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FRAME WAVELETS IN SUBSPACES OF $L^2(\mathbb{R}^d)$

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Abstract. Let A be a $d \times d$ real expansive matrix. We characterize the reducing subspaces of $L^2(\mathbb{R}^d)$ for A-dilation and the regular translation operators acting on $L^2(\mathbb{R}^d)$. We also characterize the Lebesgue measurable subsets E of $\mathbb{R}^d$ such that the function defined by inverse Fourier transform of $[1/(2\pi)^{d/2}]\chi_E$ generates through the same A-dilation and the regular translation operators a normalized tight frame for a given reducing subspace. We prove that in each reducing subspace, the set of all such functions is nonempty and is also path connected in the regular $L^2(\mathbb{R}^d)$-norm.

1. Introduction

A sequence $\{x_n\}$ in a Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|x\|^2 \leq \sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq C_2\|x\|^2, \quad \forall x \in \mathcal{H}.$$

If $C_1 = C_2 = 1$, it is called a *normalized tight* frame. It is known ([6]) that $\{x_n\}$ is a normalized tight frame for $\mathcal{H}$ if and only if $x = \sum_{n\in\mathbb{N}}\langle x, x_n\rangle x_n$ for all $x \in \mathcal{H}$, where the convergence is unconditional in norm.

In this article we will investigate a class of normalized tight frames for either $L^2(\mathbb{R}^d)$ or certain subspaces of $L^2(\mathbb{R}^d)$ which are called reducing subspaces. The normalized tight frames we will deal with are obtained by applying certain A-dilation and regular translation operators to a single Lebesgue integrable function. Let us first define the above mentioned operators.

Let A be a $d \times d$ real *invertible* matrix. It induces a unitary operator D_A acting on $L^2(\mathbb{R}^d)$ defined by

$$(D_A f)(t) \quad = \quad |\det A|^{\frac{1}{2}} f(At), \quad \forall f \in L^2(\mathbb{R}^d), t \in \mathbb{R}^d. \tag{1}$$

The matrix A is called *expansive* if all its eigenvalues have modulus greater than one. The operator D_A corresponding to a real expansive matrix A is called an *A-dilation* operator. In an analogous fashion, a vector s in $\mathbb{R}^d$ induces a unitary

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translation operator T_s defined by

$$(T_s f)(t) = f(t-s), \quad \forall f \in L^2(\mathbb{R}^d), t \in \mathbb{R}^d.$$

In this article we will only deal with expansive real matrices and translation operators T_ℓ with $\ell \in \mathbb{Z}^d$.

Throughout this article, we will use $\mathcal{F}$ to denote the Fourier-Plancherel transform on $L^2(\mathbb{R}^d)$. This is a unitary operator. If $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, then

$$(\mathcal{F}f)(s) \quad = \quad \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i(s \circ t)} f(t) dm, \tag{2}$$

where $s \circ t$ denotes the real inner product. We also write $\widehat{f}$ for $\mathcal{F}f$. For a subset X of $L^2(\mathbb{R}^d)$, $\widehat{X}$ is the set of the Fourier-Plancherel transforms of all elements in X. For a bounded linear operator S on $L^2(\mathbb{R}^d)$, we will denote $\mathcal{F}S\mathcal{F}^{-1}$ by $\widehat{S}$. It is left to the reader to verify that we have $\widehat{D}_A = D_{(A')^{-1}} = D_{A'}^{-1} = D_{A'}^*$ for any $d \times d$ real invertible matrix A (A' is the transpose of A) and $\widehat{T}_\lambda f = e^{-i(\lambda \circ s)} \cdot f$ for any $\lambda \in \mathbb{R}^d$.

A function ψ is called an *A-dilation orthonormal wavelet* if the family of functions $\{D_A^n T_\ell \psi : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$. When the above family forms a *normalized tight frame* for $L^2(\mathbb{R}^d)$, the function $\psi \in L^2(\mathbb{R}^d)$ is called an *A-dilation normalized tight frame wavelet*. The existence of A-dilation orthonormal wavelets was proved in [3] for any expansive matrix A. In fact, it was proved in [3] that there exist A-dilation orthonormal wavelets whose Fourier transforms are of the form $\frac{1}{(2\pi)^{d/2}}\chi_E$ for some measurable subset E of $\mathbb{R}^d$. Such A-dilation orthonormal wavelets are called s-elementary wavelets [2]. A function ψ defined by $\widehat{\psi} = \frac{1}{(2\pi)^{d/2}}\chi_E$ for some measurable set E of $\mathbb{R}^d$ is called an s-frame wavelet if $\{D_A^n T_\ell \psi : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}$ forms a normalized tight frame for $L^2(\mathbb{R}^d)$. The set E in this case will be called an s-frame wavelet set for short. It is proved in [7] that for any given real expansive matrix A, the set of all s-elementary wavelets is path-connected in the L^2-norm. In this paper, we are interested in the s-frame wavelets defined on a *reducing subspace* X of $L^2(\mathbb{R}^d)$ (X is a reducing subspace if $D_A X = X$ and $T_\ell X = X$ for each $\ell \in \mathbb{Z}^d$). We characterize the s-frame wavelet sets and show that the set of all s-frame wavelets (on any given reducing subspace) is also path-connected in the L^2-norm.

