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ON STABLE QUASI-HARMONIC MAPS AND LIOUVILLE TYPE THEOREMS

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ABSTRACT. We consider Liouville type problems of stable quasi-harmonic maps, by "stable" we mean that the second variation of quasi-energy functional $E_q(u)$ is nonnegative, and we prove that the stable quasi-harmonic maps must be constant under some geometry conditions.

1. Introduction and preliminaries

The classical Liouville theorem states that a bounded entire harmonic function on R^2 must be constant. There are many generalizations of this theorem considering harmonic functions on Riemannian manifold. In order to study the Liouville type theorem for harmonic maps it is natural to assume that the domain manifold has nonnegative Ricci curvature. Cheng [1] used a gradient estimate of the harmonic map to prove that the Liouville theorem is true if the target manifold has nonpositive sectional curvature and the image satisfies the sublinear growth condition. Hildebrant-Jost-Widman proved the Liouville theorem if the image of a simple Riemannian manifold is contained in a geodesic ball $B_n(Q)$ of radius $M < \frac{\pi}{2k}$, where $k^2 > 0$ is the upper bound of the sectional curvature on the target manifold. Other types of Liouville theorems consider stable harmonic maps, by "stable" we mean the second variation of energy $E(u)$ is nonnegative, i.e. $\frac{d^2}{dt^2}E(u_t) \geq 0$. Xin in [9] proved that there are no stable harmonic maps from S^k to N, where $k \geq 3$ and N is an arbitrary Riemannian manifold. Wei in [7] gives another type of Liouville theorem about stable harmonic maps.

In [2] and [4] the authors give the concept of quasi-harmonic maps which relates to the blow up of the heat flow of harmonic maps. Hence the Liouville type theorems of quasi-harmonic maps guarantee some regularities of heat flow for harmonic maps. In this paper we establish some Liouville type theorems of stable quasi-harmonic maps. Our methods are based on [7].

Suppose that N is a smooth compact Riemannian manifold. We call $\Phi : R^l \mapsto N$ a quasi-harmonic map, if Φ is a nonconstant smooth map which is a critical point of the quasi-energy functional E_q with respect to any smooth compactly supported

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variation, where

$$E_q(\Phi) = \frac{1}{2}\int_{R^l} |D\Phi|^2 e^{-\frac{|y|^2}{4}} dy. \tag{1.1}$$

The Euler-Lagrange equation of $E_q(\Phi)$ is

$$\triangle\Phi - \frac{1}{2}y \cdot D\Phi + A(\Phi)(D\Phi, D\Phi) = 0 \tag{1.2}$$

where $A(\Phi)(\cdot,\cdot)$ is the second fundamental form of N isometrically embedded in R^q.

Let us consider a smooth two parameter variation $\Phi_{s,t} : R^l \longmapsto N$ with $\Phi_{0,0} = \Phi$. Denote $V_{s,t} = \frac{\partial}{\partial s}\Phi_{s,t}, W_{s,t} = \frac{\partial}{\partial t}\Phi_{s,t}$

Lemma 1.1. *If $\Phi_{s,t}$ is a smooth two parameter variation of $\Phi : R^l \mapsto N$ with compact support, then*

$$\begin{aligned}\frac{\partial^2}{\partial t \partial s}\Big|_{t=s=0} E_q(\Phi_{s,t}) = \textstyle\int_{R^l} &[\nabla V \cdot \nabla W + \left\langle R^N \circ \Phi(W, d\Phi-)V, d\Phi- \right\rangle \cdot d\Phi] e^{-\frac{y^2}{4}} dy \\ &+ \textstyle\int_{R^l} \nabla X \cdot d\Phi e^{-\frac{y^2}{4}} dy,\end{aligned}$$

where $V = \frac{\partial}{\partial s}\big|_{s=0} \Phi_{s,t}, W = \frac{\partial}{\partial t}\big|_{t=0} \Phi_{s,t}$ and $X = \frac{\partial}{\partial t}\big|_{t=0} \frac{\partial}{\partial s}\big|_{s=0} \Phi_{s,t}$. Furthermore, if $\Phi : R^l \mapsto N$ is a quasi-harmonic map, then

