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THE SEIFERT-VAN KAMPEN THEOREM AND GENERALIZED FREE PRODUCTS OF S-ALGEBRAS

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ABSTRACT. In a Seifert-van Kampen situation a path-connected space Z may be written as the union of two open path-connected subspaces X and Y along a common path-connected intersection W. The fundamental group of Z is isomorphic to the colimit of the diagram of fundamental groups of the three subspaces. In case the maps of fundamental groups are all injective, the fundamental group of Z is a classical free product with amalgamation, and the integral group ring of the fundamental group of Z is also a free product with amalgamation in the category of rings. In this case relations among the K-theories of the group rings have been studied. Here we describe a generalization and stablization of this algebraic fact, where there are no injectivity hypotheses on the fundamental groups and where we work in the category of S-algebras. Some of the methods we use are classical and familiar, but the passage to S-algebras blends classical and new techniques. Our most important application is a description of the algebraic K-theory of the space Z in terms of the algebraic K-theories of the other three spaces and the algebraic K-theory of spaces Nil-term.

1. INTRODUCTION

Consider a situation

$$\begin{tikzcd} W \arrow[r, tail] \arrow[d, tail] & X \arrow[d, tail] \\ Y \arrow[r, tail] & Z \end{tikzcd} \tag{1.1}$$

where the space Z is given as the union of two open subsets X and Y with intersection W. If all spaces are path-connected and are given the basepoint $w \in W$, then the Seifert-van Kampen theorem [6, page 114] states that the diagram of fundamental groups

$$\begin{tikzcd} \pi_1(W,w) \arrow[r] \arrow[d] & \pi_1(X,w) \arrow[d] \\ \pi_1(Y,w) \arrow[r] & \pi_1(Z,w) \end{tikzcd} \tag{1.2}$$

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is a pushout diagram, or a generalized free product diagram, in the category of groups. Since the integral group ring functor is left adjoint to the functor which assigns to a ring with identity its group of invertible elements, passing to integral group rings yields another generalized free product diagram in the category of rings with identity. In case the maps leaving $\pi_1(W, w)$ are injective, all maps in the diagram are injective and the pushout is called a free product along an amalgamated subgroup. In [8], for example, the algebraic K-theory of the group rings arising in the injective situation is carefully studied. Generalizing the algebraic K-theory of rings, Waldhausen defined the algebraic K-theory of topological spaces in [10] and [9]. Applying the algebraic K-theory of spaces functor to diagram (1.1) yields a diagram:

$$\begin{array}{ccc} A(W) & \longrightarrow & A(X) \\ \downarrow & & \downarrow \\ A(Y) & \longrightarrow & A(Z) \end{array}$$

It is natural to ask how $A(Z)$ is related to the other algebraic K-theories $A(W)$, $A(X)$, $A(Y)$.

In [1, page 125] one of Waldhausen's definitions for the algebraic K-theory of a topological space is reformulated as the algebraic K-theory of a "group ring" formed in the category of S-algebras. Namely, for a pointed connected simplicial set X, the algebraic K-theory of X may be defined as $A(X) = K(S \wedge |G(X)|_+)$, the K-theory of the S-algebra associated to the topological group $|G(X)|$ which is the realization of the Kan loop group $G(X)$ of X. A definition of the functor $S \wedge -$ from pointed spaces to S-modules will be recalled below in Definition 2.4. Also, according to [1], the category of S-algebras is cocomplete, so there exists a generalized free product $(S \wedge |G(X)|_+) *_{S\wedge|G(W)|_+} (S \wedge |G(Y)|_+)$, or pushout, associated to the three smaller spaces in the diagram.

In [7] we began the study of the algebraic K-theory of generalized free products of S-algebras. If certain cofibration conditions on the S-algebra homomorphisms $S \wedge |G(W)|_+ \to S \wedge |G(X)|_+$ and $S \wedge |G(W)|_+ \to S \wedge |G(Y)|_+$ are satisfied, then we can calculate the K-theory of the generalized free product in terms of the K-theory of the other three rings, plus a Nil-term. Indeed, the results follow the pattern of results obtained in [8] for the K-theory of group rings. In Theorem 1.1 below we prove that, if the diagram (1.1) is a pushout diagram of reduced simplicial sets, then an isomorphism of S-algebras

$$(S \wedge |G(X)|_+) *_{S\wedge|G(W)|_+} (S \wedge |G(Y)|_+) \longrightarrow S \wedge |G(Z)|_+$$

arises from the associated diagram of S-algebras by means of the universal property of the generalized free product. We can view this result as a stabilization and globalization of the classical fact recalled above, that a Seifert-van Kampen situation gives rise to a generalized free product diagram in the category of rings. It is a stabilization in the sense that the sphere spectrum S replaces the integers as the ring of coefficients, and it is a globalization in the sense that it involves the higher homotopy groups of the spaces in the diagram, as well as their fundamental groups. In particular, interesting examples are obtained by letting X and Y be cones on W, so that Z is the suspension of W.

Moreover, we also show in Theorem 1.1 that the generalized free product diagram of S-algebras satisfies cellularity conditions that are stronger than the cofibration conditions we actually required in [7] to prove the K-theory decomposition results. Combining these results with the main result of [7] provides a decomposition theorem for the algebraic K-theory of the space Z, stated as Corollary 1.2 below.

The next section comprises the proof of Theorem 1.1. The proofs draw on some standard simplicial homotopy theory, some standard facts about the category of S-algebras, and some techniques developed in [7] for working on generalized free products of S-algebras. Our proof of the algebraic part of the result is essentially calculational; in turn, these calculations enable us to exhibit the cell structures we require for the application to K-theory.

