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# COMPLEMENTED ISOMETRIC COPIES OF $L_1$ IN DUAL BANACH SPACES

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Abstract. Let $X$ be a real or complex Banach space and $K \geq 1$. Then $X^*$ contains a $K$–complemented, isometric copy of $L_1[0,1]$ if and only if $X^*$ contains a $K$–complemented, isometric copy of $C[0,1]^*$ if and only if $X$ contains a subspace $(1,K)$-asymptotic to $\left(\ell_1 \oplus \sum_n \ell_\infty^n\right)_1$.

## Introduction

Let $X$ be a real or complex Banach space. In 1968, Pełczyński [P] showed that if $X$ contains an isomorphic copy of $\ell_1$, then its dual space $X^*$ contains an isomorphic copy of $L_1 = L_1[0,1]$. He also proved the converse under mild technical restrictions which were removed in the early 1970's by the author (cf. [H2], [H1]). Shortly thereafter, Stegall and the author [HS] showed that $X^*$ contains a *complemented* isomorphic copy $L_1$ if and only if $X$ contains an isomorphic copy of $\left(\sum_n \ell_\infty^n\right)_1$.

Recent work involving isometric copies of $L_1$ and asymptotically isometric copies of $\ell_1$ (see below for the definition) has suggested the usefulness of precise, quantitative versions of these results. For example, Dowling and Lennard [DL] showed that every nonreflexive subspace of $L_1$ fails the fixed point property for nonexpansive maps. Dowling, Lennard and Turrett [DLT] showed that if a Banach space contains asymptotically isometric copies of $\ell_1$, then its dual space fails this fixed point property. The papers of Dowling, Randrianantoanina and Turett [DRT] and Dowling, Johnson, Lennard and Turett [DJLT], both dealing with the sharpness of James' distortion theorems, use asymptotically isometric copies of $\ell_1$ in a fundamental way.

In [DGH] (see also [H1]), Dilworth, Girardi and the author showed that $X^*$ contains an isometric copy of $L_1$ if and only if $X$ contains asymptotically isometric copies of $\ell_1$. (This provides a direct link between the papers [DL] and [DLT].) It is also shown in [DGH] that this characterization is sharp: $X^*$ can contain an isometric copy of $L_1$ without $X$ containing an isometric copy of $\ell_1$.

The present article provides quantitative characterizations of Banach spaces $X$ such that $X^*$ contains a *complemented*, *isometric* copy of $L_1$. In the real case, these results appeared in the author's dissertation [H1] but have not been published

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before. We show here that $X^*$ contains $K$-complemented, isometric copies of $L_1$ if and only if $X$ contains $(1,K)$-asymptotic copies of $\left(\ell_1 \oplus \sum_n \ell_\infty^n\right)_1$. Here is the precise definition.

**Definition.** Let $X$ be a Banach space and $K \geq 1$. We say that $X$ contains $(1,K)$-*asymptotic copies of* $\left(\ell_1 \oplus \sum_n \ell_\infty^n\right)_1$ if for some (equivalently, for any) sequence $\lambda_0 > \lambda_1 > \cdots$ with $\lim_n \lambda_n = 1$, there are sequences $(w_n)_{n\geq 0}$ and $(z_{n,j})_{n\geq 0,\ 0\leq j\leq n}$ in $X$ such that for all $m$ and the scalars $(t_n : 0 \leq n \leq m)$ and $(s_{n,j} : 0 \leq n \leq m, 0 \leq j \leq n)$,

$$\sum_{n=0}^{m}\left(|t_n| + \max_j |s_{n,j}|\right) \leq \left\|\sum_{n=0}^{m}\left(t_n w_n + \sum_{j=0}^{n} s_{n,j} z_{n,j}\right)\right\| \leq \sum_{n=0}^{m} \lambda_n \left(|t_n| + K \max_j |s_{n,j}|\right).$$

If we ignore the $z_{n,j}$'s (i.e., let each $s_{n,j} = 0$), then we get

$$\sum_{n=0}^{m} |t_n| \leq \left\|\sum_{n=0}^{m} t_n w_n\right\| \leq \sum_{n=0}^{m} \lambda_n |t_n|,$$

which is precisely the condition that $X$ contains *asymptotically isometric copies of* $\ell_1$. We show below that the $(w_n)$'s give an isometric copy of $C(\Delta)^*$ in $X^*$ and the $(z_{n,j})$'s give a projection of norm $\leq K$ onto this copy of $C(\Delta)^*$. (Throughout this paper, $\Delta$ denotes the Cantor set.)

Here is the main result of this paper.

**Theorem 1.** *Let $X$ be a real or complex Banach space and $K \geq 1$. The following are equivalent:*

(1) *$X$ contains $(1,K)$-asymptotic copies of $\left(\ell_1 \oplus \sum_n \ell_\infty^n\right)_1$;*
(2) *$L_1$ is linearly isometric to a $K$-complemented subspace of $X^*$;*
(3) *$C(\Delta)^*$ is linearly isometric to a $K$-complemented subspace of $X^*$;*
(4) *$X^*$ contains an infinite set $\Gamma$ such that $\Gamma$ is isometrically equivalent to the usual basis of $\ell_1(\Gamma)$, $[\Gamma]$ is $K$-complemented in $X^*$, and $\Gamma$ is dense-in-itself in the weak$^*$ topology on $X^*$.*
*If $X$ is separable, then (1)-(4) are equivalent to:*
(5) *There exists a quotient map $T : X \to C(\Delta)$ such that $T^*\left(C(\Delta)^*\right)$ is $K$-complemented in $X^*$.*

Many of the implications in Theorem 1 are well known and easy. Before giving the proofs of the nontrivial implications, we indicate briefly the ideas underlying the arguments and the organization of the rest of the paper. We first prove (1)⇒(3). To do this, we give a property of a linear map $S : X \to Y$ (Lemma 2) which insures that $X^*$ contains a $K$-complemented subspace isometric to $Y^*$. It follows immediately from this lemma that $\left(\ell_1 \oplus \sum_n \ell_\infty^n\right)_1^*$ contains a 1-complemented subspace isometric to $C(\Delta)^*$. A standard lifting argument for $\mathcal{L}_1$ spaces (Lemma 3) finishes the argument.

