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ASYMPTOTIC DIRICHLET PROBLEM FOR THE p-LAPLACIAN ON CARTAN-HADAMARD MANIFOLDS

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ABSTRACT. We show the existence of nonconstant bounded p-harmonic functions on Cartan-Hadamard manifolds of pinched negative curvature by solving the asymptotic Dirichlet problem at infinity for the p-Laplacian. More precisely, we prove that given a continuous function h on the sphere at infinity there exists a unique p-harmonic function u on M with boundary values h.

1. INTRODUCTION

In this paper we show the existence of nonconstant bounded p-harmonic functions on Cartan-Hadamard manifolds M of pinched negative curvature by solving the asymptotic Dirichlet problem at infinity for the p-Laplacian. More precisely, we prove that given a continuous function h on the sphere at infinity there exists a unique p-harmonic function u on M with boundary values h.

Let M be a Cartan-Hadamard manifold, that is, a connected, simply connected, complete Riemannian n-manifold, $n \geq 2$, of nonpositive sectional curvature. By the Cartan-Hadamard theorem, the exponential map $\exp_o\colon T_oM \to M$ is a diffeomorphism for every point $o \in M$. In particular, M is diffeomorphic to $\mathbb{R}^n$. It is well-known that M can be compactified by adding a *sphere at infinity*, denoted by $S(\infty)$, so that the resulting space $\bar{M} = M \cup S(\infty)$ will be homeomorphic to a closed Euclidean ball. The sphere at infinity is defined as the set of all equivalent classes of geodesic rays in M; two geodesic rays γ_1 and γ_2 are equivalent if there exists a finite constant c such that $d(\gamma_1(t), \gamma_2(t)) \leq c$ for all $t \geq 0$. There is a natural topology, called the *cone topology,* on $\bar{M} = M \cup S(\infty)$ defined as follows. For any point $o \in M$ and $v \in T_oM$, let

$$C_o(v, \alpha) = \{x \in M \setminus \{o\}\colon \sphericalangle(v, \dot\gamma^x(0)) < \alpha\}$$

be the *cone about v of angle $\alpha > 0$,* where γ^x is the unique geodesic from $o = \gamma^x(0)$ to x and $\sphericalangle(v, \dot\gamma^x(0))$ is the angle between vectors v and $\dot\gamma^x(0)$ in T_oM. Then geodesic balls $B(q, r),\ q \in M, r > 0$, and *truncated cones*

$$T_o(v, \alpha, s) = C_o(v, \alpha) \setminus \bar{B}(o, s),$$

with $v \in T_0M, \alpha > 0, s > 0$, form a basis for the cone topology. Furthermore, the cone topology is independent of the choice of $o \in M$ and, equipped with this

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topology, $\bar{M}$ is homeomorphic to the closed unit ball $\bar{B}^n \subset \mathbb{R}^n$ and $S(\infty)$ to the sphere $S^{n-1} = \partial B^n$; see [7]. In particular, given $o \in M$, $S(\infty)$ may be canonically identified with the unit sphere $S^{n-1} \subset T_o M$.

It is natural to ask whether every continuous function on $S(\infty)$ has a unique harmonic extension to M. This so-called *asymptotic Dirichlet problem* was solved by Choi if the sectional curvature has a negative upper bound $K \leq -a^2 < 0$ and any two points of the sphere at infinity can be separated by convex neighborhoods; see [6]. Such appropriate convex sets were constructed by Anderson [3] for manifolds of pinched sectional curvature $-b^2 \leq K \leq -a^2 < 0$. The Dirichlet problem was independently solved by Sullivan [13] under the same curvature assumptions by using probabilistic arguments. In [4], Anderson and Schoen presented a simple and direct proof. Ancona [1] was able to replace the lower curvature bound by a bounded geometry assumption that each ball up to a fixed radius is bi-Lipschitz equivalent to an open set in $\mathbb{R}^n$. He also considered a more general class of operators. On the other hand, Ancona [2] showed that the Dirichlet problem cannot be solved, in general, if there are neither curvature lower bounds nor the bounded geometry assumption; see also [5]. In the general case of the p-Laplacian, the corresponding problem has been open so far. Pansu [11] has shown the existence of nonconstant bounded p-harmonic functions with finite p-energy on Cartan-Hadamard manifolds of pinched curvature $-b^2 \leq K \leq -a^2$ if $p > (n-1)b/a$.