In section 3 we extend our first result to cover a more usual pushout diagram

(1.3)
$$\begin{tikzcd} W \arrow[r, tail] \arrow[d] & X \arrow[d] \\ Y \arrow[r, tail] & Z \end{tikzcd}$$

of reduced simplicial sets in which only the horizontal arrows are cofibrations. We obtain in Theorem 3.1 that the diagram of S-algebras arising from this situation induces a homotopy equivalence of S-algebras

$$(S \wedge |G(X)|_+) *_{S\wedge|G(W)|_+} (S \wedge |G(Y)|_+) \longrightarrow S \wedge |G(Z)|_+.$$

Using this equivalence we explain how the K-theory fibration sequence of Corollary 1.2 generalizes to the more usual situation in Corollary 3.2. We conclude section 3 by explaining how to apply our results for pushout diagrams of reduced simplicial sets to pushout diagrams of connected pointed Kan sets and topological spaces.

In the following statement we assume that each simplicial set is reduced, which means that the set of zero-simplices consists of a single element. In particular, a reduced simplicial set is connected. This hypothesis is technically convenient and involves no real loss of generality, as we explain at the end of section 3.

Theorem 1.1. *Let*

$$\begin{tikzcd} W \arrow[r, tail] \arrow[d, tail] & X \arrow[d, tail] \\ Y \arrow[r, tail] & Z \end{tikzcd}$$

be a pushout diagram of reduced simplicial sets, where all simplicial sets are pointed by the zero-simplex w of W. Then the associated diagram of S-algebras

$$\begin{tikzcd} S \wedge |G(W)|_+ \arrow[r] \arrow[d] & S \wedge |G(X)|_+ \arrow[d] \\ S \wedge |G(Y)|_+ \arrow[r] & S \wedge |G(Z)|_+ \end{tikzcd}$$

induces an isomorphism

(1.4)
$$(S \wedge |G(X)|_+) *_{S\wedge|G(W)|_+} (S \wedge |G(Y)|_+) \longrightarrow S \wedge |G(Z)|_+$$

from the generalized free product defined in [7] *to the S-algebra $S \wedge |G(Z)|_+$.*

All the S-algebras in the diagram above are cell S-algebras. Moreover, the S-algebras $S \wedge |G(X)|_+$ and $S \wedge |G(Y)|_+$ are cell $S \wedge |G(W)|_+$-modules.

In view of the cellularity properties of the maps to $S\wedge|G(X)|_+$ and $S\wedge|G(Y)|_+$ and the isomorphism (1.4), Theorem 1.1 of ([7]) applies and there is a fibration-up-to-homotopy

$$\begin{aligned}&K(\mathcal{MV}^w_f, v)\to\\ &\quad K(S\wedge|G(W)|_+)\times K(S\wedge|G(X)|_+)\times K(S\wedge|G(Y)|_+)\to K(S\wedge|G(Z)|_+)\end{aligned}$$

where the fiber is the K-theory of the category of split Mayer-Vietoris presentations of finite cell modules associated to the generalized free product diagram of S-algebras given in the theorem. If we make the appropriate substitutions, then we obtain the following corollary.

Corollary 1.2. *Given a diagram* (1.1) *there is a fibration-up-to-homotopy*

$$K(\mathcal{MV}^w_f, v)\longrightarrow A(W)\times A(X)\times A(Y)\longrightarrow A(Z).$$

□

2. Main results

In this section we describe the passage from a pushout diagram (1.1) of reduced simplicial sets to a pushout diagram of simplicial groups and on to a pushout diagram of S-algebras. Not only are we interested in the preservation of the pushout property at each step, but we also need to know that the end result of all the constructions has the "correct" homotopy type. One way to ensure this is to see that the final diagram possesses certain cofibrancy properties. For the passage from reduced simplicial sets to simplicial groups, the verifications are very easy. We obtain the preservation of the pushout diagram by appealing to the fact that the loop group functor on reduced simplicial sets is a left adjoint, and we obtain cofibration conditions of the type we need for the second step by examining the definition of the loop group functor. For the passage from simplicial groups to S-algebras, the verifications are more difficult. We choose to handle both verifications by analysing the composite functor that takes us from simplicial groups to S-algebras. Thus, we obtain computational proofs that this step preserves pushouts and that the cofibration conditions on the diagram of S-algebras needed to apply the K-theory results of [7] are indeed satisfied.

Lemma 2.1. *Given a pushout diagram* (1.1) *of reduced simplicial sets and cofibrations, the diagram*

$$\begin{tikzcd} G(W) \arrow[r, tail] \arrow[d, tail] & G(X) \arrow[d, tail] \\ G(Y) \arrow[r, tail] & G(Z) \end{tikzcd} \tag{2.1}$$

is a pushout diagram of simplicial groups. Moreover, in each dimension n all the homomorphisms are injective, and $G_nX\to G_nZ$ and $G_nY\to G_nZ$ induce an isomorphism

$$G_n(X)*_{G_n(W)}G_n(Y)\longrightarrow G_n(Z)$$

from the free product of $G_n(X)$ and $G_n(Y)$ amalgamated over $G_n(W)$ to $G_n(Z)$.

Proof. For a reduced simplicial set X the definition of the loop group $G(X)$ is found in [3, page 291]. Moreover, it is proved in [4, Proposition 10.5, page 49] that the functor G from reduced simplicial sets to simplicial groups is a left adjoint. Since functors which are left adjoints preserve colimits, the fact that the diagram of simplicial groups is a pushout diagram of simplicial groups is just a specialization of the fact that a functor which is a left adjoint preserves colimits. Since a cofibration of simplicial sets is an injection, the injectivity of the homomorphisms follows from the definition of the loop group. The last statement is the dimensionwise interpretation of the fact that the diagram of simplicial groups is a pushout. □

Remark 2.2. The result is also stated as Theorem 20.1 of [3] with a computational proof.

Now we need a familiar result [5, Theorem 4.4, page 201] about the free product of two groups amalgamated along a common subgroup.