The central part of the paper proves assertions (2)⇒(1) and (2)⇒(5). Since this part is quite technical, we give a more detailed preview of the argument. Suppose that $Z$ is a Banach space and it is known that there is a bounded linear onto map $S : Z \to C(\Delta)$ which satisfies the assumptions of Lemma 2. Then it is possible to find $(1,K)$-asymptotic copies of $(\ell_1 \oplus \sum_n \ell_\infty^n)_1$ in $Z$. On the other hand, if $X$ is a Banach space with $X^*$ containing a $K$-complemented isometric copy of $L_1$, then there need not even be an onto map from $X$ to $C(\Delta)$, let alone a map to which Lemma 2 applies. To overcome this difficulty, we show in Lemma 4 that if there is a Banach space $Y$ containing a subspace isometric to $C(\Delta)$ and a map $T : X \to Y$ which shares enough properties with $S$, then $X$ contains $(1,K)$-asymptotic copies of $(\ell_1 \oplus \sum_n \ell_\infty^n)_1$.

We find such a space $Y$ and map $T$ by constructing a weak$^*$ compact subset $\Omega$ of $X^*$ so that Lemma 4 applies when $Y = C(\Omega)$ and $T : X \to C(\Omega)$ is the evaluation map (given by $Tx(\omega) = \omega(x)$). In Lemma 5 we use the Liapounoff convexity theorem and consequences of local reflexivity to assemble the machinery necessary to construct $\Omega$. (Lemma 5 is really the heart of this paper!) Following this lemma, $\Omega$ is built, the properties of $T$ are established and the proofs of (2)$\Rightarrow$(1) and (2)$\Rightarrow$(5) are concluded.

Finally, the assertions following from condition (4) are briefly discussed.

**Preliminaries.** The Banach space $X$ is real or complex. The dual space of $X$ is denoted by $X^*$, the closed unit ball of $X$ by $B_X$, and the closed linear span of $A \subset X$ (i.e. the smallest closed subspace of $X$ containing the set $A$) by $[A]$. A linear map $T : X \to Y$ is a *quotient map* if $\overline{T(B_X)} = B_Y$.

A subspace $Y$ of $X$ is *$K$-complemented in* $X$ if there is a projection $P : X \to X$ such that $P(X) = Y$ and $\|P\| \le K$. Equivalently, $Y$ is $K$-complemented in $X$ if there is a linear map $S : X \to Y$ such that $Sy = y$ if $y \in Y$ and $\|S\| \le K$. There is no difference in assuming below that $Y$ is a $K$–complemented subspace of $X$ or that $Y$ is the range of a projection on $X$ of norm exactly $K$. Indeed, suppose that $Y$ is a $K$–complemented, proper subspace of the Banach space $X$ and $P : X \to X$ is a projection onto $Y$ with $\|P\| \le K$. Let $f \in X^*$, $f \ne 0$, be zero on $Y$ and let $y \in Y$, $y \ne 0$. For each $t \ge 0$, let $Q_t : X \to X$ be the projection onto $Y$ given by $Q_t(x) = P(x) + tf(x)y$. It is now easy to see that $\|Q_t\| = K$ for some $t$. (Thanks to W. B. Johnson for pointing this out.)

Let $\ell_1$ (resp. $\ell_1(\Gamma)$) be the space of (real or complex) absolutely summable sequences (resp. absolutely summable functions on the set $\Gamma$) and $L_1$ (or $L_1[0,1]$) the space of (real or complex) Lebesgue–integrable functions on $[0,1]$, all with their usual norms. We denote the Lebesgue measure of the measurable set $O \subseteq [0,1]$ by $|O|$. $\ell_\infty^n = \{x = (x_1, \cdots, x_n) : x_i \in \mathcal{F} \text{ and } ||x|| = \max_i |x_i|\}$. ($\mathcal{F}$ is $\mathbb{R}$ or $\mathbb{C}$, the field of scalars). Also, $(\ell_1 \oplus \sum_n \ell_\infty^n)_1$ is the completion of the set of finitely nonzero scalars $x = (t_n, s_{n,j})_{n \ge 0, 0 \le j \le n}$ with $\|x\| = \sum_{n=0}^\infty (|t_n| + \max_j |s_{n,j}|)$.

We need the usual binary tree ordering in this work. Let $\mathcal{T} = \{(n,i) : 0 \le n, 0 \le i < 2^n\}$. If $(m,j)$,$(n,i) \in \mathcal{T}$, we say that $(n,i) \succcurlyeq (m,j)$ if $n \ge m$ and $2^{n-m} j \le i \le 2^{n-m}(j+1) - 1$.

For a compact Hausdorff space $\Omega$, $C(\Omega)$ is the Banach space of (real or complex valued) continuous functions on $\Omega$ with sup norm. Let $\Delta = \{\lambda = (\lambda_n)_{n \ge 0} : \lambda_n = 0$ or $1$ for each $n\}$ be the Cantor set with metric $\rho(\lambda, \tau) = \sum_{n=0}^\infty 2^{-n+1} |\lambda_n - \tau_n|$ for $\lambda = (\lambda_n)_{n \ge 0}$ and $\tau = (\tau_n)_{n \ge 0}$. Let $\{\Delta_{n,i} : (n,i) \in \mathcal{T}\}$ be closed open sets satisfying $\Delta_{0,0} = \Delta$, $\Delta_{n,i} = \Delta_{n+1,2i} \cup \Delta_{n+1,2i+1}$ for all $(n,i) \in \mathcal{T}$, $\Delta_{n,i} \cap \Delta_{n,j} = \emptyset$ if $i \ne j$, and such that the diameter of $\Delta_{n,i}$ is $2^{-n}$ for all $(n,i)$. Put $G_n = \left[\left\{\chi_{\Delta_{n,i}} : 0 \le i < 2^n\right\}\right]$. Then $G_n$ is isometric to $\ell_\infty^{2^n}$, $G_n \subseteq G_m$ if $n \ge m$ and $\overline{\bigcup_{n \ge 0} G_n} = C(\Delta)$. It is also well known that $C(\Delta)^*$ and $C[0,1]^*$ are linearly isometric.

