ON ANTOSIK’S LEMMA AND THE ANTOSIK-MIKUSINSKI BASIC MATRIX THEOREM

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Abstract. That Antosik’s Lemma is not a special case of the Antosik-Mikusinski Basic Matrix Theorem will be shown and, an equivalent form of the Antosik-Mikusinski Basic Matrix Theorem will also be presented in this paper.

In [1] and [2], Antosik proved two results which are called the Antosik-Mikusinski Basic Matrix Theorem and Antosik’s Lemma, respectively. The theorem and lemma have been proven to be quite effective in treating various topics in Functional Analysis and Set Function Theory [1]–[6]. In [6, 2.2], Swartz thought that Antosik’s Lemma is a special case of the Antosik-Mikusinski Basic Matrix Theorem. In [7], Li Ronglu presented the Uniform Convergent Principle. Now, we will show that Swartz’s conclusion is incorrect and, the Uniform Convergent Principle is an equivalent form of the Antosik-Mikusinski Basic Matrix Theorem.

Lemma 1 (Antosik). Let $G$ be an abelian topological group and $x_{ij} \in G$ for $i, j \in \mathbb{N}$. Suppose that each strictly increasing sequence $\{m_i\}$ in $\mathbb{N}$ has a subsequence $\{n_i\}$ such that

(i) $\lim_{i} x_{n_i n_j} = 0$ for all $j \in \mathbb{N}$, and
(ii) $\lim_{i} \sum_{j=1}^{\infty} x_{n_i n_j} = 0$.

Then $\lim_{i} x_{ii} = 0$.

Antosik observed that assumption (i) can be dropped if $G$ is a locally convex space and posed the problem ([2]) of whether (i) can also be dropped in general. In [8], Weber solved this problem showing that assumption (i) is in fact superfluous.

A direct consequence of Lemma 1 is as follows:

Corollary 1. Let $G$ be an abelian topological group and $z_{ij} \in G$ for $i, j \in \mathbb{N}$. Suppose that

(I) $\lim_{i} z_{ij} = 0$ for each $j \in \mathbb{N}$, and
(II) for each strictly increasing sequence of positive integers $\{m_j\}$ in $\mathbb{N}$ there is a subsequence $\{n_j\}$ such that $\lim_{i} \sum_{j=1}^{\infty} z_{n_j} = 0$.

Then $\lim_{i} z_{ii} = 0$. 
Now, we show that the following Antosik-Mikusinski Basic Matrix Theorem can be obtained from Corollary 1.

**Theorem 1 (Antosik-Mikusinski).** Let $G$ be an abelian topological group, $x_{ij} \in G$ for $i, j \in \mathbb{N}$. Suppose

(I) $\lim_{i} x_{ij} = x_{j}$ exists for each $j$, and

(II) for each strictly increasing sequence of positive integers $\{m_j\}$ there is a subsequence $\{n_j\}$ such that $\sum_{i=1}^{\infty} x_{im_i}$ is a Cauchy sequence.

Then $\lim_{i} x_{ij} = x_{j}$ uniformly for $j \in \mathbb{N}$. In particular, $\lim_{i} x_{ii} = 0$.

**Proof.** If the conclusion fails, then there exist a closed, symmetric neighbourhood $V_0$ of 0 in $G$ and strictly increasing sequences of positive integers $\{p_k\}$ and $\{q_k\}$ such that

$$x_{p_k q_k} - x_{q_k} \notin V_0$$

for all $k$. Pick a closed, symmetric neighbourhood $V_1$ of 0 such that $V_1 + V_1 \subseteq V_0$. Note that $x_{p_i q_i} - x_{q_i} \rightarrow 0$ as $i \rightarrow \infty$ for $j \in \mathbb{N}$. Therefore, there exists a subsequence $\{m_i\}$ of $\{p_i\}$ such that

$$x_{m_i q_i} - x_{q_i} \in V_1$$

for $i \in \mathbb{N}$. We have

$$x_{p_i q_i} - x_{q_i} = (x_{p_i q_i} - x_{m_i q_i}) + (x_{m_i q_i} - x_{q_i}).$$

Consider the matrix $(x_{p_i q_j} - x_{m_i q_j})$ and note that the matrix satisfies conditions of Corollary 1. Consequently,

$$x_{p_i q_i} - x_{m_i q_i} \rightarrow 0$$

as $i \rightarrow \infty$, and

$$x_{p_i q_i} - x_{m_i q_i} \in V_1$$

for sufficiently large $i$. Hence, by (3) and (2)

$$x_{p_i q_i} - x_{q_i} \in V_1 + V_1 \subseteq V_0$$

for sufficiently large $i$. Which contradicts (1) and we established the result.