2. Asymptotic Dirichlet problem

Let $G \subset M$ be an open set and $1 < p < \infty$. Recall that a function $u \in W^{1,p}_{\mathrm{loc}}(G)$ is a (weak) solution of the equation

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0 \tag{2.1}$$

in G if

$$\int_G \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle = 0$$

for all $\varphi \in C_0^\infty(G)$. Above $W^{1,p}_{\mathrm{loc}}(G)$ is the (local) Sobolev space of all functions $u \in L^p_{\mathrm{loc}}(G)$ whose distributional gradient ∇u belongs to $L^p_{\mathrm{loc}}(G)$. Continuous solutions of (2.1) are called *p-harmonic.* It is well-known that every solution of (2.1) has a continuous representative by the fundamental work of Serrin [12]. We say that a function $u \in W^{1,p}_{\mathrm{loc}}(G)$ is a *p-supersolution* in G if

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \geq 0 \tag{2.2}$$

weakly in G, that is,

$$\int_G \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \geq 0$$

for all nonnegative $\varphi \in C_0^\infty(G)$. Furthermore, we say that u is a *p-subsolution* if $-u$ is a p-supersolution.

In this section we show that the direct approach to solve the Dirichlet problem taken by Anderson and Schoen in [4] also works in the nonlinear setting of p-harmonic functions.

Theorem 2.1. *Let M be a Cartan-Hadamard manifold whose sectional curvature K satisfies*

$$-b^2 \leq K \leq -a^2 \tag{2.3}$$

for some constants $b \geq a > 0$. Let h be a continuous function on $S(\infty)$. Then there exists a unique function $u \in C(\bar{M})$ which is p-harmonic in M and $u = h$ on $S(\infty)$.

The proof requires some preliminaries. Let $h \in C(S(\infty))$ be given. Fix a point $o \in M$ and write $r(x) = d(x, o)$. We identify $S(\infty)$ with the unit sphere $S^{n-1} \subset T_oM$. Therefore, we may consider h as a continuous function on S^{n-1}. Assume that $h\colon S^{n-1} \to \mathbb{R}$ is Lipschitz. We extend h radially to a continuous function $\tilde{h}$ on $M \setminus \{o\}$. More precisely, we define $\tilde{h}$ in polar coordinates about o by

$$\tilde{h}(r, \vartheta) = h(\vartheta)$$

for every $r > 0$ and $\vartheta \in S^{n-1}$. The Lipschitz continuity of h and the curvature upper bound $K \leq -a^2$ imply that

$$\operatorname{osc}\big(\tilde{h}, B(x,3)\big) := \sup_{B(x,3)} \tilde{h} - \inf_{B(x,3)} \tilde{h} \leq cLe^{-ar(x)}, \tag{2.4}$$

where L is the Lipschitz constant of h; see [4]. Next we define a smooth function h on M such that

$$\lim_{x \to \xi} h(x) = h(\xi)$$

for every $\xi \in S(\infty)$ and that first and second order derivatives of h are effectively controlled. For this purpose, we fix a maximal 1-separated set $Q = \{q_1, q_2, \dots\} \subset M$, that is,

$$d(q_i, q_j) \geq 1 \tag{2.5}$$

whenever $i \neq j$ and no more points can be added to Q without breaking the condition (2.5). We may assume that $o \notin Q$. In particular, the balls $B(q_i, 1/2)$ are mutually disjoint and $M = \bigcup_i B(q_i, 1)$. For each $x \in M$, we write $Q_x = Q \cap B(x, 3)$. The curvature lower bound then implies that

$$\operatorname{card} Q_x \leq c, \tag{2.6}$$

where c is independent of x; see e.g. [10]. Then we define

$$h(x) = \sum_{q_i \in Q} \tilde{h}(q_i)\varphi_i(x), \tag{2.7}$$

where $\{\varphi_i\}$ is a partition of unity subordinate to $\{B(q_i, 3)\}$ defined as follows. First choose a C^∞ function $f\colon [0, \infty[\to [0,1]$ such that $f|[0,1] = 1$, $f|[2, \infty[= 0$, and

$$\max\{|f'(t)|, |f''(t)|\} \leq c\mathcal{X}_{[1,2]}(t) \tag{2.8}$$

for some constant c, where $\mathcal{X}_{[1,2]}$ is the characteristic function of the interval $[1, 2]$. For $q_i \in Q$ and $x \in M$, let $\eta_i(x) = f(r_i(x))$, where $r_i(x) = d(x, q_i)$. Finally we set

