgebra* is the Lie superalgebra spanned by L_H, which is a direct sum of the central ideal $F\,\mathrm{I}$ and the ideal $\mathfrak{L}_0(H) = \mathrm{L}_{H_0} \oplus [\mathrm{L}_{H_0}, \mathrm{L}_{H_0}]$. For simplicity, let us adopt the notation $S = \operatorname{Inder} H$ and $T = H_0$. Then $\mathfrak{L}_0(H)$ is isomorphic to the Lie superalgebra $\mathfrak{L} = S \oplus T$ with multiplication determined by

(1) the natural Lie multiplication on S,
(2) the natural action of S on T, that is, $[s, t] = s(t)$ for any $s \in S$ and $t \in T$, and
(3) $[t_1, t_2] = [\mathrm{L}_{t_1}, \mathrm{L}_{t_2}] \in S$, for any $t_1, t_2 \in T$.

Also, given a nonzero scalar $\nu \in F$, let $\mathfrak{L}_\nu$ be the Lie superalgebra defined over $\mathfrak{L}$, but with the new bracket given by

(i) $[x, y]^\nu = [x, y]$, for x, y both in S or one in S and the other in T,
(ii) $[x, y]^\nu = \nu[x, y]$, for $x, y \in T$.

Lemma 2.10. *Under the hypotheses above, if $\operatorname{Hom}_S(T \otimes T, T)$ is spanned by $u \otimes v \mapsto u * v$ and if any automorphism of the superalgebra $(T, *)$ is an isometry of the supersymmetric bilinear form $(u, v) \mapsto \mathfrak{t}(uv)$, then for any $0 \neq \mu, \nu \in F$, $\mathfrak{L}_\mu$ is isomorphic to $\mathfrak{L}_\nu$ if and only if $\mu^{-1}\nu \in F^2$.*

Proof. It is enough to apply the ideas of ([BDE], proof of Prop. 4.2). We include the argument for completeness.

First, since any isomorphism $\mathfrak{L}_\mu \to \mathfrak{L}_\nu$ is also an isomorphism $\mathfrak{L} \to \mathfrak{L}_{\mu^{-1}\nu}$, it is enough to assume $\mu = 1$. Suppose $\Phi : \mathfrak{L} \to \mathfrak{L}_\nu$ is an isomorphism. Then the map $T \otimes T \to T$ defined by $u \otimes v \mapsto \Phi^{-1}\Big(\Phi(u) * \Phi(v)\Big)$ is S-invariant. Consequently, there exists a nonzero scalar $\alpha \in F$ such that $\alpha\Phi(u) * \Phi(v) = \Phi(u * v)$ for any $u, v \in T$. Then $\psi : T \to T$ given by $\psi(u) = \alpha\Phi(u)$ is an automorphism of $(T, *)$ which, by hypothesis, extends by means of $\psi(1) = 1$ to an automorphism of the Jordan superalgebra H. Now the map $\Psi : \mathfrak{L} \to \mathfrak{L}$, such that $\Psi(s) = \psi s \psi^{-1}$ (composition of maps) for $s \in S$, and $\Psi(t) = \psi(t)$ for $t \in T$ is an automorphism of $\mathfrak{L}$.

Furthermore, $\tilde{\Phi} = \Phi\Psi^{-1} : \mathfrak{L} \to \mathfrak{L}_\nu$ is an isomorphism such that $\tilde{\Phi}(t) = \Phi(\psi^{-1}(t)) = \alpha^{-1}t$ for $t \in T$. Hence, for any $t_1, t_2, t_3 \in T$,

$$\begin{aligned}\tilde{\Phi}\Big([t_1, t_2]\Big) &= [\tilde{\Phi}(t_1), \tilde{\Phi}(t_2)]^\nu = \alpha^{-2}\nu[t_1, t_2],\\ \alpha^{-1}[[t_1, t_2], t_3] &= \tilde{\Phi}\Big([[t_1, t_2], t_3]\Big)\\ &= [[\tilde{\Phi}(t_1), \tilde{\Phi}(t_2)]^\nu, \tilde{\Phi}(t_3)]^\nu\\ &= \alpha^{-3}\nu[[t_1, t_2], t_3]\,,\end{aligned}$$

so that $\nu = \alpha^{-2}$ as required. (Note $[[T, T], T] \neq 0$ because $(H, H, H) \neq 0$.)