$$\frac{d^2}{dt^2}\Big|_{t=0} E_q(\Phi_t) = \int_{R^l} \left[|\nabla V|^2 + \left\langle (R^N \cdot \Phi)(V, d\Phi_-)V, d\Phi \right\rangle\right] e^{-\frac{|y|^2}{4}} dy, \tag{1.3}$$

where $(R^N \cdot \Phi(V, d\Phi_-)V, d\Phi) = -\sum_i g^N_{\Phi(y)}(R_{\Phi(y)}(\partial_i\Phi(y), V(y))V(y), \partial_i\Phi(y))$.

The proof of Lemma 1.1 is similar to the harmonic map's case and we may refer to Eells and Lemaire [3].

We say a quasi-harmonic map is stable, if

$$\frac{d^2}{dt^2} E_q(\Phi_t) \geq 0$$

for any $V \in \Phi^{-1}(TN)$ with compact support.

As in [7] we define

Definition. A complete Riemannian manifold N is called super-strongly unstable (SSU) if Q^N_y is negative definite for every y in N, where

$$\left\langle Q^N_y(x), x \right\rangle_N = \sum_{i=1}^n 2\,|A(x, \alpha_i)|^2 - \langle A(x,x), A(\alpha_i, \alpha_i)\rangle_{R^q}$$

$\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is an orthonormal basis for T_yN. It is obvious that Q^N_y is a self-adjoint linear map of tangent space T_yN.

Our main results are the following:

Theorem 1. *Assume that N is a smooth SSU manifold. Then any smooth stable quasi-harmonic map with finite quasi-energy from R^l to N is constant.*

Theorem 2. *Assume that N is a smooth compact manifold Riemannian. If N is simply-connected and is δ_n-pinched for $\frac{1}{4} < \delta_n < 1$, then any stable quasi-harmonic map with finite quasi-energy is constant, where $n = \dim N$ and δ_n will be defined in section 3.*

Theorem 3. *Let u be a quasi-harmonic map with finite energy from R^l into N, where N is a simple Riemannian manifold. Moreover, suppose that u is strict stable, i.e., $E_q(u) \le E_q(v)$ for each C^2-map v from R^l into N which agrees with u except on a compact subset of R^l. Then u is a constant map.*

Combining our results with the result in [4], we pose the following question: Without any topology restriction, can one prove that quasi-harmonic maps obtained from the blow-up of heat flow are stable?

2. A stability inequality and the proof of Theorem 1

Let Φ be a smooth stable quasi-harmonic map. Let $u(x)$ be a smooth real-valued function on R^l with compact support, let $\{e_1, e_2, \cdots, e_l\}$ be an orthonormal basis in R^l, let α be a unit vector in R^q, and let α^T be the tangential projection of α onto TN. Denote by $\Phi_t^{u\alpha^T}$ a variation of Φ with $\Phi_0^{u\alpha^T} = \Phi$, and deformation vector, $u\alpha^T \in \Phi^{-1}(TN)$, where $\Phi^{-1}(TN)$ is the pull-back bundle from the tangent bundle TN of N. Denote ∇, the Riemannian connection of TN, and ∇' the pull-back connection. By Lemma 1.1