2. Subspace s-frame wavelets

In this section, we will characterize all reducing subspaces and all s-frame wavelet sets for a given reducing subspace. We will begin with the following definition.

Definition 1. A function ψ is an s-frame wavelet on X if $\widehat{\psi} = \frac{1}{(2\pi)^{d/2}}\chi_E$ for some measurable subset E of $\mathbb{R}^d$, $\{D_A^n T_\ell \psi : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\} \subset X$ and

$$f = \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}^d} \langle f, D_A^n T_\ell \psi \rangle D_A^n T_\ell \psi, \quad \forall f \in X.$$

We also need the following lemmas in the proof of our theorems.

Lemma 1. *Let A be a $d \times d$ real expansive matrix. Then* $\lim_{k \to +\infty} \|A^{-k}\| = 0$ *and* $\lim_{k \to +\infty} \|A^k t\| = \infty$ *for every nonzero element t in* $\mathbb{R}^d$.

Proof. Since A is an expansive matrix, it follows that A is invertible and for any $\lambda \in \sigma(A^{-1})$, $|\lambda| < 1$ and thus $\lim_{k\to+\infty} \lambda^{-k} = 0$. This implies that ([5], p. 559, Theorem 9) $\lim_{k\to+\infty} \|A^{-k}\| = 0$. Now let $t \in \mathbb{R}^d$ with $t \neq 0$. The result then follows from the fact that $\|A^{-k}\|\|A^k t\| \geq \|t\|$. □

Lemma 2. *Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$ and let P be an orthogonal projection onto a subspace $\mathcal{H}_0$. Then $\{Pe_n\}$ is a normalized tight frame for $\mathcal{H}_0$.*

Lemma 3. *Let $\{\mathcal{H}_n : n \in \mathbb{N}\}$ be a family of mutually orthogonal subspaces that sum to $\mathcal{H}$. Suppose that for each $n \in \mathbb{N}$, $\{x_{nm} : m \in \mathbb{N}\}$ is a normalized tight frame for $\mathcal{H}_n$. Then $\{x_{nm} : n, m \in \mathbb{N}\}$ is a normalized tight frame for $\mathcal{H}$.*

The proofs of Lemmas 2 and 3 are left to the reader.

Lemma 4. *Let A be a $d \times d$ real expansive matrix. Then $\{A^n\mathbb{Z}^d : n \in \mathbb{Z}\}$ is dense in $\mathbb{R}^d$.*

Proof. Let $\varepsilon > 0$. By Lemma 1 we can choose $k \in \mathbb{N}$ such that $\|A^{-k}\| < \frac{\varepsilon}{d}$. Let $\{e_j : 1 \leq j \leq d\}$ be the standard orthonormal basis for $\mathbb{R}^d$. Since A is expansive hence invertible, the set $\{A^{-k}e_j : 1 \leq j \leq d\}$ is also a basis for $\mathbb{R}^d$. Let $x \in \mathbb{R}^d$, then $x = \sum_{j=1}^d x^{(j)} A^{-k} e_j$ with some $x^{(j)} \in \mathbb{R}$ for $j \in \{1, 2, \cdots, d\}$. Let $\ell = \sum_{j=1}^d [x^{(j)}] e_j \in \mathbb{Z}^d$, where $[x^{(j)}]$ denotes the largest integer among those that are no greater than $x^{(j)}$, then $\|A^{-k}\ell - x\| = \|\sum_{j=1}^d \big(x^{(j)} - [x^{(j)}]\big) A^{-k} e_j\| \leq \sum_{j=1}^d \|A^{-k}\| \cdot \|e_j\| < \varepsilon$. □

We now characterize the reducing subspaces in the following theorem.

