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CRITERIA FOR POSITIVELY QUADRATICALLY HYPONORMAL WEIGHTED SHIFTS

JU YOUN BAE, IL BONG JUNG, AND GEORGE R. EXNER

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Abstract. For bounded linear operators on Hilbert space, positive quadratic hyponormality is a property strictly between subnormality and hyponormality and which is of use in exploring the gap between these more familiar properties. Recently several related positively quadratically hyponormal weighted shifts have been constructed. In this note we establish general criteria for the positive quadratic hyponormality of weighted shifts which easily yield the results for these examples and other such weighted shifts.

1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $A, B \in \mathcal{L}(\mathcal{H})$, we let $[A, B] = AB - BA$. Recall that an n-tuple $T = (T_1, \ldots, T_n)$ of operators on $\mathcal{H}$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For $k \geq 1$ and $T \in \mathcal{L}(\mathcal{H})$, T is *k-hyponormal* if $\left(I, T, \ldots, T^k\right)$ is hyponormal. Additionally, T is *weakly k-hyponormal* if $p\,(T)$ is hyponormal for every polynomial p of degree $\leq k$. It is easy to show that k-hyponormality implies weak k-hyponormality. In particular, weak 2-hyponormality, often referred to as quadratic hyponormality, was first considered in detail by Curto in [4]. He studied the positively quadratically hyponormal shifts (which will be defined below) to determine the gap between hyponormal and subnormal operators, and he proved that for a weight sequence $\alpha(x) : \sqrt{x}, \sqrt{(n+1)/(n+2)}$ $(n \geq 1)$ with a variable x, the weight shift $W_{\alpha(x)}$ is (positively) quadratically hyponormal if and only if $0 < x \leq \frac{2}{3}$, which is an example to distinguish the two classes of 2-hyponormal and quadratically hyponormal operators. In further work several positively quadratically hyponormal weighted shifts have been discussed for their own purposes in [2], [3] and [9]. Through such examples, one knows that the detection of positive quadratic hyponormality for weighted shifts is a difficult job.

In this note we establish criteria which can easily detect the positive quadratic hyponormality of weighted shifts.

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Some calculations in Sections 3 and 4 were obtained through computer experiments using the software tool *Mathematica* [11].

2. Preliminaries

Notice that an operator T is quadratically hyponormal if $T+sT^2$ is hyponormal for every $s\in\mathbb{C}$. Let $\alpha:\alpha_0,\alpha_1,\ldots$ be a weight sequence of positive real numbers. Let W_α be a hyponormal weighted shift with a weight sequence α. Let $\{e_i\}_{i=0}^\infty$ be an orthonormal basis for $\mathcal{H}$. We may consider $\mathcal{H}$ as $\ell^2(\mathbb{Z}_+)$, where $\ell_2:=\ell^2(\mathbb{Z}_+)$ is the set of square summable sequences in $\mathbb{C}$. Let P_n be the orthogonal projection on $\bigvee_{i=0}^n\{e_i\}$. For $s\in\mathbb{C}$, let $D(s):=[(W_\alpha+sW_\alpha^2)^*,W_\alpha+sW_\alpha^2]$. For $s\in\mathbb{C}$, let

$$\begin{aligned} D_n(s) &= P_n[(W_\alpha+sW_\alpha^2)^*,W_\alpha+sW_\alpha^2]|P_n\ell^2(\mathbb{Z}_+)\\ &= \begin{bmatrix} q_0 & r_0 & 0 & \cdots & 0 & 0\\ r_0 & q_1 & r_1 & \cdots & 0 & 0\\ 0 & r_1 & q_2 & \ddots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & q_{n-1} & r_{n-1}\\ 0 & 0 & 0 & \cdots & r_{n-1} & q_n \end{bmatrix}, \end{aligned}$$