Theorem 2.3. *Fix a dimension n, and suppose that specific right coset representative systems U_X and U_Y for $G_n(W)$ in $G_n(X)$ and $G_n(Y)$, respectively, have been chosen. For example, each right coset $G_n(W)x \in G_n(X)/G_n(W) - G_n(W)$ is assigned a unique representative $x' \in G_n(W)x$, and U_X is the set of all representatives x'. Then to each element z of the generalized free product $G_n(X) *_{G_n(W)} G_n(Y)$ we can associate a unique sequence $(w, u_1, u_2, \ldots, u_r)$ such that*

1. *w is an element of $G_n(W)$,*
2. *u_i is in U_X or in U_Y, and u_i and u_{i+1} are not both in U_X or U_Y,*
3. *$z = wu_1u_2 \cdots u_r$.* □

The result has this interpretation: If we say that the word length of z is r when the product decomposition is as in the theorem, then $G_n(X) *_{G_n(W)} G_n(Y)$ carries a filtration by word length and, for $r \geq 1$, the quotient of the subset of words of length r by the words of length $r-1$ is isomorphic to the disjoint union

$$G_n(W) \times (G_n(X)/G_n(W)) \times (G_n(Y)/G_n(W)) \times \cdots \amalg \\ G_n(W) \times (G_n(Y)/G_n(W)) \times (G_n(X)/G_n(W)) \times \cdots ,$$

where there are r terms in each of the products to the right of $G_n(W)$.

Now we take up the passage to the category of S-algebras. First we define the functor $S \wedge -$ which plays a role analogous to that of the integral group ring functor in the passage from groups to algebras.

Definition 2.4. Let (spaces) be the category of pointed compactly generated weak Hausdorff spaces, and define the functor $S\wedge$: (spaces) $\longrightarrow$ (S-modules) on objects by the formula

$$S \wedge X = S \wedge_{\mathcal{L}} \mathbb{L}\Sigma^\infty X$$

where $\Sigma^\infty X$ is the spectrum associated to a suspension prespectrum generated by X, $\mathbb{L}$ is defined by [1, page 17], and $S \wedge_{\mathcal{L}} -$ is defined as in [1, Lemma II.1.3, page 32]. Make the obvious extension to define $S\wedge$ on morphisms. Since $S \wedge -$ is a functor which carries a smash product of pointed spaces to a smash product of S-modules, if G is a topological group, then $S \wedge G_+$ will be an S-algebra.

Remark 2.5. Our definition of the functor $S \wedge -$ explicitly incorporates the composite functor $S \wedge_{\mathcal{L}} \mathbb{L}\Sigma^\infty -$ in order to ensure that cellular spaces are taken to cell S-modules. According to [1, page 33] the building blocks for cellular constructions are cones attached along the sphere S-modules $S_S^{n-1} \equiv S \wedge_{\mathcal{L}} \mathbb{L}S^{n-1}$. The extra complication is necessary to get around the fact [1, page 39] that the S-module S does not have the homotopy type of a CW S-module.

Now we reach the main part of the proof of Theorem 1.1. We keep in mind that we want to prove the algebraic statement that

$$\begin{array}{ccc} S \wedge |G(W)|_+ & \longrightarrow & S \wedge |G(X)|_+ \\ \downarrow & & \downarrow \\ S \wedge |G(Y)|_+ & \longrightarrow & S \wedge |G(Z)|_+ \end{array}$$

is a generalized free product diagram of S-algebras, and the homotopic theoretic statement that the algebras $S \wedge |G(X)|_+$ and $S \wedge |G(Y)|_+$ are *cell* $S \wedge |G(W)|_+$-modules. The computational proof we give for the first statement provides input for the proof of the second statement. To give a brief outline, structural results from [7] quoted below in Theorem 2.6 apply to the diagram of simplicial S-algebras obtained from the diagram (2.1) by applying the functor $S \wedge -\colon (\text{spaces}) \longrightarrow (S\text{-modules})$ in each dimension. These results permit a computational verification of the assertions in each simplicial dimension. Then we want to exploit the fact that geometric realization commutes with the functor $S \wedge -$ from spaces to S-modules. However, the generalized free product $B *_A C$ is an algebra under A defined as a colimit of a certain diagram in the category of A-bimodules, so we must also see that $B *_A C$ can also be computed in the category of S-modules as a colimit of a different diagram. These facts allow us to assemble the dimension-wise results into the statements we want.

To state the precise result, we need the monomial word modules in A-bimodules M and N, which are the smash products

$$M \wedge_A N \wedge_A M \wedge_A \cdots \qquad \text{and} \qquad N \wedge_A M \wedge_A N \wedge_A \cdots .$$

From [7, Theorem 2.7] and its proof we obtain the following facts about the generalized free product $B *_A C$ using the construction in the category of A-bimodules.

Theorem 2.6. *Let $\beta\colon A \longrightarrow B$ and $\gamma\colon A \longrightarrow C$ be a pair of inclusions of S-algebras that are cell left A-modules relative to A. Then $B *_A C$ is a cell left A-module relative to A.*

Indeed, there is an expanding sequence of A-submodules

$$A = F_0 D \subset F_1 D \subset \cdots \subset F_n D \subset \cdots \subset B *_A C = D$$

such that, for $n \geq 1$, $F_n D / F_{n-1} D$ is the wedge of the two monomial word modules of length n in B/A and C/A. □

Let us write $A_n = S \wedge G_n(W)_+$, $B_n = S \wedge G_n(X)_+$, $C_n = S \wedge G_n(Y)_+$, and $D'_n = S \wedge G_n(Z)_+$ for the S-algebras obtained by applying the functor $S \wedge -$ to each dimension of the diagram (2.1). We also let $D_n = B_n *_{A_n} C_n$ be the generalized free product of B_n and C_n over A_n. We give D_n the filtration described in Theorem 2.6. For each simplicial dimension n, let D'_n be given the filtration by A_n-submodules induced from the filtration on the group $G_n(Z)$ recalled above in Theorem 2.3. Observe that, since $S \wedge -$ is a functor, and since the generalized free product can also be constructed functorially, we obtain simplicial S-algebras $A_\bullet$, $B_\bullet$, $C_\bullet$, $D_\bullet$, and $D'_\bullet$. Using these notations we have the following lemma.

Lemma 2.7. *Under the hypotheses of Lemma 2.1 and for any $n \geq 0$, A_n, B_n, C_n, and D_n are cell S-modules, B_n and C_n are cell left A_n-modules, and the map of S-algebras*

$$B_n *_{A_n} C_n = D_n \longrightarrow D'_n$$

induced from the S-algebra morphisms $B_n \to D'_n$ and $C_n \to D'_n$ is filtration preserving and induces an isomorphism of S-modules on filtration quotients in each simplicial dimension n. In particular, the map of S-algebras $D_n \to D'_n$ is an isomorphism.