The converse is clear, since the map Φ given by $\Phi(s) = s$ for $s \in S$ and $\Phi(t) = \alpha t$ for $t \in T$ provides an isomorphism $\mathfrak{L} \to \mathfrak{L}_{\alpha^2}$. □

In order to apply the last result to our 10-dimensional Jordan superalgebra J, notice first that J is endowed with a normalized trace $\mathfrak{t}$ such that $\mathfrak{t}(1) = 1$ and $\mathfrak{t}((J, J, J)) = 0$, as $(J, J, J) = K \otimes K$ does not intersect $F1$. Moreover, the associated bilinear form is given by $\mathfrak{t}\Big((a \otimes b)(c \otimes d)\Big) = -(-1)^{\bar{b}\bar{c}}\frac{3}{4}(a \mid c)(b \mid d)$ for any homogeneous $a, b, c, d \in K$, because of (2.1). Then all we need is the following

Lemma 2.11. *Any automorphism of $(K \otimes K, *)$ is an isometry of the bilinear trace form.*

Proof. Write $T = J_0 = K \otimes K$ and set $\langle a \otimes b \mid c \otimes d\rangle = (-1)^{\bar{b}\bar{c}}(a \mid c)(b \mid d)$ for any homogeneous elements $a, b, c, d \in K$. Then $T_{\bar{0}} = F(e \otimes e) \oplus (K_{\bar{1}} \otimes K_{\bar{1}})$. Suppose $\tilde{e} = e \otimes e$. Then the multiplication in $T_{\bar{0}}$ is given by

$$\tilde{e}^{*2} = \tilde{e}, \quad \tilde{e} * v = \frac{1}{4}v, \quad v * w = \langle v \mid w\rangle\tilde{e} \tag{2.12}$$

for all $v, w \in K_{\bar{1}} \otimes K_{\bar{1}}$.

Let φ be an automorphism of $(T, *)$ and let us first verify that $\varphi(\tilde{e}) = \tilde{e}$. This is clear if the characteristic is 3, because $\tilde{e}$ is then the unit element of $T_{\bar{0}}$. Otherwise, $\tilde{e}$ is the only idempotent of $T_{\bar{0}}$ such that the corresponding left multiplication operator has eigenvalue $\frac{1}{4}$ with multiplicity 4. This is because the eigenvalues of $\mathrm{L}_{\alpha\tilde{e}+v}$ (for $v \in K_{\bar{1}} \otimes K_{\bar{1}}$, $\alpha \in F$) include $\frac{\alpha}{4}$ with multiplicity at least 3, as any $w \in K_{\bar{1}} \otimes K_{\bar{1}}$ which is orthogonal to v relative to $\langle\,|\,\rangle$ is a corresponding eigenvector. Moreover, $\tilde{e} + v$ is an idempotent only if $v = 0$.

Now, $\varphi(K_{\bar{1}} \otimes K_{\bar{1}}) \subseteq K_{\bar{1}} \otimes K_{\bar{1}}$ because $K_{\bar{1}} \otimes K_{\bar{1}} = \{v \in T_{\bar{0}} \mid \tilde{e} \notin Fv \text{ and } v * v \in F\tilde{e}\}$. Because of (2.12), $\varphi|_{T_{\bar{0}}}$ is an isometry of the restriction of $\langle\ |\ \rangle$ to $T_{\bar{0}}$.