where $q_k:=u_k+|s|^2v_k$, $r_k:=s\sqrt{w_k}$, $u_k:=\alpha_k^2-\alpha_{k-1}^2$, $v_k:=\alpha_k^2\alpha_{k+1}^2-\alpha_{k-2}^2\alpha_{k-1}^2$, $w_k:=\alpha_k^2(\alpha_{k+1}^2-\alpha_{k-1}^2)^2$ $(k\geq 0)$, and $\alpha_{-1}=\alpha_{-2}:=0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s)\geq 0$ for every $s\in\mathbb{C}$ and every $n\geq 0$. To detect this, we consider $d_n(\cdot):=\det(D_n(\cdot))$; it follows from [7, p.390] that $d_0=q_0$, $d_1=q_0q_1-|r_0|^2$, and $d_{n+2}=q_{n+2}d_{n+1}-|r_{n+1}|^2d_n$ $(n\geq 0)$, and that d_n is actually a polynomial in $t:=|s|^2$ of degree $n+1$, with McLaurin expansion

$$d_n(t):=\sum_{i=0}^{n+1}c(n,i)\,t^i.$$

It follows from [4] that for $n\geq 0$ and $1\leq i\leq n+1$,
(1) $c(n,0)=u_0u_1\cdots u_n\geq 0$,
(2) $c(n,n+1)=v_0v_1\cdots v_n\geq 0$,
(3) $c(1,1)=u_1v_0+u_0v_1-w_0\geq 0$,
(4) $c(n,i)=u_nc(n-1,i)+v_nc(n-1,i-1)-w_{n-1}c(n-2,i-1)$ $(n\geq 2)$,
(5) $c(n,1)=u_nc(n-1,1)+(v_nu_{n-1}-w_{n-1})u_0\cdots u_{n-2}$ $(n\geq 2)$.

To detect the positivity of $d_n(t)$, the following concept was introduced.

Definition 2.1 ([7]). Let $\alpha:\alpha_0,\alpha_1,\ldots$ be a weight sequence. We say that W_α is *positively quadratically hyponormal* if $c(n,i)\geq 0$ for all $n,i\geq 0$ with $0\leq i\leq n+1$.

It is obvious that positive quadratic hyponormality is stronger than quadratic hyponormality. In particular, it is known that quadratic hyponormality need not imply positive quadratic hyponormality (see [2] or [9]). If $c(n,n+1)=0$ for some $n\geq 2$, then by (2) and [4, Th. 2] (or [1]), W_α is flat ($\alpha_1=\alpha_2=\ldots$). Thus to avoid the trivial case, we usually assume that $c(n,n+1)>0$ $(n\geq 2)$ when we consider positive quadratic hyponormality. Note that $c(0,1)$ and $c(1,2)$ are positive.

3. Criteria

We start the work with the following definitions.

Definition 3.1. Let $\alpha : \alpha_0, \alpha_1, \dots$ be a weight sequence.
(1) A weighted shift W_α has *property* $B(k)$ if $u_{n+1}v_n \geq w_n \quad (n \geq k)$.
(2) A weighted shift W_α has *property* $C(k)$ if $v_{n+1}u_n \geq w_n \quad (n \geq k)$.

We will discuss examples satisfying property $B(k)$ or $C(l)$ in Section 4.

Theorem 3.2. *Let W_α be a weighted shift with property $C(l)$ for some $l \geq 1$. If for some $m \in \mathbb{N}$, $c(l+j, j+1) \geq 0$ for $j = 0, 1, \dots, m$, then $c(l+i+j, j+1) \geq 0$ for $j = 0, 1, \dots, m$ and all $i \in \mathbb{N}$.*

Proof. We first let $\eta(1,1) = u_0 v_1 - w_0$ and

$$\eta\,(n,i) = v_n c\,(n-1, i-1) - w_{n-1} c\,(n-2, i-1) \qquad (n \geq 2, 1 \leq i \leq n).$$

Then by (3) and (4), we have

$$c(n,i) = u_n c(n-1,i) + \eta\,(n,i) \qquad (n \geq 1, \quad 1 \leq i \leq n). \tag{6}$$