(2.1)
$$\begin{aligned}
&\frac{d^2}{dt^2} E_q(\Phi_t^{u\alpha^T})\Big|_{t=0} \\
&= \int_{R^l} \left[\sum_{i=1}^{l} \left(\left|\nabla'_{e_i} u(x)\alpha^T\right|^2 - \left\langle R^N(u\alpha^T, \overline{e}_i)u\alpha^T, \overline{e}_i \right\rangle_N \right) \right] e^{-\frac{|x|^2}{4}} dx \\
&= \int_{R^l} \sum_{i=1}^{l} \left(\left|\frac{\partial u}{\partial x_i}\right|^2 \left|\alpha^T\right|^2 + u^2(x) \left|\nabla'_{e_i}(\alpha - \alpha^\perp)\right|^2 \right) e^{-\frac{|x|^2}{4}} dx \\
&\quad - \int_{R^l} \left[2\frac{\partial u}{\partial x_i} u \left\langle \alpha^T, \nabla'_{e_i}\alpha^\perp \right\rangle_N - u^2(x) \sum_{i=1}^{l} \left\langle R^N(\alpha^T, \overline{e}_i)\alpha^T, \overline{e}_i \right\rangle_N \right] e^{-\frac{|x|^2}{4}} dx \\
&= \int_{R^l} \left[|\nabla u(x)|^2 \left|\alpha^T\right|^2 + \sum_{i=1}^{l} \left(u^2(x) \left| A^{\alpha^\perp} \overline{e}_i \right|^2 \right.\right. \\
&\qquad\qquad \left.\left. + 2\frac{\partial u}{\partial x_i} u \left\langle \alpha^T, A^{\alpha^\perp}\overline{e}_i \right\rangle_N \right) \right] e^{-\frac{|x|^2}{4}} dx \\
&\quad - \int_{R^l} \left(u^2(x) \sum_{i=1}^{l} \left\langle R^N(\alpha^T, \overline{e}_i)\alpha^T, \overline{e}_i \right\rangle_N \right) e^{-\frac{|x|^2}{4}} dx
\end{aligned}$$

where $\overline{e}_i = d\Phi(e_i)$ and the Weingarten map $A^{\alpha^\perp}$ is given by $A^{\alpha^\perp}\overline{e}_i = -\nabla_{\overline{e_i}}\alpha^\perp$. Hence $\left\langle A^{\alpha^\perp}\overline{e}_i, \overline{e}_i \right\rangle = \left\langle A(\overline{e}_i, \overline{e}_i), \alpha^\perp \right\rangle_N$, where $A(\cdot,\cdot)$ denotes the second fundamental form of N.

Assume that $\{\alpha_1, \alpha_2, \cdots, \alpha_q)$ is an orthonormal basis in R^q and $\alpha_1, \alpha_2, \cdots, \alpha_k$ are tangent to N, $\alpha_{k+1}, \alpha_{k+2}, \cdots, \alpha_q$ are orthonormal to N. Taking $\alpha = \alpha_i$ ($i = 1, \cdots, q$), computing the second variational formula along $u(x)\alpha_i$ and summing over i from 1 to q, we obtain by the Gauss curvature equation and the stability assumption

(2.2)

$$
\begin{aligned}
0 &\le \sum_{j=1}^{q} \frac{d^2}{dt^2} E_q(\Phi_t^{u\alpha^T})\Big|_{t=0} \\
&= \int_{R^l} \left[n\left|\nabla u(x)\right|^2 + \sum_{i=1}^{l}\sum_{j=1}^{q} \left(u^2(x)\left|A^{\alpha_j^{\perp}}\overline{e}_i\right|^2 + 2\frac{\partial u}{\partial x_i} u \left\langle \alpha_j^T, A^{\alpha_j^{\perp}}\overline{e}_i \right\rangle_N \right)\right] e^{-\frac{|x|^2}{4}} dx \\
&\quad - \int_{R^l} \left(u^2(x) \sum_{i=1}^{l}\sum_{j=1}^{q} \left\langle R^N(\alpha_j^T, \overline{e}_i)\alpha_j^T, \overline{e}_i \right\rangle_N \right) e^{-\frac{|x|^2}{4}} dx \\
&= \int_{R^l} \left[n\left|\nabla u(x)\right|^2 + \sum_{i=1}^{l}\sum_{j=1}^{n} u^2(x) \left(2\left|A(\alpha_j, \overline{e}_i)\right|^2 - \left\langle A(\alpha_j,\alpha_j), A\left(\overline{e}_i, \overline{e}_i\right)\right\rangle \right)\right] e^{-\frac{|x|^2}{4}} dx \\
&= \int_{R^l} \left[n\left|\nabla u(x)\right|^2 + \sum_{i=1}^{l} u^2(x) \left\langle Q_y^N(\overline{e}_i), \overline{e}_i \right\rangle_N \right] e^{-\frac{|x|^2}{4}} dx
\end{aligned}
$$