Theorem 1. *Let A be a $d\times d$ real expansive matrix. A closed subspace X of $L^2(\mathbb{R}^d)$ is a reducing subspace if and only if $\widehat{X} = L^2(\mathbb{R}^d) \cdot \chi_\Omega$ for some Lebesgue measurable subset Ω of $\mathbb{R}^d$ with the property that $\Omega = A'\Omega$.*

Proof. Let X be a reducing subspace and P be the orthogonal projection from $L^2(\mathbb{R}^d)$ onto X. Then P commutes with both D_A^n and T_ℓ for all $n \in \mathbb{Z}$ and $\ell \in \mathbb{Z}^d$. Note that $D_A^{-n} T_\ell D_A^n = T_{A^n\ell}$. It follows that P commutes with $T_{A^n\ell}$ for all $n \in \mathbb{Z}$ and $\ell \in \mathbb{Z}^d$. By Lemma 4, the set $\{A^n\mathbb{Z}^d : n \in \mathbb{Z}\}$ is dense in $\mathbb{R}^d$. So P commutes with T_t for all $t \in \mathbb{R}^d$. Hence $\widehat{P}$ commutes with $\widehat{T_t}$ for all $t \in \mathbb{R}^d$. If we use M_f to denote the multiplicative operator by $f(s)$, then $\widehat{T_t} = M_{e^{-t\circ s}}$. $\{M_{e^{-t\circ s}} : t \in \mathbb{R}^d\}$ generates a maximal abelian von Neumann algebra $\mathcal{A} = \{M_f : f \in L^\infty(\mathbb{R}^d)\}$. Thus $\widehat{P}$ must be in $\mathcal{A}$ and $\widehat{P} = M_f$ for some $f \in L^\infty(\mathbb{R}^d)$. The relation $P^2 = P$ implies that $f^2 = f$. So f is χ_Ω for some measurable subset Ω of $\mathbb{R}^d$. Thus $\widehat{X} = L^2(\mathbb{R}^d) \cdot \chi_\Omega$. Since $PD_A = D_A P$, we have $\widehat{P}\widehat{D}_A = \widehat{D}_A\widehat{P}$. Equivalently $M_{\chi_\Omega} D_{A'}^{-1} = D_{A'}^{-1} M_{\chi_\Omega}$, hence $M_{\chi_\Omega} D_{A'} = D_{A'} M_{\chi_\Omega}$. Let $f \in L^2(\mathbb{R}^d)$. Observe that

$$\begin{aligned} M_{\chi_\Omega} D_{A'} f(t) &= |A'|^{\frac{1}{2}} f(A't) \cdot \chi_\Omega(t), \\ D_{A'} M_{\chi_\Omega} f(t) &= |A'|^{\frac{1}{2}} f(A't) \cdot \chi_\Omega(A't). \end{aligned}$$

This implies that $\chi_\Omega(A't) = \chi_\Omega(t)$, therefore $A'\Omega = \Omega$. The proof of the other direction is simple and is omitted. □

Remark. The above proof shows that the *commutant* of $\{\widehat{D}_A^n, \widehat{T}_\ell, \ell \in \mathbb{Z}^d\}$ is $\{M_f : f \in L^\infty(\mathbb{R}^d), f(A'x) = f(x)\}$.

Let E be a subset of $\mathbb{R}^d$. Two points $x, y \in E$ are said to be 2π-*translation equivalent* if $x - y = 2\pi\ell$ for some $\ell \in \mathbb{Z}^d$. This is an equivalence relation on E. The 2π-*translation redundancy index* of a point x in E is the cardinality of its equivalence class. We use $E(\tau, k)$ to denote the set of all points in E with 2π-translation redundancy index k. For $k \neq m$, $E(\tau, k) \cap E(\tau, m) = \emptyset$, so we have $E = E(\tau, \infty) \cup \left(\bigcup_{n\in\mathbb{N}} E(\tau, n)\right)$. Similarly, two nonzero points $x, y \in E$ are said to be A-dilation equivalent if $y = A^k x$ for some $k \in \mathbb{Z}$. This is also an equivalence relation on E. The A-*dilation redundancy index* of a point x in E is the cardinality in its equivalence class. The set of all points in E with A-dilation redundancy index k is denoted by $E(\delta_A, k)$. For $k \neq m$, $E(\delta_A, k) \cap E(\delta_A, m) = \emptyset$. So $E = E(\delta_A, \infty) \cup \left(\bigcup_{n\in\mathbb{N}} E(\delta_A, n)\right)$. For a Lebesgue measurable set E of $\mathbb{R}^d$, $E(\tau, k)$ and $E(\delta_A, k)$ are both measurable for any $k \in \mathbb{N} \cup \{\infty\}$. The proof of case $d = 1$ can be found in [1] and the general cases can be treated similarly. The details are left to the reader. Furthermore, $E(\delta_A, k)$ can be decomposed as a union of k disjoint measurable subsets $E^{(j)}(\delta_A, k)$ $(1 \leq j \leq k)$ such that each $E^{(j)}(\delta_A, k)$ contains only points of A-dilation redundancy index 1. A set E is said to be the A-dilation *generator* of another set Ω if $E = E(\delta_A, 1)$ and $\bigcup_{k\in\mathbb{Z}} A^k E = \Omega$. Two sets are said to be A-dilation equivalent if they are both A-dilation generators of the same set.