Furthermore, for $n \geq 2$, $\eta(n,1) = u_{n-2} \cdots u_0 (v_n u_{n-1} - w_{n-1})$, and for $n \geq 3$ and $2 \leq i \leq n$,

$$\begin{aligned}
\eta\,(n,i) &= v_n c(n-1,i-1) - w_{n-1} c(n-2,i-1) \\
&= v_n \left[u_{n-1} c\,(n-2,i-1) + v_{n-1} c\,(n-2,i-2) \right. \\
&\qquad \left. -\ w_{n-2} c\,(n-3,i-2)\right] - w_{n-1} c\,(n-2,i-1) \\
&= (v_n u_{n-1} - w_{n-1})\, c\,(n-2,i-1) \\
&\qquad +\ v_n \left[v_{n-1} c\,(n-2,i-2) - w_{n-2} c\,(n-3,i-2)\right] \\
&= (v_n u_{n-1} - w_{n-1})\, c\,(n-2,i-1) + v_n \eta\,(n-1,i-1).
\end{aligned} \tag{7}$$

Claim (†): If for some k, $0 \leq k \leq m$, $c(l+j, j+1) \geq 0, \quad j = 0, \dots, k$, then

$$\begin{cases} c(l+i+j, j+1) \geq 0, \quad j = 0, \dots, k \quad \text{and} \quad i \in \mathbb{N}, \\ \qquad \eta(n, k+1) \geq 0 \qquad (n \geq l+k+1). \end{cases}$$

We will prove the claim by mathematical induction on k. For $k = 0$, we assume first that $c(l,1) \geq 0$. Then by (6) we have

$$c(l+i,1) = u_{l+i} c(l+i-1,1) + \eta\,(l+i,1) \qquad (i \in \mathbb{N}). \tag{8}$$

According to property $C(l)$ and (1), we have that for $n \geq l+1$ $(l \geq 1)$

$$\eta(n,1) = u_{n-2} \cdots u_0 (v_n u_{n-1} - w_{n-1}) \geq 0. \tag{9}$$

Hence by (8) and (9), we have $c(l+1,1) \geq 0$. Continuing this process with (8), we may obtain $c(l+j,1) \geq 0$ $(j \in \mathbb{N})$.

Assume that if $c(l+j, j+1) \geq 0, \quad j = 0, \dots, p-1$, where $p \leq m$, then

$$\begin{cases} c(l+i+j, j+1) \geq 0, \quad j = 0, \dots, p-1 \quad \text{and} \quad i \in \mathbb{N}, \\ \qquad \eta(n,p) \geq 0 \qquad (n \geq l+p). \end{cases}$$

We now prove the claim (†) in the case of $k = p$. To do so, we make the assumption that $c(l+j, j+1) \geq 0$, $j = 0, \dots, p$. Applying the induction hypothesis, we have

$$\begin{cases} c(l+i+j, j+1) \geq 0, \quad j = 0, \dots, p-1 \quad \text{and} \quad i \in \mathbb{N}, \\ \qquad \eta(n,p) \geq 0 \quad (n \geq l+p). \end{cases} \tag{10}$$

Furthermore, by (6) and (7) we have

$$c(l+i+p,p+1) = u_{l+i+p}c(l+i+p-1,p+1) + \eta(l+i+p,p+1)$$

and

$$\eta\,(l+i+p,p+1) = (v_{l+i+p}u_{l+i+p-1} - w_{l+i+p-1})\,c\,(l+i+p-2,p) + v_{l+i+p}\eta\,(l+i+p-1,p)\,. \tag{11}$$

By (10), (11) and the assumption $C(l)$, $\eta\,(l+i+p,p+1) \geq 0$ $(i \in \mathbb{N})$. Hence we obtain easily that $c(l+i+p,p+1) \geq 0$ $(i \in \mathbb{N})$. $\square$

Corollary 3.3. *Let W_α be a weighted shift with property $C(l)$ for some $l \geq 2$. If $\eta\,(l,l) \geq 0$, then $c(l+i,l+i) \geq 0$ for all $i \in \mathbb{N}\cup\{0\}$. In particular, if $l = 2$, then W_α is positively quadratically hyponormal if and only if $c(n+1,n) \geq 0$ for all $n \in \mathbb{N}$.*

Proof. Since $c(n,n) = u_n c(n-1,n) + \eta\,(n,n)$, to show that $c(n,n) \geq 0$ for all $n \geq l$, by (2) it is sufficient to show that $\eta(n,n) \geq 0$. According to (7), we have

$$\eta\,(n,n) = (v_n u_{n-1} - w_{n-1})\,c\,(n-2,n-1) + v_n\eta\,(n-1,n-1) \qquad (n \geq 3).$$