$$\varphi_i(x) = \frac{\eta_i(x)}{\sum_j \eta_j(x)}. \tag{2.9}$$

To estimate first and second order derivatives of h, we first observe that

$$\nabla \eta_i(x) = f'\big(r_i(x)\big)\nabla r_i(x) \tag{2.10}$$

and

$$\begin{aligned} \Delta \eta_i(x) &= f'\big(r_i(x)\big)\Delta r_i(x) + \big\langle \nabla f'\big(r_i(x)\big), \nabla r_i(x)\big\rangle \\ &= f'\big(r_i(x)\big)\Delta r_i(x) + f''\big(r_i(x)\big) \end{aligned}$$

since $\langle \nabla r_i(x), \nabla r_i(x)\rangle = |\nabla r_i(x)|^2 \equiv 1$. Thus (2.8) and (2.10) imply that

$$|\nabla \eta_i(x)| \le c\mathcal{X}_{A(q_i;1,2)}(x),$$

where $A(y;s,t) = \bar{B}(y,t)\backslash B(y,s)$. By the Hessian comparison theorem ([8, Theorem A]),

$$(n-1)a\coth(ar_i(x)) \le \Delta r_i(x) \le (n-1)b\coth(br_i(x)). \tag{2.11}$$

Combining this with (2.8) and (2.11) yields

$$|\Delta \eta_i(x)| \le c\mathcal{X}_{A(q_i;1,2)}(x).$$

Since $\sum_j \eta_j(x) \ge 1,\ 0 \le \eta_i(x) \le 1$, and $\operatorname{card} Q \cap B(x,3) \le c$ for every $x \in M$, we get by a simple computation that

$$|\nabla \varphi_i(x)| \le c\mathcal{X}_{B(q_i,4)}(x) \tag{2.12}$$

and

$$|\Delta \varphi_i(x)| \le c\mathcal{X}_{B(q_i,4)}(x). \tag{2.13}$$

In the next lemma we collect those properties of h that are crucial in the sequel.

Lemma 2.2. *Let $r\colon M \to \mathbb{R}$ be the distance function $r(x) = d(x,o)$ and let $h\colon M \to \mathbb{R}$ be the function given by (2.7). Furthermore, let $v\colon M \setminus \{o\} \to \mathbb{R}$ be defined by*

$$v(x) = e^{-\delta r(x)}, \tag{2.14}$$

with $\delta > 0$. Then there exists a constant c_0 independent of h and δ such that

$$|\nabla h(x)| \le c_0 L e^{-ar(x)}, \tag{2.15}$$

$$|\Delta h(x)| \le c_0 L e^{-ar(x)}, \tag{2.16}$$

$$|\nabla\langle \nabla h, \nabla h\rangle(x)| \le (c_0 L)^2 e^{-2ar(x)}, \tag{2.17}$$

$$|\nabla\langle \nabla h, \nabla v\rangle(x)| \le c_0 L(1+\delta)\delta e^{-(a+\delta)r(x)} \tag{2.18}$$

for $r(x) \ge 1$. Moreover,

$$\lim_{x\to\xi} h(x) = h(\xi) \tag{2.19}$$

for every $\xi \in S(\infty)$.

Proof. Fix $x \in M \setminus B(o,1)$ and choose $q \in Q$ such that $x \in B(q,1)$. Then

$$\begin{aligned}\nabla h(x) &= \sum_{q_i\in Q} \tilde{h}(q_i)\nabla\varphi_i(x) = \sum_{q_i\in Q_x} \tilde{h}(q_i)\nabla\varphi_i(x)\\ &= \sum_{q_i\in Q_x} \big(\tilde{h}(q_i) - \tilde{h}(q)\big)\nabla\varphi_i(x)\end{aligned}$$

since $\sum_{q_i\in Q_x} \varphi_i = 1$ in a neighborhood of x and therefore

$$\sum_{q_i\in Q_x} \tilde{h}(q)\nabla\varphi_i(x) = \tilde{h}(q)\nabla\Big(\sum_{q_i\in Q_x}\varphi_i\Big)(x) = 0.$$