Before we state and prove the next theorem, we need to point out that all the statements in the theorems of this paper about a measurable set are to be modulus a null set.

Theorem 2. *Let A be a $d \times d$ real expansive matrix and X be a reducing subspace. Then a measurable subset $E \subset \mathbb{R}^d$ is an s-frame wavelet set for X if and only if $E = E(\tau, 1) = E(\delta_{A'}, 1)$ and $\bigcup_{k\in\mathbb{Z}} (A')^k E = \Omega$ satisfies the conditions in Theorem 1.*

Proof. "$\Leftarrow$" Since E is 2π-translation equivalent to a subset F of $[0, 2\pi)^d$, the set $G = E \cup ([0, 2\pi)^d \backslash F)$ is 2π-translation equivalent to $[0, 2\pi)^d$ and $\{\widehat{T}_\ell \widehat{\psi}_G : \ell \in \mathbb{Z}^d\}$ is an orthogonal basis for $L^2(G)$. Let P be the orthogonal projection from $L^2(G)$ onto $L^2(E)$. By Lemma 2, $\{P\widehat{T}_\ell \widehat{\psi}_G : \ell \in \mathbb{Z}^d\} = \{\widehat{T}_\ell \widehat{\psi}_E : \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for $L^2(E)$. Therefore for each $k \in \mathbb{Z}$, $\{\widehat{D}_A^k \widehat{T}_\ell \widehat{\psi}_E : \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for $L^2((A')^k E)$. Since Ω is the disjoint union of $\{(A')^k E : k \in \mathbb{Z}\}$, it follows that $L^2(\Omega) = L^2(\mathbb{R}^d)\chi_\Omega$ is the (orthogonal) direct sum of $L^2((A')^k E)$. Thus, by Lemma 3, $\{\widehat{D}_A^k \widehat{T}_\ell \widehat{\psi}_E : \ell \in \mathbb{Z}^d, k \in \mathbb{Z}\}$ is a normalized tight frame for $L^2(\Omega)$, i.e., E is an s-frame wavelet set for X.

"$\Rightarrow$" Assume that E is an s-frame wavelet set for X. By Theorem 1, $\widehat{X} = L^2(\mathbb{R}^d) \cdot \chi_\Omega$ for some measurable set Ω with $\Omega = A'\Omega$. Note that $\widehat{X}$ contains functions $\widehat{D}_A^n \widehat{\psi}_E$, hence it contains $\chi_{(A')^n E}$ for any $n \in \mathbb{Z}$. This implies that $\Omega \supset \bigcup_{n\in\mathbb{Z}} (A')^n E$. On the other hand, any function in $\widehat{X}$ is in the span of $\{\widehat{D}_A^n \widehat{T}_\ell \widehat{\psi}_E : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}$, hence it is supported on $\bigcup_{n\in\mathbb{Z}} (A')^n E$. Therefore $\Omega \subset \bigcup_{n\in\mathbb{Z}} (A')^n E$. In order to prove that $E = E(\delta_{A'}, 1)$, we need to prove that (modulo null sets) $\mu((A')^k E \cap (A')^j E) = 0$ for any distinct integers k, j, where μ is the Lebesgue measure in $\mathbb{R}^d$. If this is not true, then there exists an integer $j_0 > 0$ such that $\mu(E \cap (A')^{j_0} E) > 0$. We can then find a subset F of $E \cap (A')^{j_0} E$ with positive measure such that elements in the set $\{(A')^k F : k \in \mathbb{Z}\}$ are mutually disjoint. We can choose F in such a way that it is contained in the cube $\prod_{j=1}^d [2m_j\pi, 2(m_j+1)\pi)$ for some integers m_j with $j \in \{1, 2, \cdots, d\}$. Now define f by $\widehat{f} = \chi_F$. $f \in X$