Since $\eta(l,l) \geq 0$ and $v_{n+1}u_n - w_n \geq 0$ for all $n \geq l$, by (2) we have

$$\eta\,(l+1,l+1) = (v_{l+1}u_l - w_l)\,c\,(l-1,l) + v_{l+1}\eta\,(l,l) \geq 0. \tag{12}$$

Continuing this process, we obtain $\eta(n,n) \geq 0$ $(n \geq l)$. By (2), it is easy to show that $c(n,n) \geq 0$ for all $n \geq l$. Furthermore, since $\eta(2,2) \geq 0$ (*indeed*, put $\alpha_0 := 1$, $\alpha_1 = 1+h$, $\alpha_2 = \alpha_1 + k$, and $\alpha_3 = \alpha_2 + p$, where $h,k,p \geq 0$, and compute $\eta(2,2)$ directly), the second statement comes immediately from (12) and Theorem 3.2. $\square$

Since $c(1,1) \geq 0$ and $\eta(2,2) \geq 0$, the following corollary comes immediately from Theorem 3.2 and Corollary 3.3.

Corollary 3.4. *Let W_α be a weighted shift with property $C(1)$. Then W_α is positively quadratically hyponormal.*

Recall that a weighted shift W_α has property $C(1)$ if and only if W_α is 2-hyponormal ([4, Corollary 5]). Hence we immediately have the following corollary.

Corollary 3.5 ([10])**.** *If W_α is 2-hyponormal, then W_α is positively quadratically hyponormal.*

We now turn to consideration of property $B(k)$.

Lemma 3.6. *Let W_α be a weighted shift with property $B(k)$ for some $k \geq 2$. If $c(m,k) \geq 0$ for all $m \geq k$, then $c(n,i) \geq 0$ for all $i \geq k+1$ and all $n \geq i-1$.*

Proof. Claim I: We first claim that

$$c(m,k) \geq 0 \quad (m \geq k) \Longrightarrow c(n,k+1) \geq 0 \quad (n \geq k+1). \tag{13}$$

Let

$$\rho\,(n,i) = u_n c\,(n-1,i) - w_{n-1}c\,(n-2,i-1) \qquad (n \geq 2,\ 1 \leq i \leq n)\,. \tag{14}$$

Then, by (2),

$$\rho\,(n,n) = v_0 \cdots v_{n-2}(u_n v_{n-1} - w_{n-1}) \qquad (n \geq 2). \tag{15}$$

According to property $B(k)$, we have

$$\rho(n,n) \geq 0 \qquad (n \geq k+1). \tag{16}$$

By (4) and (14), we have that for $n \geq 3$,

$$\begin{aligned}\rho(n,i) &= u_n\left[u_{n-1}c(n-2,i)+v_{n-1}c(n-2,i-1)\right.\\&\quad\left.-\,w_{n-2}c(n-3,i-1)\right]-w_{n-1}c(n-2,i-1)\\&=(u_nv_{n-1}-w_{n-1})\,c(n-2,i-1)\\&\quad+\,u_n\left[u_{n-1}c(n-2,i)-w_{n-2}c(n-3,i-1)\right]\\&=(u_nv_{n-1}-w_{n-1})\,c(n-2,i-1)+u_n\rho(n-1,i).\end{aligned}$$

Since

$$\rho(n,k+1)=(u_nv_{n-1}-w_{n-1})\,c(n-2,k)+u_n\rho(n-1,k+1) \qquad (n\geq 3),$$

by (16) and the hypothesis on the k-th coefficients in (13), $\rho(k+2,k+1)\geq 0$. Similarly, we obtain $\rho(k+3,k+1)\geq 0$, and continuing recursively we obtain

$$\rho(n,k+1)\geq 0 \qquad (n\geq k+1). \tag{17}$$

Furthermore, we have

$$\begin{aligned}c(n,i)&=v_nc(n-1,i-1)+u_nc(n-1,i)-w_{n-1}c(n-2,i-1)\\&=v_nc(n-1,i-1)+\rho(n,i).\end{aligned} \tag{18}$$

By (16) and (18), we have that

$$c(k+1,k+1)=v_{k+1}c(k,k)+\rho(k+1,k+1)\geq 0.$$

Similarly, by (17) and (18) we obtain $c(k+2,k+1)\geq 0$. We continue the recursive process to obtain $c(k+i,k+1)\geq 0$ $(i\geq 1)$, which proves the claim.