Thus, we have Lemma 1 $\Rightarrow$ Corollary 1 $\iff$ Theorem 1. Now, we show that Lemma 1 is not a special case of Theorem 1.

**Example 1.** Consider the matrix $(x_{ij})$ such that $x_{ij} = 0$ if $i \neq j+1$ and $x_{ii-1} = 1$. Then $(x_{ij})$ satisfies the assumptions of Lemma 1. If Lemma 1 was a special case of Theorem 1, then the columns should converge to 0 uniformly. But they do not converge to 0 uniformly.

In this way, we have corrected the incorrect statement in [6, 2.2]).

Now, we use Theorem 1 to prove Theorem 2 below.

**Theorem 2 (Uniform Convergent Principle).** Let $G$ be an abelian topological group and let $\Omega$ be a sequentially compact topological space. Let $\{f_\omega\}$ be a sequence of sequentially continuous $G$-valued functions defined on $\Omega$. If each strictly increasing sequence $\{m_j\}$ in $\mathbb{N}$ has a subsequence $\{n_j\}$ such that for each $\omega \in \Omega$, the series $\sum_{j} f_{n_j}(\omega)$ converges and $\sum_{j} f_{n_j} : \Omega \rightarrow G$ is sequentially continuous, then $\lim_{j} f_j(\omega) = 0$ uniformly with respect to $\omega \in \Omega$. 

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Proof. We will show that \( f_j(\omega) \to 0 \) uniformly for \( \omega \) in \( \Omega \) or, equivalently, for each sequence \( \{\omega_i\} \) in \( \Omega \),

\[
f_i(\omega_i) \to 0
\]

as \( i \to \infty \). Let \( \{\omega_i\} \) be a sequence of \( \Omega \). Since \( \Omega \) is sequentially compact, there exists a subsequence \( \{\omega_{n_i}\} \) of \( \{\omega_i\} \) and \( \omega_0 \) such that \( \omega_{n_i} \to \omega_0 \). Consider the matrix \( (f_{n_j}(\omega_{n_i}) - f_{n_j}(\omega_0)) \). Note that the matrix satisfies conditions of Theorem 1. Therefore,

\[
f_{n_i}(\omega_{n_i}) - f_{n_i}(\omega_0) \to 0.
\]

Since \( f_{n_i}(\omega_0) \to 0 \), we get \( f_{n_i}(\omega_{n_i}) \to 0 \). Hence, by assumption conditions, (4) holds. Thus, Theorem 1 \( \implies \) Theorem 2.

\[\square\]

Theorem 3. Theorem 2 is an equivalent form of Theorem 1.

Proof. Since Corollary 1 \( \iff \) Theorem 1 \( \implies \) Theorem 2, we only need to show that Theorem 2 \( \implies \) Corollary 1.

Indeed, assume the conditions of Corollary 1 are satisfied. Let \( \Omega = \{\frac{1}{n}, 0\}_{n=1}^{\infty} \), for \( x, y \in \Omega \), put \( d(x, y) = |x - y| \). Then \( (\Omega, d) \) is a sequentially compact topological space. Let \( f_j : \Omega \to G \) satisfy that if \( \omega = \frac{1}{n}, f_j(\omega) = z_{ij} \); if \( \omega = 0, f_j(\omega) = 0 \). It is easily shown that each \( f_j \) is continuous, and for each strictly increasing sequence \( \{n_j\} \) in \( \mathbb{N} \) has subsequence \( \{n_{j_i}\} \) such that for \( i \in \mathbb{N} \), the series \( \sum_j z_{in_j} \) is convergent and \( \lim_i \sum_j z_{in_j} = 0 \). Thus, for each \( \omega \in \Omega \), the series \( \sum_j f_{n_j}(\omega) \) is convergent and \( \sum_j f_{n_j} : \Omega \to G \) is continuous. It follows from Theorem 2 that \( \lim_j f_j(\omega) = 0 \) uniformly with respect to \( \omega \in \Omega \). In particular, \( \lim_j f_j(\frac{1}{n}) = \lim_j z_{jj} = 0 \). So Corollary 1 is proved. \[\square\]

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References


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