since $F \subset E \subset \Omega$. By assumption, ψ_E is an s-frame wavelet for X. Thus $f = \sum_{n\in\mathbb{Z},\ell\in\mathbb{Z}^d}\langle f, D_A^n T_\ell \psi_E\rangle D_A^n T_\ell \psi_E$. It follows that

$$\begin{aligned}\|f\|^2 &= \sum_{n\in\mathbb{Z},\ell\in\mathbb{Z}^d} |\langle f, D_A^n T_\ell \psi_E\rangle|^2 \\ &\geq \sum_{\ell\in\mathbb{Z}^d} |\langle \chi_F, \widehat{T}_\ell \widehat{\psi}_E\rangle|^2 + |\langle \chi_F, \widehat{D}_A^{j_0}\widehat{\psi}_E\rangle|^2 \\ &= \|f\|^2 + \frac{|\det A|^{2j_0}}{(2\pi)^d}\|f\|^2 > \|f\|^2.\end{aligned}$$

This is the contradiction we need.

To prove that $E = E(\tau, 1)$, it suffices to show that $E \cap (E + 2\pi\ell)$ is a null set for any $\ell \in \mathbb{Z}^d\backslash\{O\}$. If there exists $\ell_0 \neq \vec{O}$ in $\mathbb{Z}^d$ such that $E \cap (E + 2\pi\ell_0)$ has positive measure, we can choose a subset J of $E \cap (E + 2\pi\ell_0)$ with positive measure in $\prod_{j=1}^d [2\pi m_j, 2(m_j + 1)\pi)$ for some integers m_j with $j \in \{1, \cdots, d\}$. Define g by $\widehat{g} = \chi_J - \chi_{J-2\pi\ell_0}$. $g \in X$ since the support of $\widehat{g}$ is contained in E. We have $\langle \widehat{g}, \widehat{D}_A^n \widehat{T}_\ell \widehat{\psi}_E\rangle = 0$ if $n \neq 0$ (since $E = E(\delta_{A'}, 1)$). Also, for any $\ell \in \mathbb{Z}^d$, $\langle \widehat{g}, \widehat{T}_\ell \widehat{\psi}_E\rangle = \langle \chi_J - \chi_{J-2\pi\ell_0}, \widehat{T}_\ell \widehat{\psi}_E\rangle = \langle \chi_J, \widehat{T}_\ell \widehat{\psi}_E\rangle - \langle \chi_{J-2\pi\ell_0}, \widehat{T}_\ell \widehat{\psi}_E\rangle = 0$, since $\widehat{T}_\ell \widehat{\psi}_E$ is a 2π-periodic function. So we have

$$\widehat{g} = \sum_{n\in\mathbb{Z},\ell\in\mathbb{Z}^d} \langle \widehat{g}, \widehat{D}_A^n \widehat{T}_\ell \widehat{\psi}_E\rangle \widehat{D}_A^n \widehat{T}_\ell \widehat{\psi}_E = 0.$$

This is again a contradiction. ◻

3. Path connectivity

In this section we will prove that for a fixed $d \times d$ real expansive matrix A, in a reducing subspace, the set of all s-frame wavelets is not empty and is path-connected in the L^2-norm.

Theorem 3. *Let A be a $d \times d$ real expansive matrix A and let X be a reducing subspace. Then for any given positive ε, there exists ε_1, $0 < \varepsilon_1 < \varepsilon$, such that the ring $\mathcal{B}(\varepsilon)\backslash\mathcal{B}(\varepsilon_1)$ contains an s-frame wavelet set for X, where $\mathcal{B}(r)$ stands for the ball centered at the original point with radius r.*

Proof. Without loss of generality, we can assume $\varepsilon < 1$. Let $C = \big(A'\mathcal{B}(1)\big)\backslash\mathcal{B}(1)$. Since A is expansive, A' is also expansive. Thus for any $x \in \mathbb{R}^d\backslash\{O\}$, by Lemma 1, we have $\lim_{k\to+\infty}\|(A')^{-k}x\| = 0$ and $\lim_{k\to+\infty}\|(A')^k x\| = \infty$. So there is an integer n such that $(A')^n x \notin \mathcal{B}(1)$ and $(A')^{n-1}x \in \mathcal{B}(1)$. It follows that $(A')^n x \in C$, hence $x \in (A')^{-n}C$. Therefore, we have

$$\mathbb{R}^d\backslash\{O\} = \bigcup_{n\in\mathbb{Z}} (A')^n C.$$

Now for any ε with $0 < \varepsilon < 1$, since C is a bounded set and $\lim_{k\to\infty}\|(A')^{-k}\| = 0$, there is an integer k_0 such that $(A')^{k_0}C \subset \mathcal{B}(\varepsilon)$. Note that 0 is an exterior point of C, hence it is also an exterior point of $(A')^{k_0}C$. So there is an $\varepsilon_1 > 0$ ($\varepsilon_1 < \varepsilon$) such that

$$E = (A')^{k_0}C \subset \mathcal{B}(\varepsilon)\backslash\mathcal{B}(\varepsilon_1) \subset [-\pi, \pi)^d.$$