Claim II: $c(m,k)\geq 0 \quad (m\geq k{+}1) \ \Rightarrow c(n,k+2)\geq 0 \quad (n\geq k{+}2)$.

Since property $B(k)$ implies property $B(k{+}1)$, we may repeat the proof of Claim I to prove Claim II.

Continuing the recursive process, we may obtain the lemma. □

The following comes immediately from Theorem 3.2 and Lemma 3.6.

Theorem 3.7. *Let W_α be a weighted shift with properties $C(l)$ and $B(k)$ for some $l\geq 1$, $k\geq 2$. Then W_α is positively quadratically hyponormal if and only if $c(n+i-1,i)\geq 0$, for $n=1,2,\ldots,l$ and $i=1,2,\ldots,k$, which is equivalent to $c(n,i)\geq 0$ for all $(n,i)\in\mathcal{D}\cap(\mathbb{N}\times\mathbb{N})$, where $\mathcal{D}$ is in Figure 1.*

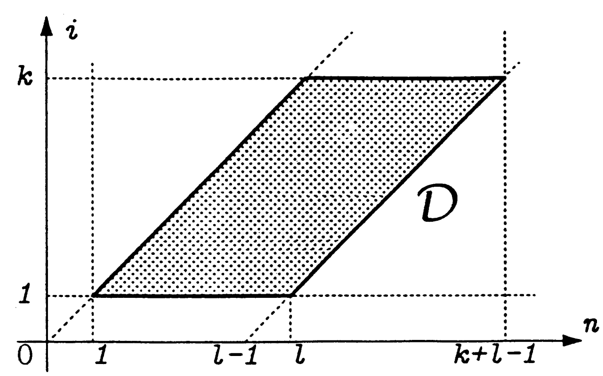

FIGURE 1.

Lemma 3.8. *If W_α has property $B(n+1)$ for some $n \geq 1$, then W_α has property $C(n)$.*

Proof. Without loss of generality, for any $j \geq n$ we may assume that $\alpha_{j-1} = 1$, $\alpha_j = \sqrt{1+h}$, $\alpha_{j+1} = \sqrt{1+h+k}$, $\alpha_{j+2} = \sqrt{1+h+k+p}$, where $h, k, p \geq 0$. Since W_α has property $B(n+1)$, by direct computation we have

$$u_{n+2}v_{n+1} - w_{n+1} = -k^2 - hk^2 - k^3 + hp + h^2p - k^2p \geq 0,$$

which implies that

$$v_{n+1}u_n - w_n = -k^2 + hp + h^2p + hkp \geq hk^2 + k^3 + k^2p + hkp \geq 0.$$

Thus the proof is complete. □

The converse implication is not always true (see Examples 4.1 and 4.2).

The following theorem is immediate from Theorem 3.7 and Lemma 3.8.

Theorem 3.9. *Let W_α be a weighted shift with property $B(k)$ for some $k \geq 2$. Then W_α is positively quadratically hyponormal if and only if $c(n+i-1, i) \geq 0$, for $n = 1, 2, \ldots, k-1$ and $i = 1, 2, \ldots, k$.*

The following is immediate from Corollary 3.3 and Theorem 3.9.

Corollary 3.10. *Let W_α be a weighted shift with $\alpha_0 = \alpha_1$. If W_α has property $B(3)$, then W_α is positively quadratically hyponormal if and only if $c(3,2) \geq 0$ and $c(4,3) \geq 0$.*

Corollary 3.11. *Let k be any fixed number in $\mathbb{N}$ and let $\alpha_n = \sqrt{\frac{n+k}{n+k+1}}$ $(n \geq 1)$. Let $\widehat{\alpha} : \alpha_1, \alpha_1, \alpha_2, \alpha_3, \ldots$. Then $W_{\widehat{\alpha}}$ is positively quadratically hyponormal.*