where $n = \dim N$, $y = \Phi(x)$. If we assume that Q_x^N is negative definite, then there exists a positive function $\Psi(x)$ defined in R^l such that $\left\langle Q^N_{\Phi(x)}(X), X \right\rangle_N \le -\Psi(x)|X|^2$ where $X = d\Phi(z)$ and z is an arbitrary vector in R^l. In order to prove Theorem 1 we need the following:

Lemma 2.1. *Assume that* $\Phi : R^l \longmapsto N$ *is a smooth map, and* $\Psi(x)$ *is a positive function of* R^l*. If for every smooth real-valued function* $u(x)$ *of* R^l *with compact support,* Φ *satisfies the inequality*

$$
\int_{R^l} \Psi(x) u^2(x) \left|\nabla\Phi\right|^2 e^{-\frac{|x|^2}{4}} dx \le c \int_{R^l} \left|\nabla u\right|^2 e^{-\frac{|x|^2}{4}} dx, \tag{2.3}
$$

then Φ *is a constant map.*

Proof. We choose $u(x)$ to be

$$
u(x) = u_t(x) = \begin{cases} 1, & \text{on } B_t(0), \\ \ln(t^2/|x|)/\ln t, & \text{on } B_{t^2}(0)\setminus B_t(0), \\ 0, & |x| > t^2. \end{cases}
$$

Substituting $u_t(x)$ into (2.2), we have

$$
\begin{aligned}
&\int_{R^l} \Psi(x) u_t^2(x) \left|\nabla\Phi\right|^2 e^{-\frac{|x|^2}{4}} dx \\
&\le c \int_{B_{t^2}\setminus B_t} \frac{1}{(\ln t)^2 |x|^2} e^{-\frac{|x|^2}{4}} dx \\
&\le \frac{c}{(\ln t)^2} \int_{B_{t^2}\setminus B_t} \frac{1}{|x|^2} e^{-\frac{|x|^2}{4}} dx \le \frac{c}{(\ln t)^2},
\end{aligned}
$$

and letting t tend to $+\infty$, we conclude that Φ is constant, which completes Lemma 2.1. □

Combining Lemma 2.1 and (2.2), we obtain the result of Theorem 1.

As an application of Theorem 1, we immediately have

Corollary 2.2. *There are no stable quasi-harmonic maps of* R^l *to any of the following Riemannian manifolds:*

(A) a sphere S^n, $n > 2$.

(B) Grassmann manifolds $sp(p+q)/\,Sp(p) \times Sp(q), 1 \le p \le q, 3 \le p+q$*;*

(C) $SU(2n)/\,sp(n)$, $n > 2$*;*

(D) Cayley Plane $F_4/\,Spin(9)$*;*

(E) E_6/F_4*;*

(F) compactly simply connected simple Lie groups of type $A_n, n \geq 1$*,* B_n *and* $C_n, n \geq 3$*;*

(G) arbitrary finite product of any manifold from (A), (B), (C), (D), (E) and (F).

For the proof we refer to the result of the classification of a compact connected symmetric space N with Q_y^N being negative definite for every y in N in [6] and [8].

3. Another stability inequality, proof of Theorem 2 and Theorem 3

Assume that Φ is a smooth stable quasi-harmonic map from R^l to N, g is a real-valued C^1 function on R^1, and $V_y(g)$ is the gradient of $g \circ \rho_y$ where $\rho_y(y')$ is the geodesic distance of y' from y in N.