By (2.4), (2.6), and (2.12),

$$|\nabla h(x)| \le c\big(\operatorname{card} Q_x\big)\operatorname{osc}\big(\tilde{h}, B(x,3)\big) \le cLe^{-ar(x)}$$

which proves (2.15). By a similar argument using (2.13) instead of (2.12) we obtain (2.16). For the proof of the estimate (2.17), we first observe that

$$\begin{aligned}\langle \nabla h, \nabla h\rangle(x) &= \Big\langle \sum_{q_i\in Q_x}\big(\tilde h(q_i)-\tilde h(q)\big)\nabla\varphi_i, \sum_{q_j\in Q_x}\big(\tilde h(q_j)-\tilde h(q)\big)\nabla\varphi_j\Big\rangle(x)\\ &= \sum_{q_i,q_j\in Q_x}\big(\tilde h(q_i)-\tilde h(q)\big)\big(\tilde h(q_j)-\tilde h(q)\big)\langle\nabla\varphi_i,\nabla\varphi_j\rangle(x),\end{aligned}$$

and so

$$\nabla\langle \nabla h, \nabla h\rangle(x) = \sum_{q_i,q_j\in Q_x}\big(\tilde h(q_i)-\tilde h(q)\big)\big(\tilde h(q_j)-\tilde h(q)\big)\nabla\langle\nabla\varphi_i,\nabla\varphi_j\rangle(x).$$

By (2.4) and (2.6) it suffices to prove that

$$|\nabla\langle\nabla\varphi_i,\nabla\varphi_j\rangle(x)|\le c$$

for all $q_i, q_j \in Q_x$ which reduces to establishing that

$$|\nabla\langle\nabla r_i,\nabla r_j\rangle(x)|\le c \tag{2.20}$$

whenever $x\in A(q_i;1,2)\cap A(q_j;1,2)$. Let $X_1,\dots,X_n$ be an orthonormal frame in a neighborhood of x. Then

$$\begin{aligned}\nabla\langle\nabla r_i,\nabla r_j\rangle &= \sum_k\big(X_k\langle\nabla r_i,\nabla r_j\rangle\big)X_k\\ &= \sum_k\big(\langle\nabla_{X_k}\nabla r_i,\nabla r_j\rangle+\langle\nabla r_i,\nabla_{X_k}\nabla r_j\rangle\big)X_k.\end{aligned}$$

On the other hand,

$$\langle\nabla_{X_k}\nabla r_i,\nabla r_j\rangle=\nabla^2 r_i(X_k,\nabla r_j),$$

where $\nabla^2 r_i$ is the Hessian of r_i. By the Hessian comparison theorem all eigenvalues of $\nabla^2 r_i$ are nonnegative and bounded from above by $b\coth(br_i)$. Hence

$$|\langle\nabla_{X_k}\nabla r_i,\nabla r_j\rangle(x)|\le b\coth(br_i(x))|X_k(x)||\nabla r_j(x)|=b\coth(br_i(x))\le c$$

if $r_i(x)\ge 1$. Similarly, $|\langle\nabla_{X_k}\nabla r_j,\nabla r_i\rangle(x)|\le c$ if $r_j(x)\ge 1$, and so (2.20) follows. This proves (2.17). The estimate (2.18) can be established similarly since

$$\begin{aligned}|\nabla\langle\nabla h,\nabla v\rangle(x)| &\le \delta e^{-\delta r(x)}\sum_{q_i\in Q_x}|\tilde h(q_i)-\tilde h(q)||\nabla\langle\nabla\varphi_i,\nabla r\rangle(x)|\\ &+ \delta^2 e^{-\delta r(x)}|\nabla r(x)|\sum_{q_i\in Q_x}|\tilde h(q_i)-\tilde h(q)||\langle\nabla\varphi_i,\nabla r\rangle(x)|.\end{aligned}$$

Now $|\nabla\langle\nabla\varphi_i,\nabla r\rangle(x)|\le c$ if $r(x)\ge 1$ by a similar argument as above, and thus (2.18) follows. Finally, (2.19) follows easily from the definition (2.7) and from the continuity of $h|S(\infty)$. □

Lemma 2.3. *Suppose that $h\colon S^{n-1}\to\mathbb{R}$ is L-Lipschitz, where S^{n-1} is the unit sphere in T_oM. Define $h\colon M\to\mathbb{R}$ by (2.7) and let $v=e^{-\delta r}$. Then there exist $\delta_0\in]0,a[$ such that, for every $\delta\in]0,\delta_0]$, $h+v$ is a p-supersolution and $h-v$ is a p-subsolution in $M\setminus\bar B(o,R_\delta)$, where $R_\delta=R_\delta(a,\delta,c_0,L)$.*