Proof. By direct computation, $u_{n+1}v_n - w_n = 0$ $(n \geq 3)$, so $W_{\widehat{\alpha}}$ has property $B(3)$. Since

$$c(3,2) = \frac{k(k+1)^2}{(k+2)^4(k+3)^2(k+4)} > 0$$

and

$$c(4,3) = \frac{(k+1)^2(3k^2+13k-6)}{(k+2)^4(k+3)^2(k+4)^2(k+5)(k+6)} > 0,$$

by Corollary 3.10 we have the result. □

Corollary 3.12. *Let k be any fixed number in $\mathbb{N}$ and let $\alpha_n = \sqrt{\frac{kn+1}{kn+2}}$ $(n \geq 1)$. Let $\widehat{\alpha} : \alpha_1, \alpha_1, \alpha_2, \alpha_3, \ldots$. Then $W_{\widehat{\alpha}}$ is positively quadratically hyponormal.*

Proof. By direct computation, we have that for $n \geq 3$,

$$u_{n+1}v_n - w_n = \frac{2k^3(k-1)}{(kn+2)^2(kn-2k+2)(kn-k+2)^2(kn+k+2)} \geq 0,$$

so $W_{\widehat{\alpha}}$ has property $B(3)$. Since

$$c(3,2) = \frac{k^2(2k+1)(k^2+k+2)}{8(k+1)(k+2)^5(3k+2)} > 0$$

and

$$c(4,3) = \frac{k^3(39k^4+12k^3+73k^2+64k+12)}{16(k+1)(k+2)^5(3k+2)^3(5k+2)} > 0,$$

by Corollary 3.10 we have the corollary. □

4. Examples

The following first two examples show that properties $B(k)$ and $C(l)$ are different.

Example 4.1. Let α be a sequence with

$$\alpha_n = \begin{cases} \sqrt{\frac{3n+1}{3n+2}} & \text{if} \quad n \neq k, \\ \sqrt{x} & \text{if} \quad n = k \end{cases}$$

$(n \geq 0)$. Then we have the following:

(i) If $k \geq 1$, then

$$W_\alpha \text{ has property } B(k+2) \Longleftrightarrow \frac{3k-2}{3k-1} \leq x \leq \frac{27k^3+144k^2+231k+98}{27k^3+153k^2+276k+160}.$$

(ii) If $k = 0$, then W_α has property $B(2) \Longleftrightarrow 0 < x \leq \frac{49}{80}$.

(iii) If $k \geq 1$, then

$$W_\alpha \text{ has property } C(k+1) \Longleftrightarrow \frac{3k-2}{3k-1} \leq x \leq \frac{27k^3+144k^2+249k+140}{27k^3+153k^2+294k+208}.$$

(iv) If $k = 0$, then W_α has property $C(1) \Longleftrightarrow 0 < x \leq \frac{35}{52}$.

Let $\alpha : \alpha_0, \alpha_1, \dots, \alpha_{k-2}, (\alpha_{k-1}, \alpha_k, \alpha_{k+1})\widehat{\ }$ with $0 < \alpha_{k-1} < \alpha_k < \alpha_{k+1}$ $(k \geq 1)$, where $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})\widehat{\ }$ is a subnormal completion (cf [6], [7] or [8]). Then it follows from [8, Lemma 2.1] that

$$v_{n+1} = \Psi_1(u_{n+1} + u_n) \quad (n \geq k), \tag{19}$$

$$w_n = u_n v_{n+1} \quad (n \geq k), \tag{20}$$

$$u_n = -\Psi_0 \frac{u_{n-1}}{\alpha_{n-2}^2 \alpha_{n-1}^2} \quad (n \geq k+1), \tag{21}$$

where

$$\Psi_0 = -\frac{\alpha_k^2 \alpha_{k-1}^2 (\alpha_{k+1}^2 - \alpha_k^2)}{\alpha_k^2 - \alpha_{k-1}^2} \quad \text{and} \quad \Psi_1 = \frac{\alpha_k^2 (\alpha_{k+1}^2 - \alpha_{k-1}^2)}{\alpha_k^2 - \alpha_{k-1}^2}.$$