Denote by $\Phi_t^{uV_y(g)}$ a variation of Φ with $\Phi_0^{uV_y(g)} = \Phi$, and deformation vector $u(x)V_y(g) \in \Phi^{-1}(TN)$, where $u(x)$ is a smooth compactly supported function on R^l. Denote by ∇' the pull-back connection of Φ. Then from (1.3)

(3.1)

$$\begin{aligned}
&\frac{d^2}{dt^2}E_q(\Phi_t^{u\alpha^T})\Big|_{t=0}\\
&= \int_{R^l}\left[\sum_{i=1}^{l}\left(\left|\nabla'_{e_i}(u(x)V_y(g))\right|^2 - \left\langle R^N(uV_y(g),\overline{e}_i)uV_y(g),\overline{e}_i\right\rangle\right)\right]e^{-\frac{|x|^2}{4}}dx\\
&= \int_{R^l}\sum_{i=1}^{l}u^2(x)\left(\left|\nabla'_{e_i}V_y(g)\right|^2 - \left\langle R^N(V_y(g),\overline{e}_i)V_y(g),\overline{e}_i\right\rangle_N\right)e^{-\frac{|x|^2}{4}}dx\\
&\quad + \int_{R^l}\left[\left|\nabla u\right|^2\left|V_y(g)\right|^2 + u(x)\frac{\partial u}{\partial u_i}\overline{e}_i\left|V_y(g)\right|^2\right]e^{-\frac{|x|^2}{4}}dx
\end{aligned}$$

where $\{e_i, e_i, \cdots, e_i\}$ is the orthonormal basis of R^l and $\overline{e_i} = d\Phi(e_i)$. Integrating the second variational formula along $V_y(g)$ over N for any $y \in N$ with volume element dV of N, and applying the Fubini theorem and the stability assumption, we obtain from (3.1)

(3.2)

$$\begin{aligned}
0 &\leq \int_N \frac{d^2}{dt^2}E_q(\Phi_t^{u\alpha^T})\Big|_{t=0}dV\\
&= \int_{R^l}\left|\nabla u\right|^2 e^{-\frac{|x|^2}{4}}dx\int_N\left|V_y(g)\right|^2dV + \sum_{i=1}^{l}\int_{R^l}u(x)\frac{\partial u}{\partial u_i}e^{-\frac{|x|^2}{4}}dx\int_N\overline{e}_i\left|V_y(g)\right|^2dV\\
&\quad + \int_{R^l}u^2(x)e^{-\frac{|x|^2}{4}}dx\int_N\sum_{i=1}^{l}\left(\left|\nabla'_{e_i}V_y(g)\right|^2 - \left\langle R^N(V_y(g),\overline{e}_i)V_y(g),\overline{e}_i\right\rangle_N\right)dV.
\end{aligned}$$

Proof of Theorem 2. Now we assume that N is simply connected, δ-pinched for $\frac{1}{4} < \delta < 1$; then the radius of injectivity $> \pi$ by a famous theorem of Klingenberge. Letting

$$g(t) = \begin{cases} -\cos t, & |t| \leq \pi,\\ \quad 1, & |t| > \pi,\end{cases}$$

we have

$$V_y(g) = \begin{cases} \sin(\rho_y)\nabla e_y, & \rho_y(y') \leq \pi,\\ 0, & \text{otherwise.}\end{cases}$$

It follows from (8) and the Chauchy-Schwarz inequality that

$$\begin{aligned}
&\sum_{i=1}^{l} \int_{R^l} u(x) \frac{\partial u}{\partial u_i} e^{-\frac{|x|^2}{4}} dx \int_N \overline{e_i} \left|V_y(g)\right|^2 dV \\
&= \sum_{i=1}^{l} \int_{R^l} u(x) \frac{\partial u}{\partial u_i} \left|\overline{e_i}\right| e^{-\frac{|x|^2}{4}} dx \int_N \widehat{e_i} \left|V_y(g)\right|^2 dV \\
&\leq \sum_{i=1}^{l} \int_{R^l} c \left(\left|\frac{\partial u}{\partial u_i}\right|^2 \Big/ 2\varepsilon + u^2(x) \left|\overline{e_i}\right|^2 \frac{\varepsilon}{2} \right) e^{-\frac{|x|^2}{4}} dx \\
&= \frac{c}{2\varepsilon} \int_{R^l} |\nabla u|^2 e^{-\frac{|x|^2}{4}} dx + \frac{c\varepsilon}{2} \int_{R^l} u^2(x) \left|d\Phi\right|^2 e^{-\frac{|x|^2}{4}} dx
\end{aligned} \tag{3.3}$$