Proof. In what follows $R_1, \dots, R_5$ are constants depending only on a, δ, c_0, and L. Since h and v are smooth in $M \setminus \{o\}$, we can prove the claims by direct computation using the properties of h and v given by Lemma 2.2. Write $u = h + v$ and note that $\nabla u = \nabla h - \delta e^{-\delta r} \nabla r \neq 0$ if $\delta < a$ and $r > R_1$ by (2.15). Hence

$$\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big) = |\nabla u|^{p-2} \Delta u + \frac{p-2}{2} |\nabla u|^{p-4} \big\langle \nabla\big(|\nabla u|^2\big), \nabla u \big\rangle \tag{2.21}$$

in $M \setminus \bar{B}(o, R_1)$. Next we deduce from (2.11) that

$$\Delta v = -\delta e^{-\delta r} \Delta r + \delta^2 e^{-\delta r} \le \delta e^{-\delta r}\big(\delta - (n-1)a\big) \le -c_1 \delta e^{-\delta r} < 0,$$

with $c_1 = (n-1)a/2$ whenever $\delta \le (n-1)a/2$; cf. [4]. Given $\delta < a$ there exists R_2 such that

$$\begin{aligned} \delta^2 e^{-2\delta r} \le |\nabla h + \nabla v|^2 &= |\nabla h|^2 + 2\langle \nabla h, \nabla v \rangle + |\nabla v|^2 \\ &\le (c_0 L)^2 e^{-2ar} + 2 c_0 L \delta e^{-(a+\delta) r} + \delta^2 e^{-2\delta r} \\ &\le 2\delta^2 e^{-2\delta r} \end{aligned}$$

as soon as $r \ge R_2$. Hence

$$d_p^{-1} \delta^{p-2} e^{-\delta(p-2) r} \le |\nabla h + \nabla v|^{p-2} \le d_p \delta^{p-2} e^{-\delta(p-2) r},$$

where $d_p = 2^{|p-2|/2}$. If $\delta < a \wedge c_1$, we get an estimate

$$|\nabla h + \nabla v|^{p-2} (\Delta h + \Delta v) \le d_p^{-1} \delta^{p-2} e^{-\delta(p-2) r} \big[d_p^2 c_0 L e^{-ar} - c_1 \delta e^{-\delta r} \big]$$

for the first term in the right-hand side of (2.21). To estimate the second term in (2.21) we write

$$\begin{aligned} \big\langle \nabla\big(|\nabla u|^2\big), \nabla u \big\rangle &= \big\langle \nabla\big(|\nabla h|^2\big), \nabla u \big\rangle + \big\langle \nabla\big(|\nabla v|^2\big), \nabla u \big\rangle + 2\big\langle \nabla \langle \nabla h, \nabla v \rangle, \nabla u \big\rangle \\ &= A + B + C. \end{aligned}$$

By (2.15) and (2.17),

$$\begin{aligned} A &= \big\langle \nabla\big(|\nabla h|^2\big), \nabla h + \nabla v \big\rangle \\ &\le |\nabla \langle \nabla h, \nabla h \rangle| |\nabla h + \nabla v| \\ &\le (c_0 L)^2 e^{-2ar} \big(c_0 L e^{-ar} + \delta e^{-\delta r} \big) \\ &\le \delta^4 e^{-3\delta r} \end{aligned}$$

if $r \ge R_3$. Similarly,

$$\begin{aligned} B &= \big\langle \nabla\big(|\nabla v|^2\big), \nabla h + \nabla v \big\rangle \\ &\le 2 c_0 L \delta^3 e^{-2\delta r} e^{-ar} + 2\delta^4 e^{-3\delta r} \\ &\le 3\delta^4 e^{-3\delta r} \end{aligned}$$

if $r \ge R_4$. Finally, (2.18) and (2.15) imply that

$$\begin{aligned} C &= 2\big\langle \nabla\big(\langle \nabla h, \nabla v \rangle\big), \nabla h + \nabla v \big\rangle \\ &\le 2|\nabla \langle \nabla h, \nabla v \rangle| |\nabla h + \nabla v| \\ &\le 2 c_0 L (1+\delta) \delta e^{-(a+\delta) r} \big(c_0 L e^{-ar} + \delta e^{-\delta r} \big) \\ &\le \delta^4 e^{-3\delta r} \end{aligned}$$

whenever $r \geq R_5$. Putting these estimates together yields

$$\begin{aligned}\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) &= |\nabla u|^{p-2}\Delta u + \frac{p-2}{2}|\nabla u|^{p-4}\big\langle\nabla\big(|\nabla u|^2\big),\nabla u\big\rangle \\ &\leq d_p^{-1}\delta^{p-2}e^{-\delta(p-2)r}\big[d_p^2 c_0 L e^{-ar} - (c_1-\delta C_p)\delta e^{-\delta r}\big],\end{aligned}$$

where

$$C_p = 3|p-2|2^{\frac{|p-2|+|p-4|}{2}}.$$

Choosing $\delta_0 < \min\{a, c_1, c_1/(2C_p)\}$, with an obvious interpretation $c_1/(2C_p) = \infty$ if $p=2$, finally gives an estimate