Example 4.2. Let $\alpha : 1, (1, \sqrt{a}, \sqrt{b})\widehat{\ }$ with $1 < a < b$. By (20), W_α has property $C(2)$. But it does not have property $B(n)$ for any $n \in \mathbb{N}$. *Indeed*, first recall that $\widehat{\alpha}_n^2 = \Psi_1 + \frac{\Psi_0}{\widehat{\alpha}_{n-1}^2}$ $(n \geq 2)$ (cf. [6], [7] or [8]). For $n \geq 3$, since

$$\begin{aligned} u_{n+1}v_n - w_n &= u_{n+1}v_n - v_{n+1}u_n && \text{by (20)} \\ &= u_{n+1}\Psi_1(u_n + u_{n-1}) - \Psi_1(u_{n+1} + u_n)u_n && \text{by (19)} \\ &= \Psi_1 u_{n+1}u_{n-1} - \Psi_1 u_n^2 \\ &= \Psi_1 \left(-\Psi_0 \frac{u_n u_{n-1}}{\alpha_{n-1}^2 \alpha_n^2} \right) - \Psi_1 u_n^2 && \text{by (21)} \\ &= \Psi_1 u_n \left(-\Psi_0 \frac{u_{n-1}}{\alpha_{n-1}^2 \alpha_n^2} + \Psi_0 \frac{u_{n-1}}{\alpha_{n-2}^2 \alpha_{n-1}^2} \right) \\ &= \Psi_0 \Psi_1 u_n u_{n-1} \frac{1}{\alpha_{n-1}^2} \left(\frac{1}{\alpha_{n-2}^2} - \frac{1}{\alpha_n^2} \right) < 0, \end{aligned}$$

W_α does not have property $B(n)$ for any $n \geq 3$. Also it is easy to show that W_α does not have property $B(2)$.

We may easily recapture some well-known examples in [4], [2] and [9] by criteria in the previous section. Finally we consider some of them here.

Example 4.3 ([2, Theorem 2])**.** Let $\alpha(x) : \sqrt{x}, \sqrt{\frac{n}{n+1}}$ $(n \geq 1)$. Then:

(i) $W_{\alpha(x)}$ has property $C(1) \Longleftrightarrow 0 < x \leq \frac{1}{3}$.
(ii) $W_{\alpha(x)}$ is positively quadratically hyponormal$\Longleftrightarrow 0 < x \leq \frac{22}{47}$.

Proof. Notice that $W_{\alpha(x)}$ has property $B(3)$ and $C(2)$. Since

$$c(2,1) = \frac{1}{12}x(1-2x), \quad c(3,2) = \frac{11}{720}x(1-2x), \quad c(4,3) = \frac{1}{8640}x(22-47x),$$

by applying Corollary 3.3 and Theorem 3.9 we obviously obtain (ii). □

Example 4.4 ([9, Theorem 3.7])**.** Let $\alpha(x) : \sqrt{x}, \sqrt{\frac{5}{8}}, \sqrt{\frac{n+1}{n+2}}$ $(n \geq 2)$. Then:

(i) $W_{\alpha(x)}$ has property $C(1) \Longleftrightarrow 0 < x \leq \frac{15}{34}$.
(ii) $W_{\alpha(x)}$ has property $B(2) \Longleftrightarrow 0 < x \leq \frac{9}{40}$.
(iii) $W_{\alpha(x)}$ is positively quadratically hyponormal$\Longleftrightarrow 0 \leq x \leq \frac{1945}{3136}$.

Proof. Notice that $W_{\alpha(x)}$ has property $B(3)$. Since $c(2,1) = \frac{21}{1280}x(5-8x)$, $c(3,2) = \frac{1}{40960}x(425-644x)$ and $c(4,3) = \frac{1}{1433600}x(1945-3136x)$, apply Corollary 3.3 and Theorem 3.9 again to obtain (iii). Other statements are obvious. □

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Department of Mathematics, College of Natural Sciences, Kyungpook National University, Taegu 702-701, Korea
E-mail address: baejuyoun@hanmir.com

Department of Mathematics, College of Natural Sciences, Kyungpook National University, Taegu 702-701, Korea
E-mail address: ibjung@kyungpook.ac.kr

Department of Mathematics, Bucknell University, Lewisburg, Pennsylvania 17837
E-mail address: exner@bucknell.edu