*Proofs of the results.* We first turn to the proof of (1)⇒(3) of Theorem 1. We state two preparatory lemmas. The first is a minor variant of a result of Johnson [J, Proposition 1]. An earlier version is due to Stegall [S, Lemma 1] who refers to the local lifting property for $T$ described in the lemma as "$T$ has local selections."

**Lemma 2.** *Let $X$ and $Y$ be Banach spaces, $T : X \to Y$ a quotient map and $K \geq 1$. Suppose that $\{B_\alpha : \alpha \in \mathcal{A}\}$ is a directed (by inclusion) family of nonzero finite dimensional subspaces of $Y$ such that*

$$\overline{\bigcup_{\alpha\in\mathcal{A}} B_\alpha} = Y.$$

*Suppose also that for each $\alpha \in \mathcal{A}$ and $\varepsilon > 0$ the inclusion map $i_\alpha : B_a \to Y$ can be lifted to a linear map $I_\alpha : B_a \to X$ such that*

(1) $||I_\alpha|| \leq K + \varepsilon$.
(2) $TI_\alpha = i_\alpha$.

*Then there exists a projection $P : X^* \to X^*$ such that $P(X^*) = T^*Y^*$ and $||P|| \leq K$.*

Conclusion (1) of the next lemma is also proven by a Lindenstrauss compactness argument. The proof may be found in [HS, Lemma 2] (although the statement of that result is not as precise as the lemma given here). See [LR] for the definition and properties of $\mathcal{L}_1$ spaces.

**Lemma 3.** *Let Banach spaces $X$ and $Y$ and an isometry $u : X \to Y$ be given. Let $B$ be an $\mathcal{L}_{1,1+\varepsilon}$ space for every $\varepsilon > 0$. (In particular, $B$ may be any of the spaces $L_1$, $C[0,1]^*$ or $\ell_1(\Gamma)$.)*

(1) *For any operator $T : B \to X^*$ there exists an operator $\tilde{T} : B \to Y^*$ with $u^*\tilde{T} = T$ and $\left\|\tilde{T}\right\| = \|T\|$.*
(2) *If $B$ is isometric to a $K$-complemented subspace of $X^*$, then $B$ is isometric to a $K$-complemented subspace of $Y^*$.*

*Proof of* (2). Let $T : B \to X^*$ be an isometric embedding and $S : X^* \to B$ a linear map with $ST = id_B$ and $\|S\| \leq K$. Apply (1) to get $\tilde{T}$. It is clear that $\tilde{T}$ is an isometry. Put $\tilde{S} = Su^* : Y^* \to B$. Then $\left\|\tilde{S}\right\| = \|S\| \leq K$ and $\tilde{S}\tilde{T} = (Su^*)\tilde{T} = S\left(u^*\tilde{T}\right) = ST = id_B$. □

We are now ready to prove (1)⇒(3) of Theorem 1.

*Proof of Theorem 1,* (1) ⇒ (3). Let $g_{n,i} = \chi_{\Delta_{n,i}} \in C(\Delta)$ for $(n,i) \in \mathcal{T}$ and $G_n = [\{g_{n,i} : 0 \leq i < 2^n\}]$ for each $n$.

Now suppose that $X$ contains $(1, K)$-asymptotic copies of $(\ell_1 \oplus \sum_n \ell_\infty^n)_1$ and let $(w_n)_{n\geq 0}$ and $(z_{n,j})_{n\geq 0,\ 0\leq j\leq n}$ be as in the definition. By selecting a subsequence of the $z_{n,j}$'s and renaming we may assume that we have $(w_n)_{n\geq 0}$ and $(v_{n,j})_{n\geq 0,\ 0\leq j<2^n}$ satisfying

$$\sum_{n=0}^{m}\left(|t_n| + \max_j |s_{n,j}|\right) \leq \left\|\sum_{n=0}^{m}\left(t_n w_n + \sum_{j=0}^{2^n-1} s_{n,j} v_{n,j}\right)\right\|$$
$$\leq \sum_{n=0}^{m} \lambda_n \left(|t_n| + K \max_j |s_{n,j}|\right)$$

for appropriate scalars. Let $X_0 = \left[(w_n)_{n\geq 0}, (v_{n,j})_{n\geq 0,\ 0\leq j<2^n}\right]$. Define a map $T$ by letting $\{Tw_n : n \geq 0\}$ be a dense sequence on the unit sphere of $C(\Delta)$, defining $Tv_{n,j} = g_{n,j}$ and extending $T$ by linearity to the linear span of the $w_n$'s and $v_{n,j}$'s. Then given $m$ and scalars $(t_n : 0 \leq n \leq m)$ and $(s_{n,j} : 0 \leq n \leq m, 0 \leq j < 2^n)$,

$$\begin{aligned}
\left\| T \sum_{n=0}^{m} \left( t_n w_n + \sum_{j=0}^{2^n-1} s_{n,j} v_{n,j} \right) \right\| &\leq \sum_{n=0}^{m} \left( |t_n| \, \|Tw_n\| + \left\| \sum_{j=0}^{2^n-1} s_{n,j} T v_{n,j} \right\| \right) \\
&\leq \sum_{n=0}^{m} \left( |t_n| + \left\| \sum_{j=0}^{2^n-1} s_{n,j} g_{n,j} \right\| \right) \\
&= \sum_{n=0}^{m} \left( |t_n| + \max_j |s_{n,j}| \right) \\
&\leq \left\| \sum_{n=0}^{m} \left( t_n w_n + \sum_{j=0}^{2^n-1} s_{n,j} v_{n,j} \right) \right\|
\end{aligned}$$

so $T$ extends to a norm one operator $T : X_0 \to C(\Delta)$. Since $\|T\| = 1$ and the closure of the set $\{Tw_n / \|w_n\| : n \geq 0\}$ contains the unit sphere of $C(\Delta)$, it follows that $T$ is a quotient map.