Proof. It is easy to see that $A_n = S \wedge G_n(W)_+ = S \wedge_{\mathcal{L}} \mathbb{L}\Sigma^\infty G_n(W)_+$ is a cell S-module, for example. Each element $g \in G_n(W)$ defines an inclusion $S^0 \to G_n(W)_+$ taking the basepoint of S^0 to $+$ and the nonbasepoint element to g. Consequently, g induces a morphism of S-modules $S \wedge S^0 \to S \wedge G_n(W)_+$. Assembling these for all elements of $G_n(W)$ we obtain an isomorphism $\bigvee_{g \in G_n(W)} S \wedge S^0 \xrightarrow{\cong} A_n$ which exhibits a one-stage cellular filtration on A_n.

To see that B_n is a cell left A_n-algebra, we once again return to the definitions. Let us write $H = G_n(W)$ and $G = G_n(X)$ so that the notation is compact. In the category of spaces we can display the discrete space G_+ as a pushout by means of the diagram

$$\begin{tikzcd} + \arrow[r, tail] \arrow[d] & H_+ \wedge (G/H)^0_+ \arrow[d] \\ H_+ \arrow[r, tail] & G_+ \end{tikzcd}$$

where $(G/H)^0$ denotes the set of cosets $\{Hg\}$ with the identity coset omitted. As a space, $(G/H)^0_+$ is just a wedge of 0-dimensional spheres, and on the summand corresponding to Hg the set H is mapped into G in the obvious way, by $h \mapsto hg$. Apply the composite functor $S \wedge - = S \wedge_{\mathcal{L}} \mathbb{L}\Sigma^\infty -$ and note that all these functors are left adjoints by [1, pages 19 and 32], so that the resulting diagram is a pushout diagram of S-modules. Also note that, thanks to the behavior of $\mathbb{L}$ with respect to smash products described by [1, page 22], the resulting diagram may be placed into the final form

$$\begin{tikzcd} + \arrow[r, tail] \arrow[d] & ((S \wedge_{\mathcal{L}} \mathbb{L}\Sigma^\infty H_+) \wedge_S (S \wedge_{\mathcal{L}} \mathbb{L}\Sigma^\infty (G/H)^0_+) \arrow[d] \\ S \wedge_{\mathcal{L}} \mathbb{L}\Sigma^\infty H_+ \arrow[r, tail] & S \wedge_{\mathcal{L}} \mathbb{L}\Sigma^\infty G_+ \end{tikzcd}$$

which displays $S \wedge G_+$ as $S \wedge H_+$ with a free $S \wedge H_+$-module [1, Definition III.1.2] trivially adjoined. Thus, we obtain that $S \wedge G_n(X)_+$ is a cell left $S \wedge G_n(W)_+$-module for each n. Exactly the same argument handles the case of $S \wedge G_n(Y)_+$.

To conclude the proof, take the results of Theorem 2.3, apply $S \wedge -$, and compare what appears with the conclusion of Theorem 2.6. $\square$

Now that we have cell S-module structures in each simplicial dimension, we have only to assemble them. For this recall that the functor $S\wedge$: (spaces) $\longrightarrow$ (S-modules) commutes with geometric realization, according to [1, Proposition X.1.3, page 181]. We have just used a definition of $B *_A C$ as a colimit of a certain diagram in the category of A-bimodules. We now show how to create a diagram in the category of S-modules whose colimit also calculates $B *_A C$.

To calculate $B*_A C$ as a colimit in the category of S-modules, let $\mathbb{T}\colon \mathcal{M}_S \longrightarrow \mathcal{M}_S$ be the tensor algebra monad on the category of S-modules [1, page 47], and follow the recipe given in [1, Proposition II.7.4, page 48] in the case at hand. In detail, note that by the definition of an S-algebra there are S-module maps $\mathbb{T}A \longrightarrow A$, $\mathbb{T}B \longrightarrow B$, and $\mathbb{T}C \longrightarrow C$ codifying the S-algebra structures on A, B, and C, respectively. Let

$$e\colon \mathbb{T}(\mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C) \longrightarrow \mathbb{T}(B \cup_A C)$$

denote the map obtained by applying $\mathbb{T}$ to the obvious map induced by the algebra structures. Denote the canonical maps to the pushout by $i_A\colon A \longrightarrow B \cup_A C$, $i_B\colon B \longrightarrow B \cup_A C$, and $i_C\colon C \longrightarrow B \cup_A C$ and let

$$\alpha\colon \mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C \longrightarrow \mathbb{T}(B \cup_A C)$$

be the unique map such that $\mathbb{T}(B) \longrightarrow \mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C \xrightarrow{\alpha} \mathbb{T}(B \cup_A C)$ agrees with $\mathbb{T}(i_B)$ and similarily for A and C. Then define

$$f\colon \mathbb{T}(\mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C) \longrightarrow \mathbb{T}(B \cup_A C)$$

to be the composite $\mathbb{T}(\mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C) \xrightarrow{\mathbb{T}(\alpha)} \mathbb{T}(\mathbb{T}(B \cup_A C)) \xrightarrow{\mu} \mathbb{T}(B \cup_A C)$. Finally, define $B \amalg_A C$ by the following coequalizer diagram of S-modules.

$$\mathbb{T}(\mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C) \underset{f}{\overset{e}{\rightrightarrows}} \mathbb{T}(B \cup_A C) \longrightarrow B \amalg_A C.$$

According to [1, Proposition II.7.4, page 48], the S-module $B \amalg_A C$ is an S-algebra, and, according to the following result [7, Proposition 2.9], it is the generalized free product S-algebra $B *_A C$.