It is clear that $E = E(\tau, 1)$. Now define $F = \bigcup_{k\geq 1} E^{(1)}(\delta_{A'}, k)$. We see that $F = F(\tau, 1)$, $F = F(\delta_{A'}, 1)$ and $\mathbb{R}^d\backslash\{O\} = \bigcup_{n\in\mathbb{Z}}(A')^n F$.

Since X is a reducing subspace, by Theorem 1, there is a measurable subset Ω of $\mathbb{R}^d$ such that $A'\Omega = \Omega$, and $\widehat{X} = L^2(\mathbb{R}^d)\cdot\chi_\Omega$. If we define $W = F \cap \Omega$, then W is an s-frame wavelet set for X. Indeed, it is clear that $W = W(\tau, 1) = W(\delta_{A'}, 1)$ and that $\{(A')^n W : n \in \mathbb{Z}\}$ is a disjoint family of subsets in Ω. Also from the definition of W and the facts that $\mathbb{R}^d\backslash\{O\} = \bigcup_{n\in\mathbb{Z}}(A')^n F$ and that $A'\Omega = \Omega$, it follows that $\Omega\backslash\{O\} = \bigcup_{n\in\mathbb{Z}}(A')^n W$. □

Theorem 4. *Let A be a $d\times d$ real expansive matrix and let X be a reducing subspace. Then the set of all s-frame wavelet sets for X is path-connected in the L^2-norm.*

Proof. Since $\lim_{k\to\infty}\|(A')^{-k}\| = 0$ (Lemma 1), the sequence $\{\|(A')^{-k}\|\}_{k\geq 0}$ is bounded by some $M > 0$. Let $\eta_0 > 0$ be a number less than $\frac{1}{M}$. If we have $\|x\| \geq 1$ and $\|(A')^k x\| < \eta_0$, then we must have $k < 0$ since otherwise we would have $\|(A')^k x\| \geq \frac{\|x\|}{\|(A')^{-k}\|} > \eta_0$. By Theorem 1, there exists a measurable subset Ω of $\mathbb{R}^d$ such that $\widehat{X} = L^2(\mathbb{R}^d)\cdot\chi_\Omega$ and $\Omega = A'\Omega$. By the same argument used in the proof of Theorem 3, there exist a η_1 such that $0 < \eta_1 < \eta_0$ and an s-frame wavelet set G for X such that $G \subset \mathcal{B}(\eta_0)\backslash\mathcal{B}(\eta_1)$. Let E be any s-frame wavelet set for X (so that $E = E(\delta_{A'}, 1)$, $\Omega = \bigcup_{k\in\mathbb{Z}}(A')^k E$ and $E = E(\tau, 1)$ by Theorem 2). It suffices to show that χ_E is path-connected to χ_G in L^2-norm. For simplicity, we will assume that E and G are disjoint in the following proof. The general case can be obtained by applying the following proof to $E_1 = E\backslash(E\cap G)$ and $G_1 = G\backslash(E\cap G)$ while keeping $E\cap G$ intact.

Let $F_n = (A')^{-n}(G)\cap E$. F_n's are mutually disjoint and $\bigcup_{n\in\mathbb{Z}} F_n = E$ as one can easily check. $(A')^n(F_n) \subset G$ for each integer n, $(A')^n(F_n)$'s are also disjoint and $\bigcup_{n\in\mathbb{Z}}(A')^n(F_n) = G$. For each integer n, F_n is bounded and the original point is an exterior point of it. Hence there exist real numbers r_n and q_n with $0 < r_n < q_n$, such that $F_n \subset \mathcal{B}(q_n)\backslash\mathcal{B}(r_n)$.