To see that $T$ satisfies the hypotheses of Lemma 2, recall that $\{G_n : n \geq 0\}$ is a nested sequence of nonzero finite dimensional subspaces of $C(\Delta)$ such that $\overline{\bigcup_{n\geq 0} G_n} = C(\Delta)$. For each $n$, let $i_n : G_n \to C(\Delta)$ be the inclusion map. Define $I_n : G_n \to X_0$ by $I_n(g_{n,j}) = v_{n,j}$ and extending by linearity. It is clear that $TI_n = i_n$. Now given scalars $(s_j)_{0\leq j<2^n}$

$$\left\| I_n \left( \sum_{j=0}^{2^n-1} s_j g_{n,j} \right) \right\| \leq \left\| \sum_{j=0}^{2^n-1} s_j v_{n,j} \right\| \leq K\lambda_n \max_j |s_j|$$

so $\|I_n\| \leq K\lambda_n$. So, by Lemma 2, $X_0^*$ contains a $K$-complemented subspace isometric to $C(\Delta)^*$. Now, by (2) of Lemma 3, $X^*$ contains a $K$-complemented subspace isometric to $C(\Delta)^*$ as well. $\square$

We next turn to the implications in Theorem 1 which follow from condition (2), that $X^*$ contains a $K$-complemented subspace isometric to $L_1$. In the proof of Theorem 1, (1)$\Rightarrow$(3), we construct a quotient map from $(\ell_1 \oplus \sum_n \ell_\infty^n)_1$ onto $C(\Delta)$ which satisfies the assumptions of Lemma 2. The next lemma shows that $X$ contains $(1, K)$-asymptotic copies of $(\ell_1 \oplus \sum_n \ell_\infty^n)_1$ if there is a Banach space $Y$ and a linear map $T : X \to Y$ with $C(\Delta)$ isometric to a subspace of $Y$ whose range is "near enough" to this copy of $C(\Delta)$ and which "almost" satisfies the assumptions of Lemma 2. To avoid introducing more notation in this lemma, we suppress explicit mention of the isometric map from $C(\Delta)$ into $Y$. Rather, we assume that $C(\Delta) \subseteq Y$. Throughout this lemma, we let $g_{n,i} = \chi_{\Delta_{n,i}}$ and $G_n = [\{g_{n,i} : 0 \leq i < 2^n\}]$.

**Lemma 4.** *Let $X$ and $Y$ be Banach spaces such that $Y$ contains a subspace isometric to $C(\Delta)$ and $K \geq 1$. Suppose that there exists a norm one operator $T : X \to Y$ with the following property: for any decreasing sequence $(\delta_n)$ with $\delta_0 < 1/2$ and*

$\lim_{n\to\infty}\delta_n = 0$ *there are a sequence of finite sets* $(A_n)$ *and* $(x_{n,i})_{(n,i)\in\mathcal{T}}$ *in* $X$ *such that*

(1) $\|a\| \le 1+\delta_n$ *if* $a \in A_n$.
(2) *If* $g \in G_n$, $\|g\| = 1$, *there exists* $a \in A_n$ *with* $\|Ta - g\| \le \delta_n$.
(3) $\|Tx_{n,i} - g_{n,i}\| \le \delta_n$ *for* $0 \le i \le 2^n - 1$.
(4) $\max_i |r_i| \le \left\|\sum_{i=0}^{2^n-1} r_i x_{n,i}\right\| \le K(1+\delta_n)\max_i |r_i|$ *for all* $n$ *and scalars* $(r_i)_{i=0}^{2^n-1}$.

*Then* $X$ *contains* $(1,K)$*-asymptotic copies of* $\left(\ell_1 \oplus \sum_n \ell_\infty^n\right)_1$.

*Proof.* A standard construction yields sequences of integers $(p_n)$ and $(q_n)$ with $0 \le p_0 < q_0 < p_1 < q_1 < \cdots$, and sequences $(d_n \in G_{p_n} : n \ge 0)$ and $(e_{n,j} \in G_{q_n} : n \ge 0, 0 \le j \le n)$ such that $(d_n)$,$(e_{n,j})$ is isometrically equivalent to the normalized unit vector basis of $\left(\ell_1 \oplus \sum_n \ell_\infty^n\right)_1$. That is, given $m$ and scalars $(t_n)$ and $(s_{n,j})$ for $0 \le n \le m$, $0 \le j \le n$,

$$\left\|\sum_{n=0}^{m}\left(t_n d_n + \sum_{j=0}^{n} s_{n,j} e_{n,j}\right)\right\| = \sum_{n=0}^{m}\left(|t_n| + \max_j |s_{n,j}|\right).$$

Let $\lambda_0 > \lambda_1 > \cdots$ with $\lim_n \lambda_n = 1$. Let $(\delta_n)$ be a decreasing sequence with

$$\max\left\{(1-\delta_n)^{-1}, \left(1-(n+1)\left(2^{q_n}+1\right)\delta_{q_n}\right)^{-1}, 1+\delta_n\right\} \le \lambda_n^{1/2}$$

and pick $A_n$ and $(x_{n,i})_{(n,i)\in\mathcal{T}}$ satisfying conditions (1)–(4).

First, by conditions (1) and (2) there exists $w_n' \in A_{p_n}$ with $\|w_n'\| \le 1+\delta_n$ and $\|Tw_n' - d_n\| \le \delta_n$. Next, let $(a_{i,j})$ be scalars such that $e_{n,j} = \sum_{i=0}^{2^{q_n}-1} a_{i,j} g_{q_n,i}$. Then

$$(\bigstar)\qquad \begin{aligned}\max_j |s_j| &= \left\|\sum_{j=0}^{n} s_j e_{n,j}\right\| = \left\|\sum_{i=0}^{2^{q_n}-1}\left(\sum_{j=0}^{n} s_j a_{i,j}\right) g_{q_n,i}\right\| \\ &= \max_i \left|\sum_{j=0}^{n} s_j a_{i,j}\right| \quad \text{for all scalars } (s_j).\end{aligned}$$