Proposition 2.8. *The two diagrams of S-algebras*

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d] \\ C \arrow[r] & B \amalg_A C \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d] \\ C \arrow[r] & B *_A C \end{tikzcd}$$

are isomorphic by a map of diagrams respecting the corners A, B and C. □

Proof of Theorem 1.1. Recall that we are working with a diagram of reduced simplicial sets

$$\begin{tikzcd} W \arrow[r, tail] \arrow[d, tail] & X \arrow[d, tail] \\ Y \arrow[r, tail] & Z \end{tikzcd}$$

where all the arrows are cofibrations. Then the results of Lemma 2.7 apply and the map of simplicial S-algebras

$$D_\bullet \longrightarrow D'_\bullet$$

is an isomorphism in each dimension. Therefore, the realization is an isomorphism of S-algebras. To describe the realization of the target of the arrow, we mentioned immediately following the proof of Lemma 2.7 that $S \wedge -$ commutes with geometric realization. Since $D'_n = S \wedge G_n(Z)_+$, we have

$$D' := |D'_\bullet| \cong S \wedge |G(Z)|_+.$$

To describe the realization on the domain of the arrow above, note that both geometric realization [1, page 180] and free product construction using the monad $\mathbb{T}$ are coends in the category of S-modules. Applying the general principle that coends commute, geometric realization and the free product construction commute, so that

$$D := |D_\bullet| \cong B *_A C$$

since $D_n := B_n *_{A_n} C_n$ and where we have written $B = |B_\bullet|$, and so on. Applying once again the fact [1, Proposition X.1.3, page 181] that realization commutes with $S \wedge -$, we obtain $B \cong S \wedge |G(X)|_+$, and similarly for the other terms. This proves the first statement of Theorem 1.1.

To obtain the cell S-algebra structures on A, B, C, and D, we apply [1, Theorem X.2.7, page 184], which gives conditions under which geometric realization produces a cell object. Concerning $A = |A_\bullet|$, we have already verified that each A_n is a cell S-module in Lemma 2.7. We also need to confirm that each degeneracy operator is the inclusion of a subcomplex. But this follows from the fact [4, Proposition 10.2, page 48] that $G(W)$ is a free simplicial group in the sense of [4, Definition 5.1, page 43]. The point is that the degeneracies of $G(W)$ carry generators in one dimension to generators in another dimension. Finally, we need to confirm that each face operator $A_n \to A_{n-1}$ is sequentially cellular. But here we have constructed A_n and A_{n-1} as cellular objects with one-stage filtrations, so this condition is trivially satisfied.

For the very last statement in Theorem 1.1, we want to see that B is a cell A-module. (The argument for C is, of course, similar.) We need to see that there is a filtration

$$A = F_0 B \subset \cdots \subset F_{n-1}B \subset F_n B \subset \cdots \subset B$$

such that F_nB is the cofiber of a map $W_{n-1} \to F_{n-1}B$, where W_{n-1} is a wedge of sphere modules $A \wedge_{\mathcal{L}} \mathbb{L}\Sigma^\infty S^q$ of varying dimensions.

We have already observed that the cofibration hypothesis $W \rightarrowtail X$ implies that $H = G(W) \to G = G(X)$ is dimensionwise injective. Then, for each n, G_n is a free H_n-set. We conclude by [2, Lemma V.2.8] that there is a filtration

$$\emptyset = F_{-1}G \subset \cdots \subset F_{n-1}G \subset F_nG \subset \cdots \subset G$$

such that for each $n \geq 0$ there is a pushout diagram

$$\begin{tikzcd} \coprod_\alpha H \times \partial\Delta^n \arrow[r] \arrow[d, tail] & F_{n-1}G \arrow[d, tail] \\ \coprod_\alpha H \times \Delta^n \arrow[r] & F_nG \end{tikzcd}$$

in the category of H-sets. (The result follows from an analysis of the pullback to G of the skeleton filtration of G/H.) As geometric realization of H-sets is still left adjoint to the singular functor, realization of the diagram above yields a pushout diagram of $|H|$-spaces. Now add disjoint basepoints and apply the functor $S \wedge - = S \wedge_{\mathcal{L}} \mathbb{L}\Sigma^\infty -$ from the category of pointed spaces to the category of S-modules. We have also observed earlier that all the functors in this composite are left adjoints, so there results a pushout diagram

$$\begin{tikzcd} \bigvee_\alpha (S \wedge |H|_+) \wedge S_S^{n-1} \arrow[r] \arrow[d, tail] & S \wedge |F_{n-1}G|_+ \arrow[d, tail] \\ \bigvee_\alpha (S \wedge |H|_+) \wedge CS_S^{n-1} \arrow[r] & S \wedge |F_n G|_+ \end{tikzcd}$$

in the category of S-modules. Here we have set $S_S^{n-1} \equiv S \wedge_{\mathcal{L}} \mathbb{L}S^{n-1}$, using the abbreviation suggested in [1, page 33] for a standard sphere S-module. But this diagram is, in fact, a diagram of modules over $A = S \wedge |H|_+$. According to [1, Theorem III.1.1, page 51] colimits of diagrams of A-modules and A-module maps are created in the category of S-modules, so the diagram is actually a pushout diagram of A-modules. Thus we obtain a filtration of $B = S \wedge |G|_+$ that exhibits B as a cell A-module. This completes the proof of Theorem 1.1. □

Remark 2.9. In this section the use of the commutative S-algebra S as the ground ring has played no special role. In fact, we may substitute an arbitrary commutative S-algebra R for S and work in the categories of R-algebras and R-modules. However, to obtain an application to K-theory using the results of [7], we would need to require R to be connective.

3. An extension and concluding remarks

One may also ask what may be achieved if one starts with a standard pushout diagram of reduced simplicial sets

(3.1)
$$\begin{tikzcd} W \arrow[r, tail] \arrow[d, "g"'] & X \arrow[d] \\ Y \arrow[r, tail] & Z \end{tikzcd}$$

in which only the horizontal arrows are cofibrations. As before, the loop group construction carries the pushout diagram of reduced simplicial sets to a pushout diagram of simplicial groups, and the issue is, therefore, the passage from simplicial groups to S-algebras. In the following paragraphs we indicate one way of extending the results of section 1 to cover this case, obtaining the following result.