Finally, $(K_{\bar{1}} \otimes e) \cup (e \otimes K_{\bar{1}}) = \{z \in T_{\bar{1}} \mid z * z \in F\tilde{e}\}$, and $K_{\bar{1}} \otimes e$ and $e \otimes K_{\bar{1}}$ are the only 2-dimensional subspaces contained in this union. Therefore, either φ preserves $K_{\bar{1}} \otimes e$ and $e \otimes K_{\bar{1}}$ or it interchanges them. Since $(u \otimes e) * (v \otimes e) = \frac{1}{2}\langle u \otimes e \mid v \otimes e\rangle\tilde{e}$ and $(e \otimes u) * (e \otimes v) = \frac{1}{2}\langle e \otimes u \mid e \otimes v\rangle\tilde{e}$ for any $u, v \in K_{\bar{1}}$, it follows that $\varphi|_{T_{\bar{1}}}$ is an isometry too. □

We are ready to determine the Lie multiplication superalgebra $\mathfrak{L}_0(J)$ for $J = F1 \oplus (K \otimes K)$. When the underlying field F has characteristic 0, the next result is stated in [Sh] but without the assumption of -1 being a square in F.

Theorem 2.13. $\mathfrak{L}_0(J)_{-1} \cong \mathfrak{osp}(K \oplus K)$ $\Big(\cong \mathfrak{osp}(2, 4)\Big)$. *In particular, $\mathfrak{L}_0(J)$ is a form of $\mathfrak{osp}(2, 4)$ and $\mathfrak{L}_0(J) \cong \mathfrak{osp}(2, 4)$ if and only if -1 is a square in F.*

Proof. On any vector superspace V equipped with a supersymmetric bilinear form $(\,|\,)$, the transformation $\gamma_{u,v} = u(v \mid -) - (-1)^{\bar u \bar v} v(u \mid -)$ belongs to $\mathfrak{osp}(V)$ for all homogeneous $u, v \in V$. Because of (2.3) and (1.7), $\mathrm{Der}\, J$ is embedded in $\mathfrak{osp}(K \oplus K)$ (with the natural supersymmetric bilinear form on $K \oplus K$ that makes this an orthogonal sum) by means of

$$\Phi\Big([\text{Ł}_{a\otimes b}, \text{Ł}_{c\otimes d}]\Big) = (-1)^{\bar b \bar c}\frac{1}{4}\Big(\gamma_{(a,0),(c,0)}(b \mid d) + (a \mid c)\gamma_{(0,b),(0,d)}\Big) \in \mathfrak{osp}(K \oplus K)\,.$$

This can be extended to a linear bijection $\Phi : \mathfrak{L}_0(J)_{-1} \to \mathfrak{osp}(K \oplus K)$ by defining

$$\Phi(\text{Ł}_{a\otimes b}) = \frac{1}{2}\gamma_{(a,0),(0,b)}$$

for $a, b \in K$. Since

$$\begin{aligned}
&\Big[\gamma_{(a,0),(0,b)}, \gamma_{(c,0),(0,d)}\Big] \\
&\qquad = \gamma_{\gamma_{(a,0),(0,b)}(c,0),(0,d)} + (-1)^{(\bar a + \bar b)\bar c}\gamma_{(c,0),\gamma_{(a,0),(0,b)}(0,d)} \\
&\qquad = -(-1)^{\bar a \bar b}(-1)^{(\bar a + \bar c)\bar b}(a \mid c)\gamma_{(0,b),(0,d)} + (-1)^{(\bar a + \bar b)\bar c}\gamma_{(c,0),(a,0)}(b \mid d) \\
&\qquad = -(-1)^{\bar b \bar c}\Big((a \mid c)\gamma_{(0,b),(0,d)} + \gamma_{(a,0),(c,0)}(b \mid d)\Big)\,,
\end{aligned}$$

it follows that Φ is an isomorphism of Lie superalgebras. The last part is a consequence of Lemmata 2.10 and 2.11. $\square$