In particular, each $|a_{i,j}| \le 1$. Now put $z_{n,j}' = \sum_{i=0}^{2^{q_n}-1} a_{i,j} x_{q_n,i}$. Then

$$\left\|Tz_{n,j}' - e_{n,j}\right\| \le \sum_{i=0}^{2^{q_n}-1} |a_{i,j}|\,\|Tx_{q_n,i} - g_{q_n,i}\| \le \left(2^{q_n}+1\right)\delta_{q_n}.$$

Next, given scalars $(s_j)_{j=0}^{n-1}$ we have (by condition (4) and $(\bigstar)$)

$$\begin{aligned}\left\|\sum_{j=0}^{n} s_j z_{n,j}'\right\| &= \left\|\sum_{j=0}^{n} s_j \sum_{i=0}^{2^N-1} a_{i,j} x_{q_n,i}\right\| \le \left\|\sum_{i=0}^{2^N-1}\left(\sum_{j=0}^{n} s_j a_{i,j}\right) x_{q_n,i}\right\| \\ &\le K\left(1+\delta_{q_n}\right)\max_i\left|\sum_{j=0}^{n} s_j a_{i,j}\right| \le K\left(1+\delta_{q_n}\right)\max_j |s_j|.\end{aligned}$$

Now let $m$ and scalars $(t_n)$ and $(s_{n,j})$ for $0 \le n \le m$, $0 \le j \le n$ be given. Then

$$\begin{aligned}\left\|\sum_{n=0}^{m}\left(t_n w_n' + \sum_{j=0}^{n} s_{n,j} z_{n,j}'\right)\right\| &\le \sum_{n=0}^{m} |t_n| \, \|w_n'\| + \sum_{n=0}^{m} \left\|\sum_{j=0}^{n} s_{n,j} z_{n,j}'\right\| \\ &\le \sum_{n=0}^{m} (1+\delta_n)\,|t_n| + K\sum_{n=0}^{m} (1+\delta_{q_n}) \max_j |s_{n,j}| \\ &\le \sum_{n=0}^{m} \lambda_n^{1/2}\left(|t_n| + K\max_j |s_{n,j}|\right).\end{aligned}$$

For the reverse inequality,

$$\begin{aligned}&\left\|\sum_{n=0}^{m}\left(t_n w_n' + \sum_{j=0}^{n} s_{n,j} z_{n,j}'\right)\right\| \\ &\ge \frac{1}{\|T\|}\left\|\sum_{n=0}^{m}\left(t_n T w_n' + \sum_{j=0}^{n} s_{n,j} T z_{n,j}'\right)\right\| \\ &\ge \left\|\sum_{n=0}^{m}\left(t_n d_n + \sum_{j=0}^{n} s_{n,j} e_{n,j}'\right)\right\| \\ &\qquad - \sum_{n=0}^{m}\left(|t_n|\,\|T w_n' - d_n\| + \sum_{j=0}^{n} |s_{n,j}|\,\left\|T z_{n,j}' - e_{n,j}\right\|\right) \\ &\ge \sum_{n=0}^{m}\left(|t_n| + \max_j |s_{n,j}|\right) - \sum_{n=0}^{m}\left(|t_n|\,\delta_n + (n+1)\,(2^{q_n}+1)\,\delta_{q_n}\max_j |s_{n,j}|\right) \\ &\ge \sum_{n=0}^{m}\left((1-\delta_n)\,|t_n| + (1-(n+1)\,(2^{q_n}+1)\,\delta_{q_n})\max_j |s_{n,j}|\right) \\ &\ge \sum_{n=0}^{m} \lambda_n^{-1/2}\left(|t_n| + \max_j |s_{n,j}|\right).\end{aligned}$$

Put $w_n = \lambda_n^{1/2} w_n'$ and $z_{n,j} = \lambda_n^{1/2} z_{n,j}'$. It is now routine to verify that the sequences $(w_n)_{n\ge 0}$ and $(z_{n,j})_{n\ge 0,\ 0\le j<n}$ satisfy the conclusions of the lemma. $\square$

The key to constructing a Banach space $Y$ and an operator $T : X \to Y$ to which Lemma 4 can be applied is the following technical lemma, which is an amplified version of [HS, Lemma 3]. In this lemma, the isometric copy of $L_1$ in $X^*$ is used to construct the sets $A_n$, which in turn readily yield an asymptotically isometric copy of $\ell_1$ in $X$. The fact that this copy of $L_1$ in $X^*$ is $K$-complemented is used to construct the $x_{n,i}$'s, which, together with the $A_n$'s, give the additional structure needed to produce a $(1,K)$-asymptotic copy of $(\ell_1 \oplus \sum_n \ell_\infty^n)_1$ in $X$. (So, if the parts of the argument involving the $x_{n,i}$'s are simply ignored, then an alternate proof of the following result [DGH, Theorem 2, (d)$\Rightarrow$(a)] can be given: *$X$ contains an asymptotically isometric copy of $\ell_1$ in $X$ if $L_1$ is isometric to a subspace of $X^*$.*)

Recall that for a function $f \in L_1$, the support of $f$, denoted *supp*$(f)$, is defined (almost everwhere) by *supp*$(f) = \bigcup_{n=1}^{\infty}\{t : |f(t)| > 1/n\}$.

**Lemma 5.** *Let $X$ be a Banach space and $u : L_1 \to X^*$ an isometry. Let $S : X^* \to L_1$ be a linear map with $\|S\| = K$ and $Su = id_{L_1}$. Let $(\delta_n)$ be a decreasing sequence with $\delta_0 < 1/2$ and $\lim_{n\to\infty} \delta_n = 0$.*

*Then there exist $(f_{n,i})$ in $L_1$ and $(x_{n,i})$ in $X$ for $(n,i) \in \mathcal{T}$ and finite subsets $A_n$ in $X$ for $n \geq 0$ such that*