where

$$\widehat{e_i} = \begin{cases} \frac{\overline{e_i}}{|\overline{e_i}|}, & \text{if } \overline{e_i} \neq 0, \\ 0, & \text{otherwise,} \end{cases} \qquad \varepsilon > 0 \text{ is a constant and } c = \max_N \left|\nabla \left|V_y(g)\right|^2\right| vol(N).$$

Furthermore, an estimate of Howard [5], followed from manipulation of the volume comparison theorem of Bishop and Crittenden, and the Hessian comparison theorem of Greene and Wu, implies

$$\begin{aligned}
&\int_{R^l} u^2(x) e^{-\frac{|x|^2}{4}} dx \int_N \sum_{i=1}^{l} \left(\left|\nabla'_{e_i} V_y(g)\right|^2 - \left\langle R^N(V_y(g), \overline{e}_i) V_y(g), \overline{e}_i \right\rangle_N \right) dV \\
&\leq vol(S^{n-1}) F_n(\delta) \int_{R^l} u^2(x) \left|\nabla\Phi\right|^2 e^{-\frac{|x|^2}{4}} dx
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
F_n(\delta) = \int_0^\pi [&\max\left\{\cos^2 t, \delta \sin^2 t \cos^2 \sqrt{\delta} t \Big/ \sin^2 \sqrt{\delta} t\right\} \\
&\times \left(\sin \sqrt{\delta} t \Big/ \sqrt{\delta}\right)^{n-1} - (n-1)\delta \cos^2 t \sin^{n-1} t] dt.
\end{aligned} \tag{3.5}$$

Combining (3.2)–(3.5) we obtain

$$\begin{aligned}
&\left(-vol(S^{n-1}) F_n(\delta) - \tfrac{c\varepsilon}{2}\right) \int_{R^l} u^2(x) \left|\nabla\Phi\right|^2 e^{-\frac{|x|^2}{4}} dx \\
&\leq \left(1 + \tfrac{c}{2\varepsilon}\right) \int_{R^l} |\nabla u|^2 e^{-\frac{|x|^2}{4}} dx.
\end{aligned} \tag{3.6}$$

Now let $\delta_n = \inf\left\{\delta : \frac{1}{4} < \delta \text{ and } \mathrm{F}_n(\delta) < 0\right\}$, and choose ε sufficiently small, then (3.6) implies

$$\int_{R^l} u^2(x) \left|\nabla\Phi\right|^2 e^{-\frac{|x|^2}{4}} dx \leq c \int_{R^l} |\nabla u|^2 e^{-\frac{|x|^2}{4}} dx \tag{3.7}$$

So the proof of Theorem 2 is completed by combining (3.7) and Lemma 2.1. □

Proof of Theorem 3. Since N is simple, we can introduce local coordinates $u = (u_1, u_2, \cdots, u_n)$ such that, for any map $u \in C^2\left(R^l, N\right)$ with its representation $u = (u_1, u_2, \cdots, u_n)$, the quasi-energy integral is of the form

$$E_q(u) = \int_{R^l} f(x, u, \nabla u) e^{-\frac{|x|^2}{4}} dx$$

where there exist two numbers λ and μ with $0 < \lambda < \mu$ such that

$$\lambda |p|^2 \leq f(x, u, p) \leq \mu |p|^2$$

for all $(x, u, p) \in R^l \times R^n \times R^{nl}$.