3. The Kac superalgebra

We may identify the simple Jordan superalgebra J with the 10-dimensional simple Jordan superalgebra K_{10} discovered by Kac [K] (see also [HK]) when the characteristic is not 2 or 3, and $(J, J, J) = K \otimes K$ with the 9-dimensional degenerate Kac superalgebra K_9 if $\mathrm{char} F = 3$. This realization may be described explicitly as follows. In the notation of [MZ] or [RZ], we suppose $\{\check e, v_1, v_2, v_3, v_4, \check f, x_1, y_1, x_2, y_2\}$ is a basis of K_{10} with $(\mathrm{K}_{10})_{\bar 0} = \mathrm{span}_F\{\check e, v_1, v_2, v_3, v_4, \check f\}$ and $(\mathrm{K}_{10})_{\bar 1} = \mathrm{span}_F\{x_1, y_1, x_2, y_2\}$ and with multiplication given by

$$\begin{gathered}
\check e^2 = \check e, \quad \check e v_i = v_i, \quad v_1 v_2 = 2\check e = v_3 v_4, \quad \check f^2 = \check f, \\
\check e x_j = \frac{1}{2}x_j = \check f x_j, \qquad \check e y_j = \frac{1}{2}y_j = \check f y_j, \\
v_1 y_1 = x_2, \quad v_1 y_2 = -x_1, \quad v_2 x_1 = -y_2, \quad v_2 x_2 = y_1, \\
v_3 x_2 = x_1, \quad v_3 y_1 = y_3, \quad v_4 x_1 = x_2, \quad v_4 y_2 = y_1, \\
x_j y_j = \check e - 3\check f, \quad x_1 x_3 = v_1, \quad x_1 y_2 = v_3, \quad x_2 y_1 = v_4, \quad y_1 y_2 = v_2
\end{gathered} \tag{3.1}$$

for $i = 1, 2, 3, 4$ and $j = 1, 2$. All other products are either 0 or obtained from those in (3.1) by supercommutativity. Then a straightforward computation using (3.1) shows that the assignment

$$\begin{array}{rclcrcl}
\check e & \leftrightarrow & \frac{3}{2}1 - 2e \otimes e = e_2 & & & & \\
\check f & \leftrightarrow & -\frac{1}{2}1 + 2e \otimes e = e_1 & & x_1 & \leftrightarrow & 4x \otimes e \\
v_1 & \leftrightarrow & -4x \otimes x & & x_2 & \leftrightarrow & -4e \otimes x \\
v_2 & \leftrightarrow & -y \otimes y & & y_1 & \leftrightarrow & -2y \otimes e \\
v_3 & \leftrightarrow & 2x \otimes y & & y_2 & \leftrightarrow & 2e \otimes y \\
v_4 & \leftrightarrow & -2y \otimes x & & & &
\end{array}$$

gives an isomorphism between K_{10} and J. Additionally, if $\mathrm{char}F = 3$, then $\check{e} \leftrightarrow -2e \otimes e$, so that the isomorphism restricts to an isomorphism $\mathrm{K}_9 \leftrightarrow K \otimes K$.

Concluding remarks. As the reader may have guessed, the actual process for deriving our results was the reverse path of the presentation above. First it was verified that $\mathfrak{g} := \mathrm{Der}\,\mathrm{K}_{10} = \mathfrak{osp}(1,2) \oplus \mathfrak{osp}(1,2)$, (compare Shtern [Sh]), and that as module for the Lie superalgebra $\mathfrak{g}$, $\mathrm{K}_{10} = F1 \oplus (V \otimes V)$, where V is the natural 3-dimensional module for $\mathfrak{osp}(1,2)$. The first copy of $\mathfrak{osp}(1,2)$ operates on the first tensor slot of $V \otimes V$, and the second copy on the second slot. The spaces $\mathrm{Hom}_{\mathfrak{osp}(1,2)}(V \otimes V, F)$ and $\mathrm{Hom}_{\mathfrak{osp}(1,2)}(V \otimes V, \mathfrak{osp}(1,2))$ are 1-dimensional. By identifying V with the tiny Kaplansky superalgebra K, we may describe the generators of these spaces nicely by the maps $u \otimes v \mapsto (u \mid v)$ and $u \otimes v \mapsto [\text{Ł}_u, \text{Ł}_v] \in \mathfrak{osp}(1,2)$ for $u, v \in K$. From this, the multiplication in the Kac superalgebra K_{10} is almost uniquely determined in terms of K.

In [S1], Shestakov proposed quite a different realization of the superalgebra K_{10} in characteristic $\neq 2, 3$, but the same description of K_9; however, he has not published this work [S2].

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