(1) *$\|f_{n,i}\| = 1$ and $f_{n,i} \geq 0$ almost everywhere for all $(n,i) \in \mathcal{T}$.*
(2) *$|supp\,(f_{n,i}) \cap supp\,(f_{n,j})| = 0$ if $i \neq j$.*
(3) *If $a \in A_m$, then $uf_{n,i}\,(a) = uf_{m,j}\,(a)$ if $(n,i) \succcurlyeq (m,j)$.*
(4) *For each $n$, let $F_n = [\{f_{n,i} : i = 0, \cdots, 2^n - 1\}]$. Then $u^*\,(A_n)|_{F_n}$ is a $\delta_n$-dense set in the unit sphere of $F_n^*$ such that*
   (a) *$\|a\| \leq 1 + \delta_n$ for all $a \in A_n$, and*
   (b) *$u^*\,(A_n)|_{F_n}$ contains the norm one functionals $y^*_{n,i}$ in $F_n^*$ biorthogonal to the $f_{n,i}$'s.*
(5) *$\max_i |t_i| \leq \left\|\sum_{i=0}^{2^n-1} t_i x_{n,i}\right\| \leq K\,(1 + \delta_n) \max_i |t_i|$ for all $n$ and scalars $(t_i)_{i=0}^{2^n-1}$.*
(6) *If $n \geq m$,*

$$uf_{n,i}\,(x_{m,j}) = \begin{cases} 1 & \text{if} \quad (n,i) \succcurlyeq (m,j)\,, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* When $n = 0$, we let $f_{0,0} = \chi_{[0,1]}$. Choose $x_{0,0} \in X$, $\|x_{0,0}\| \leq 1 + \delta_0$ with $uf_{0,0}\,(x_{0,0}) = 1$. Put $A_0 = \{x_{0,0}\}$.

Now assume that $f_{n,i}$, $x_{n,i}$ and $A_n$ have been chosen for $n = 0, \cdots, k-1$ satisfying (1)-(6). For each $a \in \bigcup_{n=0}^{k-1} A_n$, define $\Phi_a = u^*\,(a)$. Also, for $n = 0, \cdots, k-1$, $j = 0, \cdots, 2^n - 1$ define $\Psi_{n,j} = u^*\,(x_{n,j})$. Since each $\Phi_a$ and each $\Psi_{n,j} \in L_\infty$ and the product of an $L_\infty$ function and an $L_1$ function is an $L_1$ function, it follows that

$$\left\{\Phi_a f_{k-1,i} : a \in \bigcup_{n=0}^{k-1} A_n,\ 0 \leq i \leq 2^{k-1} - 1\right\}$$

and

$$\left\{\Psi_{n,j} f_{k-1,i} : 0 \leq n \leq k-1,\ 0 \leq j \leq 2^n - 1,\ 0 \leq i \leq 2^{k-1} - 1\right\}$$

are finite sets of $L_1$ functions. So, by considering the real and imaginary parts of each of these functions, the Liapounoff convexity theorem (see [L] for a nice proof) yields the existence of measurable sets $O_i \subseteq supp\,(f_{k-1,i})$ for $0 \leq i \leq 2^{k-1} - 1$ such that

$$|O_i| = \frac{1}{2}\,|supp\,(f_{k-1,i})|\,,$$
$$\int_{O_i} f_{k-1,i}\,dt = \frac{1}{2}\int f_{k-1,i}\,dt,$$
$$\int_{O_i} \Phi_a f_{k-1,i}\,dt = \frac{1}{2}\int \Phi_a f_{k-1,i}\,dt \quad \text{for all } a \in \bigcup_{n=0}^{k-1} A_n,$$
$$\int_{O_i} \Psi_{n,j} f_{k-1,i}\,dt = \frac{1}{2}\int \Psi_{n,j} f_{k-1,i}\,dt \quad \text{for } 0 \leq n \leq k-1,\ 0 \leq j \leq 2^n - 1.$$

Define $f_{k,2i} = 2\chi_{O_i} f_{k-1,i}$ and $f_{k,2i+1} = 2\chi_{[supp(f_{k-1,i})\setminus O_i]} f_{k-1,i}$.

We next choose the set $A_k$. Let $i_k : F_k \to L_1$ denote the inclusion map. Then since $F_k$ is finite dimensional, $i_k^* \circ u^* : X^{**} \to F_k^*$ is weak*-norm continuous. Also, since $u^*$ is a quotient map and the unit ball of $X$ is weak* dense in the unit ball of

$X^{**}$, $\overline{i_k^* \circ u^*(B_X)} = B_{F_k^*}$. It is now standard that $i_k^* \circ u^*(1+\delta_k) B_X \supseteq B_{F_k^*}$. Let $C$ be a finite $\delta_k$-dense set in $B_{F_k^*}$ which contains the elements $y_{k,i}^*$, $0 \le i \le 2^k - 1$. For each $c \in C$, let $a_c \in (1+\delta_k) B_X$ with $i_k^* \circ u^*(a_c) = c$ and $A_k = \{a_c : c \in C\}$. It is immediate that the set $A_k$ satisfies property (4) of the lemma.

To pick the elements $x_{k,i}$ for $0 \le i \le 2^k - 1$, let $S_k : X^* \to F_k$ be defined by $S_k = P_k \circ S$, where $P_k$ is a norm one projection of $L_1$ onto $F_k$. Then by [JRZ, Lemma 3.1] there exists an operator $W_k : F_k^* \to X$ such that $\|W_k\| \le K(1+\delta_k)$ and such that $W_k^* \circ u \circ i_k = S_k \circ u \circ i_k = id_{F_k}$. (This is a local reflexivity argument.) Let $x_{k,i} = W_k\left(y_{k,i}^*\right)$ for $0 \le i \le 2^k - 1$.

We now verify that the $f_{k,i}$'s, $x_{k,i}$'s and the set $A_k$ in $X$ satisfy properties (1)-(6). Properties (1) and (2) are clear and (4) has already been checked. To establish (3), let $a \in A_m$ for some $m < k$. We show that $uf_{k,2i}(a) = uf_{m,j}(a)$ if $(k,2i) \succcurlyeq (m,j)$. (The proof for $(k,2i+1)$ will be the same.) Since $m < k$, $(k,2i) \succcurlyeq (m,j) \Leftrightarrow (k-1,i) \succcurlyeq (m,j)$.

$$\begin{aligned} uf_{k,2i}(a) = u^*a(f_{k,2i}) &= \int \Phi_a f_{k,2i}\, dt = 2\int_{O_i} \Phi_a f_{k-1,i}\, dt \\ &= \int \Phi_a f_{k-1,i}\, dt = u^*a(f_{k-1,i}) = uf_{m,j}(a) \end{aligned}$$

by the induction hypotheses and the construction of the set $O_i$.