Theorem 3.1. *The arrow*

$$S \wedge |G(X)|_+ *_{S \wedge |G(W)|_+} S \wedge |G(Y)|_+ \longrightarrow S \wedge |G(Z)|_+$$

arising from a standard pushout diagram (3.1) *through the universal property of the generalized free product is a homotopy equivalence of S-algebras.*

This result is sufficient to draw a useful conclusion about the algebraic K-theory of topological spaces, since algebraic K-theory is not sensitive to homotopy equivalences.

Corollary 3.2. *Given a diagram* (3.1) *which displays a pushout of reduced simplicial sets, there is a natural fibration-up-to-homotopy*

$$F \longrightarrow A(W) \times A(X) \times A(Y) \xrightarrow{e} A(Z),$$

where F and e are described below.

Before we prove Theorem 3.1 we sketch the strategy, which aims to exploit the fact that the category of S-algebras is tensored over the category of topological spaces. First we replace diagram (3.1) with a homotopy equivalent diagram of the type discussed in section 2. Adequate input to exploit the existence of tensors in the category of S-algebras is obtained by observing that the loop group construction promotes homotopy equivalences to loop homotopy equivalences [4, page 41], a notion we recall in some detail following the proof of Lemma 3.3 below. (In modern terminology, Kan [4] is exploiting the fact that certain tensors over simplicial sets exist in the category of simplicial groups.) This step rests on particular techniques for working with free simplicial groups developed by Kan [4]. In turn, these techniques exploit the fact that in passing from simplicial sets to their loop groups we gain two extra niceness properties—the extension condition and freeness.

To improve diagram (3.1), we apply the pointed mapping cylinder construction to g, obtaining a reduced simplicial set Y' and the following diagram:

$$\begin{tikzcd} W \arrow[r, tail, "j_1"] \arrow[dr, "g"'] & Y' \arrow[d, "p"] & Y \arrow[l, tail, "j_2"'] \arrow[dl, "="] \\ & Y & \end{tikzcd}$$

Then we replace the given diagram with a pushout diagram

$$\begin{tikzcd} W \arrow[r, tail] \arrow[d, tail, "j_1"'] & X \arrow[d, tail] \\ Y' \arrow[r, tail] & Z' \end{tikzcd} \tag{3.2}$$

of the type previously considered. The universal property of the pushout induces a morphism of diagrams from diagram (3.2) to diagram (3.1) and, in particular, there is a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, tail] \arrow[d, "p"'] & Z' \arrow[d, "q"] \\ Y \arrow[r, tail] & Z \end{tikzcd}$$

in which both p and q are weak homotopy equivalences. (In the category of pointed topological spaces, p and q would be homotopy equivalences, of course. In a category of reduced simplicial sets, which are not necessarily Kan sets, we easily obtain only that p and q are weak homotopy equivalences.) The morphism of diagrams induces a commuting diagram

$$\begin{tikzcd} S \wedge |G(X)|_+ *_{S\wedge|G(W)|_+} S \wedge |G(Y')|_+ \arrow[r, "\cong"] \arrow[d, "\mathrm{id} *(S\wedge|G(p)|_+)"'] & S \wedge |G(Z')|_+ \arrow[d, "S\wedge|G(q)|_+"] \\ S \wedge |G(X)|_+ *_{S\wedge|G(W)|_+} S \wedge |G(Y)|_+ \arrow[r] & S \wedge |G(Z)|_+ \end{tikzcd} \tag{3.3}$$

in which the horizontal arrows arise by means of the universal property of the generalized free product. The upper horizontal arrow is an isomorphism by Theorem 1.1. We are going to prove that p and q induce S-algebra homotopy equivalences $S \wedge |G(Y')|_+ \to S \wedge |G(Y)|_+$ and $S \wedge |G(Z')|_+ \to S \wedge |G(Z)|_+$ in Lemmas 3.3 and 3.4, and, to complete the proof of Theorem 3.1, we will then show that left-hand vertical arrow is also a homotopy equivalence of S-algebras. Granting that we are able to do all this, we can now explain the proof of the corollary.

Proof of Corollary 3.2. For this proof write $A = S \wedge |G(W)|_+$, $B = S \wedge |G(X)|_+$, $C = S \wedge |G(Y)|_+$, $C' = S \wedge |G(Y')|_+$, and $D = S \wedge |G(Z)|_+$ for the S-algebras arising from spaces. Let us also set $R = S \wedge |G(X)|_+ *_{S\wedge|G(W)|_+} S \wedge |G(Y)|_+$ and $R' = S \wedge |G(X)|_+ *_{S\wedge|G(W)|_+} S \wedge |G(Y')|_+$. We can form the following diagram:

$$\begin{tikzcd}
K(\mathcal{MV}^w_f, v) \arrow[r] & K(A)\times K(B)\times K(C') \arrow[r, "e''"] \arrow[d] & K(R') \arrow[d] \\
 & K(A)\times K(B)\times K(C) \arrow[r, "e'"] \arrow[d] & K(R) \arrow[d] \\
 & K(A)\times K(B)\times K(C) \arrow[r, "e"] & K(D)
\end{tikzcd}$$

According to Theorem 1.1 of [7] the first row is a fibration up to homotopy, where the map e'' is induced by the functor from the product of module categories $\mathcal{M}(A)_f \times \mathcal{M}(B)_f \times \mathcal{M}(C')_f$ to the module category $\mathcal{M}(R')_f$ whose value on a triple $(M_A, M_B, M_{C'})$ is the R'-module $\Sigma M_A \wedge_A R' \vee M_B \wedge_B R' \vee M_{C'} \wedge_{C'} R'$, where Σ denotes suspension. Define the maps e' and e similarly. Since the comparison diagram (3.3) commutes, easy arguments show that all the squares commute. Algebraic K-theory preserves the homotopy equivalences supplied by Lemma 3.4 and Theorem 3.1, so the vertical arrows are all homotopy equivalences. Since the upper row is a fibration-up-to-homotopy, it follows that the lowest row is also a fibration-up-to-homotopy, and that the fiber of the lower row has the homotopy type of $K(\mathcal{MV}^w_f, v)$. If we again make appropriate interpretations, we obtain Corollary 3.2, including a description of F. $\square$

As the next step in the proof of Theorem 3.1 we observe the following lemma.