Define

$$F_n^s = \mathcal{B}(r_n + (q_n - r_n)s) \cap F_n,$$
$$I_0^s = \bigcup_{n\in\mathbb{Z}} F_n^s.$$

We have $I_0^0 = \emptyset$ and $I_0^1 = E$. For subsets S, P and Q of $\mathbb{R}^d$, define

$$\begin{aligned} g(S) &= \bigcup_{n\in\mathbb{Z}}(A')^n\big((A')^{-n}(G)\cap S\big) = G\cap(\bigcup_{n\in\mathbb{Z}}(A')^n S), \\ \tau(P) &= \bigcup_{\ell\in\mathbb{Z}^d}(P - 2\pi\ell), \\ h(P,Q) &= \tau(P)\cap Q. \end{aligned}$$

The following elementary properties of g, h are needed in the proof later. We will prove (v) and (vi) and leave the rest to the reader.

- **(i)** If $S \subset E$, then $g(S) \subset G$. Moreover, S and $g(S)$ are A'-dilation equivalent.
- **(ii)** If $\{S_n\}$ is a family of disjoint subsets of E, then $\{g(S_n)\}$ is a family of disjoint subsets of G. Furthermore, $\big(\bigcup g(S_n)\big)\cup\big(E\backslash\bigcup S_n\big)$ is an A'-dilation generator for Ω.

(iii) For subsets $P \subset G$ and $Q \subset E$ the set $h(P,Q)$ is a subset of Q which does not intersect $\mathcal{B}(1)$. Furthermore, $\mu\big(h(P,Q)\big) \le \mu(P)$ and $\mu\big(g(h(P,Q))\big) \le \frac{\mu(h(P,Q))}{|\det(A)|} \le \frac{\mu(P)}{|\det(A)|}$.

(iv) If $\{P_n\}$ is a family of disjoint subsets of G and Q is a subset of E, then $\{h(P_n,Q)\}$ is a family of disjoint subsets of E.

(v) The symmetric difference of two sets P and Q, denoted by $P\Delta Q$ is $(P\backslash Q)\cup(Q\backslash P)$. For any $\varepsilon > 0$, there exists $\delta > 0$, such that for any subsets P, Q of E, if $\mu(P\Delta Q) < \delta$, then $\mu\big(g(P)\Delta g(Q)\big) < \varepsilon$.

(vi) For any $\varepsilon > 0$, there exists $\delta > 0$, such that for any subsets C, D of G, and any subsets P, Q of E, if $\mu\big(C\Delta D\big) < \delta$ and $\mu\big(P\Delta Q\big) < \delta$, then $\mu\big(h(C,P)\Delta h(D,Q)\big) < \varepsilon$.

Proof of property (v). Since $g(P)$ and $g(Q)$ are subsets of G and $\mu(G) < \infty$, there exists a natural number N so that

$$\mu\Big(\bigcup_{|n|\ge N}(G\cap (A')^nP)\Big) < \varepsilon/4 \quad\text{and}\quad \mu\Big(\bigcup_{|n|\ge N}(G\cap (A')^nQ)\Big) < \varepsilon/4.$$

On the other hand, for each n such that $1 \le n \le N$, it is apparent that

$$\mu\big(G\cap((A')^nP\Delta(A')^nQ)\big) < \frac{\varepsilon}{2N}$$

if $\mu\big((P\Delta Q)\big)$ is small enough, say less than $\delta_n > 0$. Let $\delta = \min\{\delta_1,\delta_2,\cdots,\delta_N\}$, the result then follows by the definition of g and the triangle inequality. □

Proof of property (vi). Using the fact that $(A\cap B)\Delta(A'\cap B') \subset (A\Delta A')\cup(B\Delta B')$ for any subsets A, B, A' and B' of $\mathbb{R}^d$, we have

$$\begin{aligned}
h(C,P)\Delta h(D,Q) &= \big((\tau(C)\cap E)\cap P\big)\Delta\big((\tau(D)\cap E)\cap Q\big)\\
&\subset \big((\tau(C)\Delta\tau(D))\cap E\big)\cup(P\Delta Q)\\
&= \big((\bigcup_{\ell\in\mathbb{Z}}(C-2\pi\ell)\Delta\bigcup_{\ell'\in\mathbb{Z}}(D-2\pi\ell'))\cap E\big)\cup(P\Delta Q)\\
&\subset \big(\bigcup_{\ell\in\mathbb{Z}}(C\Delta D-2\pi\ell)\cap E\big)\cup(P\Delta Q)\\
&= \big(\bigcup_{\ell\in\mathbb{Z}}E_\ell\big)\cup(P\Delta Q),
\end{aligned}$$

where $E_\ell = (C\Delta D - 2\pi\ell)\cap E$. Since $\{E_\ell\}$ are disjoint subsets of E and $\mu(E_\ell) < \mu(C\Delta D)$ for each ℓ, it follows that $\mu\big(h(C,P)\Delta h(D,Q)\big) < \varepsilon$ if δ is small enough, using an argument similar to the one used in the proof of (v) above. □