To establish (5), we let scalars $t_0, t_1, \cdots, t_{2^k-1}$ be given. Then

$$\left\|\sum_{i=0}^{2^k-1} t_i x_{n,i}\right\| = \left\|\sum_{i=0}^{2^k-1} t_i W_k\left(y_{k,i}^*\right)\right\| \le \|W_k\| \left\|\sum_{i=0}^{2^k-1} t_i y_{k,i}^*\right\| \le K(1+\delta_n) \max_i |t_i|$$

since the $y_{k,i}^*$'s are isometrically equivalent to the usual basis of $\ell_\infty^{2^k}$. For the reverse inequality, first observe that if $0 \le i,j \le 2^k - 1$, then

$$uf_{k,i}(x_{k,j}) = uf_{k,i}\left(W_k\left(y_{k,j}^*\right)\right) = \left(W_k^* \circ u \circ i_k\right) f_{k,i}\left(y_{k,j}^*\right) = f_{k,i}\left(y_{k,j}^*\right) = \delta_{i,j}.$$

(This is (6) when $k = m$.) So,

$$\left\|\sum_{i=0}^{2^k-1} t_i x_{k,i}\right\| \ge \left|\sum_{i=0}^{2^k-1} t_i x_{k,i}(uf_{k,j})\right| = |t_j|$$

which gives $\left\|\sum_{i=0}^{2^k-1} t_i x_{k,i}\right\| \ge \max_j |t_j|$.

Finally, we establish the remainder of (6). If $k > m$, then

$$\begin{aligned} uf_{k,2i}(x_{m,j}) = u^*(x_{m,j}) f_{k,2i} &= 2\int_{O_i} \Psi_{m,j} f_{k-1,i}\, dt = \int \Psi_{m,j} f_{k-1,i}\, dt \\ &= \begin{cases} 1 & \text{if} \quad (k-1,i) \succcurlyeq (m,j), \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

by the construction and induction hypotheses. Now $(k,2i) \succcurlyeq (m,j) \Leftrightarrow (k-1,i) \succcurlyeq (m,j)$. (The argument for $(k,2i+1)$ is the same.) $\square$

*Proof of Theorem 1,* (2) $\Rightarrow$ (1). Let $u : L_1 \to X^*$ be an isometry and $S : X^* \to L_1$ an operator with $Su = id_{L_1}$ and $\|S\| = K$. Let $(\delta_n)$ be a strictly decreasing sequence such that that $\delta_n < 1/4^{n+1}$ for all $n$. Select $(f_{n,i})$, $(x_{n,i})$ and $A_n$ as in Lemma 5 for the sequence $(\delta_n/2)$. We shall construct a weak$^*$ compact subset $\Omega$

of $B_{X^*}$ such that the evaluation map $T : X \to C(\Omega)$ given by $Tx(x^*) = x^*(x)$ for $x \in X$, $x^* \in \Omega$, the $A_n$'s and the $x_{n,i}$'s satisfy the assumptions of Lemma 4.

For each $(n,i) \in \mathcal{T}$, define relatively weak* open subsets $O_{n,i}$ in $B_{X^*}$ by

$$\begin{aligned} O_{n,i} = & \bigcap_{m=0}^{n} \bigcap_{a \in A_m} \{x^* \in B_{X^*} : |x^*(a) - uf_{n,i}(a)| < \delta_n/2\} \\ & \cap \bigcap_{j=0}^{2^n-1} \{x^* \in B_{X^*} : |x^*(x_{n,j}) - uf_{n,i}(x_{n,j})| < \delta_n/2\}. \end{aligned}$$

Put $K_{n,i} = cl^*(O_{n,i})$ (where $cl^*$ denotes closure in the weak* topology). Then the following assertions are easy consequences of (3) and (6) of Lemma 5:

- $K_{n,i} \subseteq O_{m,j}$ if $n > m$ and $(n,i) \succeq (m,j)$, and
- $K_{n,i} \cap K_{n,j} = \emptyset$ if $i \neq j$.

Indeed, suppose that $(n,i) \succeq (m,j)$. Then it follows from (3) of Lemma 5 that if $a \in \bigcup_{k=0}^{m} A_k$, then $uf_{n,i}(a) = uf_{m,j}(a)$. Also, (6) of Lemma 5 yields that $uf_{n,i}(x_{m,k}) = uf_{m,j}(x_{m,k})$ for all $k$. These show that $f_{n,i} \in O_{m,j}$. If additionally $n > m$, and $x^* \in K_{n,i}$, $a \in \bigcup_{k=0}^{m} A_k$, then $|x^*(a) - uf_{n,i}(a)| \leq \delta_n/2 < \delta_m/2$ and $|x^*(x_{m,j}) - uf_{n,i}(x_{m,j})| \leq \delta_n/2 < \delta_m/2$, so $x^* \in O_{m,j}$. Thus $K_{n,i} \subseteq O_{m,j}$.

Now fix $n \geq 1$ and suppose that $i \neq j$. If $x^* \in K_{n,i} \cap K_{n,j}$, then

$$|x^*(x_{n,j})| = |x^*(x_{n,j}) - uf_{n,i}(x_{n,j})| \leq \frac{\delta_n}{2} < \frac{1}{2}$$

and

$$|x^*(x_{n,j}) - 1| = |x^*(x_{n,j}) - uf_{n,j}(x_{n,j})| < \frac{1}{2}$$

yield $\frac{1}{2} < |x^*(x_{n,j})| < \frac{1}{2}$, a contradiction.