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \leq -c_2\delta^{p-1}e^{-\delta(p-1)r} < 0$$

if $\delta \leq \delta_0$ and $r \geq R_\delta$. Similarly, we obtain an estimate

$$\operatorname{div}\big(|\nabla h - \nabla v|^{p-2}(\nabla h - \nabla v)\big) \geq c_2\delta^{p-1}e^{-\delta(p-1)r} > 0$$

if $\delta \leq \delta_0$ and $r \geq R_\delta$. □

Lemma 2.4. *Identify $S(\infty)$ with the unit sphere $S^{n-1} \subset T_oM$. Assume that $h\colon S^{n-1} \to \mathbb{R}$ is L-Lipschitz. Then there exists a p-harmonic function u in M satisfying*

$$\lim_{x\to\xi} u(x) = h(\xi) \tag{2.22}$$

for every $\xi \in S(\infty)$.

Proof. Define $h\colon M \to \mathbb{R}$ by (2.7) and let $\delta \in]0,\delta_0]$ and R_δ be given by Lemma 2.3. First we note that h is bounded, and therefore we can choose a constant $\lambda \in]0,1]$ such that

$$\lambda \operatorname{osc}(h, M) \leq e^{-\delta R_\delta}.$$

Since $\lambda h \mid S^{n-1}$ is also L-Lipschitz, $\lambda h + v$ is a p-supersolution and $\lambda h - v$ is a p-subsolution in $M \setminus \bar{B}(o,R_\delta)$. For $i = 1,2,\dots$, let $u_i \in C(M)$ be the unique function such that u_i is p-harmonic in $B(o,2^iR_\delta)$ and $u_i \equiv \lambda h$ in $M \setminus B(o,2^iR_\delta)$. Now $\lambda h - v \leq u_i \leq \lambda h + v$ on $\partial\big(B(o,2^iR_\delta)\setminus \bar{B}(o,R_\delta)\big)$, and hence the same holds in $B(o,2^iR_\delta)\setminus\bar{B}(o,R_\delta)$ by the comparison principle; see [9, 3.18 and 7.6]. Hence there exists a subsequence, denoted again by (u_i) and a function $u \in C(M)$ such that $\lambda^{-1}u_i \to u$ locally uniformly in M. Furthermore, the function u is p-harmonic in M and satisfies (2.22) for every $\xi \in S(\infty)$. □

Proof of Theorem 2.1. Fix $o \in M$ and identify $S(\infty)$ with $S^{n-1} \subset T_oM$. Let (h_i) be a sequence of Lipschitz functions on S^{n-1} such that $h_i \to h$ uniformly on S^{n-1}. By Lemma 2.4 there are p-harmonic functions $u_i \in C(\bar{M})$ with $u_i = h_i$ in $S(\infty)$. The sequence (u_i) converges uniformly in $\bar{M}$ to a function $u \in C(\bar{M})$ which is p-harmonic in M and $u = h$ in $S(\infty)$. To prove the uniqueness, suppose that u and w are both p-harmonic in M, continuous in $\bar{M}$, with $u = w$ in $S(\infty)$, and $u(y) > w(y)$ for some $y \in M$. Let $\varepsilon = \big(u(y)-w(y)\big)/2$. Since u and w are continuous in $\bar{M}$ and they coincide on the compact set $S(\infty)$, there exists $R > 0$ such that $|u(x)-w(x)| < \varepsilon$ for every $x \in M \setminus B(o,R)$. Let D be the y-component of $\{x \in M\colon u(x) > w(x)+\varepsilon\}$. It follows that D is a relatively compact domain in M and $u = w+\varepsilon$ on ∂D. Hence $u = w+\varepsilon$ in D which leads to a contradiction since $y \in D$. This proves the uniqueness and thus the whole theorem is proved. □

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