Lemma 3.3. *The weak homotopy equivalences $p : Y' \longrightarrow Y$ and $q : Z' \longrightarrow Z$ induce loop homotopy equivalences*

$$G(p)\colon G(Y') \longrightarrow G(Y) \quad \text{and} \quad G(q)\colon G(Z') \longrightarrow G(Z), \tag{3.4}$$

respectively.

Before we give the proof, we recall that if $G(q)$ is a loop homotopy equivalence [4, page 41], then the following conditions hold. First, there is a homomorphism $\ell\colon G(Z) \longrightarrow G(Z')$ and, second, there are loop homotopies between $\ell \circ G(q)$ and $G(q) \circ \ell$ and the respective identities on $G(Z')$ and $G(Z)$. To express the concept of loop homotopy diagrammatically, let

$$\tilde{\delta} : G(Z) \times G(Z) \times \Delta[1] \longrightarrow G(Z) \times \Delta[1] \times G(Z) \times \Delta[1]$$

be the composite of the morphism $G(Z)\times G(Z)\times\Delta[1] \to G(Z)\times G(Z)\times\Delta[1]\times\Delta[1]$ that is the diagonal on the one-simplex followed by the transposition of the second and third factors. Then a loop homotopy between $G(q)\circ\ell$ and $\mathrm{id}_{G(Z)}$ is a simplicial homotopy $L\colon G(Z) \times \Delta[1] \longrightarrow G(Z)$ between these maps such that the diagram

$$\begin{array}{ccc} G(Z) \times G(Z) \times \Delta[1] & \xrightarrow{\tilde{\delta}} & G(Z) \times \Delta[1] \times G(Z) \times \Delta[1] \\ {\scriptstyle \mu\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mu\circ(L\times L)} \\ G(Z) \times \Delta[1] & \xrightarrow[L]{} & G(Z) \end{array}$$

commutes. Here μ denotes the multiplication on $G(Z)$. Taking geometric realizations clearly converts the simplicial notion into a topological homotopy with the property that the morphism at each level t is a homomorphism, that is, into a topological homotopy through homomorphisms. Thus, loop homotopies are the simplicial analogues of homotopies through homomorphisms.

Proof of Lemma 3.3. To economize on notation, we discuss the case of q; the argument for p is the same. Since Z and Z' are reduced simplicial sets, and since q induces isomorphisms on the homotopy groups of the geometric realizations, the induced homomorphism $G(q)\colon G(Z') \to G(Z)$ also induces isomorphisms on the homotopy groups of the geometric realizations. This is easily seen by comparing the long exact sequence of homotopy groups [3, page 286] associated with the principal bundle

$$G(Z) \longrightarrow G(Z) \times_t Z \longrightarrow Z$$

with the long exact sequence associated with the similar bundle defined for Z'. These bundles have weakly contractible total spaces, so applying the five lemma yields the claim about $G(q)$. In the terminology of Definition 6.4 of [4, page 44] the homomorphism $G(q)$ is a weak loop homotopy equivalence.

Now we will see that $G(q)$ is actually a loop homotopy equivalence, exploiting one of Kan's other results. We observe that $G(Z')$ and $G(Z)$ are free c.s.s. groups in the sense of Definition 5.1 of [4, page 43]. This means that not only are the groups $G_n(Z')$, for instance, free, but also that a degeneracy operator $\alpha^*\colon G_n(Z') \longrightarrow G_{n'}(Z')$ carries a basis element of the first group into a basis element of the second. That this is true is most easily seen by examining the formulas of [3, page 291]. Then Proposition 6.5 of [4, page 44] applies and we find that $G(q)\colon G(Z') \longrightarrow G(Z)$ is actually a loop homotopy equivalence [4, page 41]. □

Consider the loop homotopy equivalences (3.4) of the preceding lemma.

Lemma 3.4. *The loop homotopy equivalence $G(p)\colon G(Y') \to G(Y)$ induces a homotopy equivalence of S-algebras $S \wedge |G(p)|\colon S \wedge |G(Y')|_+ \longrightarrow S \wedge |G(Y)|_+$. Similarly, the loop homotopy equivalence $G(q)$ induces another homotopy equivalence of S-algebras $S \wedge |G(q)|\colon S \wedge |G(Z')|_+ \longrightarrow S \wedge |G(Z)|_+$.*

Proof. The S-module maps $S \wedge |G(p)|$ and $S \wedge |G(q)|$ are clearly maps of S-algebras, since the algebra structures arise from the group multiplications. If we let $k\colon G(Y) \longrightarrow G(Y')$ be our choice of a loop homotopy inverse to $G(p)$, then homotopy inverses to $S \wedge |G(q)|$ and $S \wedge |G(p)|$ are, respectively, the S-algebra morphisms $S \wedge |\ell|$ and $S \wedge |k|$. Moreover, realizing the loop homotopies L and K between $G(q) \circ \ell$ and $G(p) \circ k$ and the respective identities, making the standard identification $|\Delta[1]| = I$, and applying the construction $S \wedge -$ produces homotopies

$$S \wedge |L|\colon S \wedge |G(Z)|_+ \wedge I_+ \longrightarrow S \wedge |G(Z)|_+$$

and, analogously,

$$S \wedge |K| \colon S \wedge |G(Y)|_+ \wedge I_+ \longrightarrow S \wedge |G(Y)|_+$$

which are S-algebra morphisms at each stage t. We also have parallel results for the composites of the morphisms in the reversed order, hence the claim. □

To understand the implications for the generalized free product of the existence of such a nice homotopy equivalence of S-algebras $S \wedge |G(p)| \colon S \wedge |G(Y')| \longrightarrow S \wedge |G(Y)|$, we recall that the category of S-algebras is tensored over the category of spaces [1, pages 130–134]. Continuing with the identification $|\Delta[1]| = I$, we have the following consequences of Propositions VII.2.10 and VII.2.11 of [1, pages 133–134]. Given any S-algebra A there is another S-algebra $A \otimes I$, computed as a certain coend [1, Proposition VII.2.10], together with a map $\omega \colon A \wedge I_+ \longrightarrow A \otimes I$ which has the following universal property. Letting $i_t \colon A \longrightarrow A \wedge I_+$ denote the map induced by the inclusion $\{t\} \longrightarrow I$, if a homotopy $h \colon A \wedge I_+ \longrightarrow B$ has the property that each composite $h \circ i_t$ is a map of S-algebras, then there is a unique map of S-algebras $\tilde{h} \colon A \otimes I \longrightarrow B$ such that $\tilde{h} \circ \omega = h$. In other words, the appropriate way to view a homotopy $h \colon A \wedge I_+ \longrightarrow B$ through S-algebra homomorphisms is as an S-algebra homomorphism $\tilde{h} \colon A \otimes I \longrightarrow B$.