Now put

$$\Omega = \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{2^n-1} K_{n,i} \qquad \text{and} \qquad \Omega_{n,i} = \Omega \cap K_{n,i}.$$

Since each $\Omega_{n,i}$ is a closed open subset of $\Omega$, $\Omega_{n,i} = \Omega_{n+1,2i} \cup \Omega_{n+1,2i+1}$ for all $(n,i)$ and $\Omega_{n,i} \cap \Omega_{n,j} = \emptyset$ if $i \neq j$, it follows that $\left[\left\{\chi_{\Omega_{n,i}} : (n,i) \in \mathcal{T}\right\}\right]$ is a subspace of $C(\Omega)$ isometric to $C(\Delta)$. For the rest of this proof, let us abuse notation slightly and assume that $\left[\left\{\chi_{\Omega_{n,i}} : (n,i) \in \mathcal{T}\right\}\right] = C(\Delta)$. Put $g_{n,i} = \chi_{\Omega_{n,i}}$ and $G_n = [\{g_{n,i} : 0 \leq i < 2^n\}]$.

We now show that the operator $T : X \to C(\Omega)$ given by $Tx(x^*) = x^*(x)$ for $x \in X$, $x^* \in \Omega$ satisfies (1)–(4) of Lemma 4. First, $\|a\| \leq 1 + \delta_n/2 \leq 1 + \delta_n$ if $a \in A_n$ by construction, so (1) is established. To establish (2), let $g \in G_n$, $\|g\| = 1$. Write $g = \sum_{j=0}^{2^n-1} s_j g_{n,j}$. By (4) of Lemma 5, there is an $a \in A_n$ with $|(uf_{n,j})\, a - s_j| < \delta_n/2$ for $0 \leq j \leq 2^n - 1$. Now observe that if $x^* \in O_{n,j}$, then

$$|x^*(a) - s_j| \leq |x^*(a) - uf_{n,j}(a)| + |uf_{n,j}(a) - s_j| < \delta_n$$

for all $j$. So, $\|Ta - g\| = \max_j \max_{x^* \in K_{n,j}} |x^*(a) - s_j| \leq \delta_n$, establishing (2). Similarly, if $x^* \in K_{n,j}$,

$$|uf_{n,j}(x_{n,i}) - x^*(x_{n,i})| \leq \delta_n$$

and so $\|Tx_{n,i} - g_{n,i}\| \leq \delta_n$, establishing (3). Property (4) follows from (5) of Lemma 5. $\square$

*Proof of Theorem 1,* (2) $\Rightarrow$ (5). If $X$ is separable, let $d$ be a metric giving the relative weak* topology on $B_{X^*}$. Now repeat the proof of (2)$\Rightarrow$(1) insuring in addition that the $d$–diameter of $O_{n,i} \leq 2^{-n}$ for each $i$. Then $\Omega$ is a compact, totally disconnected metric space, hence homeomorphic to $\Delta$, and $T : X \to C(\Omega)$ satisfies the assumptions of Lemma 2. So, $T^*\left(C(\Omega)^*\right)$ is isometric to a $K$–complemented subspace of $X^*$. $\square$

Finally, we turn to the assertions involving $\ell_1(\Gamma)$ (i.e. (4) of Theorem 1). The proof that (1)-(3)$\Rightarrow$(4) is essentially given in [HS] (see also [DGH]). The proof that (4)$\Rightarrow$(1) is similar to that of Theorem 1, (iv)$\Rightarrow$(i), in [HS], which in turn used techniques introduced by Stegall [S] and Rosenthal [R]. For the sake of completeness, we state without proof the crucial lemma, which can be viewed as the "$\ell_1(\Gamma)$ version" of Lemma 5. Then (4)$\Rightarrow$(1) is proved by mimicking the proof of (2)$\Rightarrow$(1) above.

In this lemma we identify points $\gamma \in \Gamma$ with points $e_\gamma$ in $\ell_1(\Gamma)$ via $e_\gamma(\tau) = 1$ if $\gamma = \tau$ and 0 otherwise.

**Lemma 6.** *Let $X$ be a Banach space and $u : \ell_1(\Gamma) \to X^*$ an isometry such that $u(\Gamma)$ is weak* dense in itself. Let $S : X^* \to u(\ell_1(\Gamma))$ be a linear map with $\|S\| = K$ and $Su = id_{\ell_1(\Gamma)}$. Let $(\delta_n)$ be a decreasing sequence with $\delta_0 < 1/2$ and $\lim_{n\to\infty} \delta_n = 0$.*

*Then there exist $(\gamma_{n,i})$ in $\ell_1(\Gamma)$, $(x_{n,i})$ in $X$ and relatively weak* open sets $O_{n,i}$ in $B_{X^*}$ for $(n,i) \in \mathcal{T}$, and finite subsets $A_n$ in $X$ for $n \geq 0$ such that*

(1) *If $F_n = [\{\gamma_{n,i} : i = 0, \cdots, 2^n - 1\}]$, then $u^*(A_n)|_{F_n}$ is a $\delta_n$-dense set in the unit sphere of $F_n^*$ such that*
  (a) *$\|a\| \leq 1 + \delta_n$ for all $a \in A_n$, and*
  (b) *$u^*(A_n)|_{F_n}$ contains the norm one functionals $y_{n,i}^*$ in $F_n^*$ biorthogonal to the $\gamma_{n,i}$'s.*

(2)

$$O_{n,i} \subseteq \bigcap_{a \in A_n} \{x^* \in B_{X^*} : |x^*(a) - u\gamma_{n,i}(a)| < \delta_n\}$$
$$\cap \bigcap_{j=0}^{2^n-1} \{x^* \in B_{X^*} : |x^*(x_{n,j}) - u\gamma_{n,i}(x_{n,j})| < \delta_n\}$$

*and $u\gamma_{n,i} \in O_{n,i}$.*

(3) *$O_{n+1,2i} \cup O_{n+1,2i+1} \subseteq O_{n,i}$ for all $(n,i) \in \mathcal{T}$ .*

(4) *$\max_i |t_i| \leq \left\|\sum_{i=0}^{2^n-1} t_i x_{n,i}\right\| \leq K(1+\delta_n)\max_i |t_i|$ for all $n$ and scalars $(t_i)_{i=0}^{2^n-1}$.*

(5) *$u\gamma_{n,i}(x_{n,j}) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{otherwise.} \end{cases}$*

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