Set $B_r = \left\{x \in R^l : |x| \leq r\right\}, T_{2r} = \left\{x \in R^l : r \leq |x| \leq 2r\right\}$. Then, for each $r > 1$, there is a function $\eta \in C_0^\infty(B_{2r}, R)$ with $0 \leq \eta \leq 1$ and $\eta(x) = 1$ on B_r such that $|\nabla\eta| \leq C \cdot r^{-1}$ where $C > 0$ does not depend on r . Let

$$\omega = \frac{1}{vol\, T_{2r}} \int_{T_{2r}} u(x)\, dx$$

and set $v(x) = (1 - \eta(x))\left[u(x) - \omega(x)\right]$. Since

$$E_q(u) \leq E_q(v),$$

we infer from above that

$$\begin{aligned} \lambda \int_{B_{2r}} |\nabla u|^2 e^{-\frac{|x|^2}{4}} dx &\leq 2\mu \int_{T_{2r}} (1-\eta)^2 |\nabla u|^2 e^{-\frac{|x|^2}{4}} dx \\ +2\mu C^2 r^2 \int_{T_{2r}} |u-\omega|^2 e^{-\frac{|x|^2}{4}} dx &\leq 2\mu e^{-\frac{r^2}{4}} \int_{T_{2r}} |\nabla u|^2 dx \\ &\quad +2\mu e^{-\frac{r^2}{4}} C r^{-2} \int_{T_{2r}} |u-\omega|^2 dx. \end{aligned}$$

By the Poincaré inequality, there is a number C_0 independent of r such that

$$\int_{T_{2r}} |u-\omega|^2 dx \leq C_0 r^{-2} \int_{T_{2r}} |\nabla u|^2 dx.$$

Thus we obtain that

$$\int_{B_{2r}} |\nabla u|^2 e^{-\frac{|x|^2}{4}} dx \leq c e^{-\frac{r^2}{4}} \int_{T_{2r}} |\nabla u|^2 dx.$$

If we assume that $\lim_{r\to+\infty} e^{-\frac{r^2}{4}} \int_{T_{2r}} |\nabla u|^2 dx = 0$, we can obtain that $u \equiv$constant. If u has finite energy, the above assumption is obviously true, and this completes the proof. □

Remark 1. Although the assumption of finite energy for u can be considerably relaxed, we cannot prove Theorem 3 under the assumption that $E_q(u) < \infty$. We will consider this problem in a future paper. For some results on the finiteness of energy of quasi-harmonic maps we refer to [10].

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References

1. CHENG S. Y., Liouville theorem for harmonic maps, *Pure Math., Vol. 36, American Math. Soc., Providence, R.I.*, 147-151(1980). MR **81i:**58021
2. DING W. Y. & LIN F. H., A generalization of Eells-Sampson's theorem, *J. Partial Differential Equations* 5, no. 4, 13-22(1992). MR **93j:**58032
3. EELLS J. & LEMAIRE K., A report on harmonic maps, *Bull. London Math. Soc.* 10 1-68(1978). MR **82b:**58033
4. LIN F. H. & WANG C. Y., Harmonic and quasi-harmonic spheres, *Comm. Anal. Geom.* 7, 397-429(1999). MR **2000b:**58028
5. HOWARD R., The nonexistence of stable submanifolds, varifolds, and harmonic maps in sufficiently pinched simply connected Riemannian manifold, *Michigan Math. J.* 32, 321-334(1985). MR **87h:**58040
6. HOWARD R. & WEI S. W., Nonexistence of stable harmonic maps to and from certain homogeneous spaces and submanifolds of Euclidean space, *Trans. Amer. Math. Soc.* 294, 319-331(1986). MR **87c:**58033

7. WEI S. W., Liouville theorem for stable harmonic maps into either strongly unstable, or δ-pinched manifolds. *Proc. of Symp. in Pare Math. Vol* 44, 406-412(1986).
8. WEI S. W., Liouville Theorems and regularity of minimizing harmonic maps into super-strongly unstable manifolds, *Contemporary Mathematics, Vol.* 127, 131-154(1992). MR **92m:**58030
9. XIN Y. L., Some results on stable harmonic maps, *Duke. Math. J.* 47, 609-613(1980). MR **81j:**58041
10. HSU D. L. & ZHOU C Q., On the finiteness of energy of quasi-harmonic sphere, preprint.

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