Finally, we define

$$\begin{aligned}
&J_0^s = g(I_0^s), I_1^s = h(J_0^s, E\backslash I_0^s),\\
&\cdots\\
&J_n^s = g(I_n^s), I_{n+1}^s = h\big(J_n^s, E\backslash \bigcup_{k=0}^n I_k^s\big),\\
&\cdots\\
&E^s = \big(\bigcup_{k=0}^\infty J_k^s\big)\cup\big(E\backslash\bigcup_{k=0}^\infty I_k^s\big).
\end{aligned}$$

It is clear that $E^0 = E$ and $E^1 = G$. $\{I_n^s\}$ is a family of disjoint subsets of E by the construction. It follows that E^s is an A-dilation generator for Ω by property

(ii) above. To see that $E^s = E^s(\tau, 1)$, assume the contrary. Since $G = G(\tau, 1)$ and $E = E(\tau, 1)$, there must exist some $k_0 \geq 0$ so that $J^s_{k_0}$ contains a measurable subset J' of positive measure with the property $J' + 2\pi\ell_0 \subset E \setminus \bigcup_{k=0}^{\infty} I^s_k$ for some $\ell_0 \in \mathbb{Z}^d$. This leads to $J' \subset I^s_{k_0+1} = h(J^s_{k_0}, E \setminus \bigcup_{k=0}^{k_0} I^s_k)$, which is a contradiction. This proves that E^s is an s-frame wavelet set for X for each $s \in [0, 1]$.

Finally, we will show that χ_{E^s} is continuous. Given $\varepsilon > 0$ and $0 \leq s \leq 1$, we need to find $\delta > 0$, such that for each $0 \leq t \leq 1$, if $|t - s| \leq \delta$, then $\|\chi_{E^t} - \chi_{E^s}\|^2 < \varepsilon$. First, by property (iii) above, we have $\mu(I^s_{k+1}) \leq \mu(J^s_k) \leq \frac{\mu(I^s_k)}{|\det(A)|}$ for any s and $k \geq 0$. It follows that $\mu(I^s_{k+1}) \leq \mu(J^s_k) \leq \frac{\mu(E)}{|\det(A)|^{k+1}}$. Since A is expansive, we have $|\det(A)| > 1$. So if k_1 is large enough, we will have

$$\sum_{k=k_1}^{\infty} \left(\|\chi_{J^t_k} - \chi_{J^s_k}\|^2 + \|\chi_{I^t_k} - \chi_{I^s_k}\|^2\right) < \frac{\varepsilon}{2}.$$

On the other hand, it is clear that $\chi_{I^s_0}$ is continuous. Property (v) then ensures that $\chi_{J^s_0}$ is continuous and property (vi) in turn guarantees the continuity of $\chi_{I^s_1}$. It follows that each $\chi_{J^s_k}$ and $\chi_{I^s_k}$ are continuous by induction. Hence, there exists $\delta > 0$, such that

$$\sum_{k=0}^{k_1-1} \left(\|\chi_{J^t_k} - \chi_{J^s_k}\|^2 + \|\chi_{I^t_k} - \chi_{I^s_k}\|^2\right) < \frac{\varepsilon}{2}$$

if $|s - t| < \delta$. We then have

$$\begin{aligned}\|\chi_{E^t} - \chi_{E^s}\|^2 &\leq \|\chi_{\bigcup_{k=0}^{\infty} J^t_k} - \chi_{\bigcup_{k=0}^{\infty} J^s_k}\|^2 + \|\chi_{E\setminus \bigcup_{k=0}^{\infty} I^t_k} - \chi_{E\setminus \bigcup_{k=0}^{\infty} I^s_k}\|^2 \\ &\leq \sum_{k=0}^{\infty} \|\chi_{J^t_k} - \chi_{J^s_k}\|^2 + \sum_{k=0}^{\infty} \|\chi_{I^t_k} - \chi_{I^s_k}\|^2 < \varepsilon.\end{aligned}$$

□

Remark. In any given reducing subspace X, an s-frame wavelet set E for X is an s-elementary wavelet set if and only if E has measure $(2\pi)^d$. Since we cannot control the measures of the s-frame wavelet sets in our proof, we are not able to recapture D. Speegle's connectivity result in the case when X is $L^2(\mathbb{R}^d)$ itself. On the other hand, our approach does provide a more elementary way of proving the path connectivity property for a larger class of basis functions.

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