Proof of Theorem 3.1. Now we will see that the vertical arrows in the comparison diagram (3.3) are homotopy equivalences of S-algebras. We have already seen the right-hand vertical arrow is such a homotopy equivalence. Applying the preceding remark to the induced homotopy $S \wedge |K|$ between $(S \wedge |G(p)|) \circ (S \wedge |k|)$ and the identity on $S \wedge |G(Y)|_+$, the homotopy can be understood as an S-algebra morphism $(S \wedge |G(Y)|_+) \otimes I \longrightarrow S \wedge |G(Y)|_+$. Since both the tensor construction and the generalized free product construction are coends, the tensor construction and the generalized free product construction commute with one another. That is,

$$\begin{aligned} &S \wedge |G(X)|_+ *_{S \wedge |G(W)|_+} ((S \wedge |G(Y)|_+) \otimes I) \\ &\qquad \cong (S \wedge |G(X)|_+ *_{S \wedge |G(W)|_+} S \wedge |G(Y)|_+) \otimes I \end{aligned}$$

and, similarly with Y' replacing Y. Putting all these facts and homotopies together, we see that the left-hand vertical arrow

$$S \wedge |G(X)|_+ *_{S \wedge |G(W)|_+} S \wedge |G(Y')|_+ \longrightarrow S \wedge |G(X)|_+ *_{S \wedge |G(W)|_+} S \wedge |G(Y)|_+$$

induced from $G(p)$ is also a homotopy equivalence, and it is compatible with the equivalence $S \wedge |G(Z')|_+ \longrightarrow S \wedge |G(Z)|_+$ induced from $G(q)$. This completes the proof. □

We have chosen to work thus far with pushout diagrams of reduced simplicial sets. Suppose one wishes to consider diagrams such as (3.1), but where the simplicial sets are only assumed to be connected. There is a functor E_1 from connected pointed simplicial sets to reduced simplicial sets which associates to K the subcomplex $E_1 K$ whose n-simplices are the n-simplices of K which have their vertices at the basepoint. If K is a Kan set, $E_1 K$ is also, and $E_1 K \to K$ is a weak homotopy equivalence, as is easily seen by using [3, Definition 2.6, page 294] as the definition of the homotopy groups of both complexes. Since the algebraic K-theory of spaces is insensitive to weak homotopy equivalences, limiting the discussion to reduced simplicial sets is not a real loss of generality.

However, even if the restriction to reduced simplicial sets is not a loss of generality, the restriction may represent a loss of convenience. Therefore, it is worth pointing out that there is a version of the Kan loop group for pointed connected simplicial sets that are not necessarily reduced. This can be found in [3], but there is an alternative description due to Waldhausen [11]. Waldhausen's description is functorial, whereas Kan's original definition depends on a choice of a maximal tree in the one-skeleton of a simplicial set. Given a pointed connected simplicial set, Waldhausen constructs a simplicial space of pointed graphs and obtains the loop group by applying the fundamental group functor dimensionwise. When the construction is applied to a pushout diagram (1.1) of pointed connected simplicial sets in which all arrows are cofibrations, it is easy to see that each graph associated to the large space Z is the union of the corresponding graphs associated to the simplicial sets X and Y along a common subgraph which is the corresponding graph associated to W. Then we see that Lemma 2.1 remains true by dimensionwise application of the ordinary Seifert-van Kampen theorem to the diagram of simplicial graphs.

Before we leave the category of simplicial sets, let us also point out that the results of this section permit us to remove all explicit cofibration requirements from the arrows in diagram (3.1) and replace them by the requirement that the square diagram be homotopy equivalent to a standard cofibration diagram while the upper left corner space remains the same. The continuity property, or homotopy invariance property, of the generalized free product we verified in the course of the proof of Theorem 3.1 again implies a homotopy equivalence from the generalized free product of the three "smaller" S-algebras to the S-algebra associated to the "largest" space.

Looking ahead to possible geometric applications, one may start with a diagram of connected d-dimensional manfolds

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d, tail] & L \arrow[d, tail] \\ M \arrow[r, tail] & N \end{tikzcd}$$

representing a d-manifold N split along a two sided codimension 1 submanifold with a collar neighborhood K. The singular complex functor Δ applied to the diagram produces a diagram of connected simplicial sets

$$\begin{tikzcd} \Delta(K) \arrow[r, tail] \arrow[d, tail] & \Delta(L) \arrow[d, tail] \\ \Delta(M) \arrow[r, tail] & \Delta(N) \end{tikzcd}$$

in which the canonical map $Z' = \Delta(M) \cup_{\Delta(K)} \Delta(L) \to \Delta(N)$ is a weak homotopy equivalence. The algebraic K-theory of spaces is not sensitive to weak homotopy equivalences, so the study of the algebraic K-theory of the original diagram of manifolds is reduced to the study of the K-theory of a pushout diagram of Kan sets. Further, if we select a base vertex in $\Delta(K)$ and define $W = E_1\,\Delta(K)$, $X = E_1\,\Delta(L)$, $Y = E_1\,\Delta(M)$, and $Z = E_1\,Z'$, then we reduce the geometric situation to exactly the formal situation studied in sections 1 and 2 of this paper.

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