ASYMPTOTIC DIRICHLET PROBLEM FOR THE $p$-LAPLACIAN ON CARTAN-HADAMARD MANIFOLDS

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Abstract. We show the existence of nonconstant bounded $p$-harmonic functions on Cartan-Hadamard manifolds of pinched negative curvature by solving the asymptotic Dirichlet problem at infinity for the $p$-Laplacian. More precisely, we prove that given a continuous function $h$ on the sphere at infinity there exists a unique $p$-harmonic function $u$ on $M$ with boundary values $h$.

1. Introduction

In this paper we show the existence of nonconstant bounded $p$-harmonic functions on Cartan-Hadamard manifolds $M$ of pinched negative curvature by solving the asymptotic Dirichlet problem at infinity for the $p$-Laplacian. More precisely, we prove that given a continuous function $h$ on the sphere at infinity there exists a unique $p$-harmonic function $u$ on $M$ with boundary values $h$.

Let $M$ be a Cartan-Hadamard manifold, that is, a connected, simply connected, complete Riemannian $n$-manifold, $n \geq 2$, of nonpositive sectional curvature. By the Cartan-Hadamard theorem, the exponential map $\exp_o : T_o M \to M$ is a diffeomorphism for every point $o \in M$. In particular, $M$ is diffeomorphic to $\mathbb{R}^n$. It is well-known that $M$ can be compactified by adding a sphere at infinity, denoted by $S(\infty)$, so that the resulting space $\bar{M} = M \cup S(\infty)$ will be homeomorphic to a closed Euclidean ball. The sphere at infinity is defined as the set of all equivalent classes of geodesic rays in $M$; two geodesic rays $\gamma_1$ and $\gamma_2$ are equivalent if there exists a finite constant $c$ such that $d(\gamma_1(t), \gamma_2(t)) \leq c$ for all $t \geq 0$. There is a natural topology, called the cone topology, on $\bar{M} = M \cup S(\infty)$ defined as follows. For any point $o \in M$ and $v \in T_o M$, let

$$C_o(v, \alpha) = \{ x \in M \setminus \{ o \} : \angle(v, \dot{\gamma}(0)) < \alpha \}$$

be the cone about $v$ of angle $\alpha > 0$, where $\gamma$ is the unique geodesic from $o = \gamma(0)$ to $x$ and $\angle(v, \dot{\gamma}(0))$ is the angle between vectors $v$ and $\dot{\gamma}(0)$ in $T_o M$. Then geodesic balls $B(q, r), q \in M, r > 0$, and truncated cones

$$T_o(v, \alpha, s) = C_o(v, \alpha) \setminus \bar{B}(o, s),$$

with $v \in T_o M, \alpha > 0, s > 0$, form a basis for the cone topology. Furthermore, the cone topology is independent of the choice of $o \in M$ and, equipped with this
topology, \( \tilde{M} \) is homeomorphic to the closed unit ball \( \tilde{B}^n \subset \mathbb{R}^n \) and \( S(\infty) \) to the sphere \( S^{n-1} = \partial B^n \); see [7]. In particular, given \( o \in M \), \( S(\infty) \) may be canonically identified with the unit sphere \( S^{n-1} \subset T_o M \).

It is natural to ask whether every continuous function on \( S(\infty) \) has a unique harmonic extension to \( M \). This so-called asymptotic Dirichlet problem was solved by Choi if the sectional curvature has a negative upper bound \( K \leq -a^2 < 0 \) and any two points of the sphere at infinity can be separated by convex neighborhoods; see [6]. Such appropriate convex sets were constructed by Anderson [3] for manifolds of pinched sectional curvature \( -b^2 \leq K \leq -a^2 < 0 \). The Dirichlet problem was independently solved by Sullivan [13] under the same curvature assumptions by using probabilistic arguments. In [4], Anderson and Schoen presented a simple and direct proof. Ancona [1] was able to replace the lower curvature bound by a bounded geometry assumption that each ball up to a fixed radius is bi-Lipschitz equivalent to an open set in \( \mathbb{R}^n \). He also considered a more general class of operators. On the other hand, Ancona [2] showed that the Dirichlet problem cannot be solved, in general, if there are neither curvature lower bounds nor the bounded geometry assumption; see also [5]. In the general case of the \( p \)-Laplacian, the corresponding problem has been open so far. Pansu [11] has shown the existence of nonconstant bounded \( p \)-harmonic functions with finite \( p \)-energy on Cartan-Hadamard manifolds of pinched curvature \( -b^2 \leq K \leq -a^2 \) if \( p > (n-1)b/a \).

2. Asymptotic Dirichlet Problem

Let \( G \subset M \) be an open set and \( 1 < p < \infty \). Recall that a function \( u \in W^{1,p}_{\text{loc}}(G) \) is a (weak) solution of the equation

\[
- \text{div} (|\nabla u|^{p-2} \nabla u) = 0
\]

in \( G \) if

\[
\int_G (|\nabla u|^{p-2} \nabla u, \nabla \varphi) = 0
\]

for all \( \varphi \in C_0^\infty(G) \). Above \( W^{1,p}_{\text{loc}}(G) \) is the (local) Sobolev space of all functions \( u \in L^p_{\text{loc}}(G) \) whose distributional gradient \( \nabla u \) belongs to \( L^p_{\text{loc}}(G) \). Continuous solutions of (2.1) are called \( p \)-harmonic. It is well-known that every solution of (2.1) has a continuous representative by the fundamental work of Serrin [12]. We say that a function \( u \in W^{1,p}_{\text{loc}}(G) \) is a \( p \)-supersolution in \( G \) if

\[
- \text{div} (|\nabla u|^{p-2} \nabla u) \geq 0
\]

weakly in \( G \), that is,

\[
\int_G (|\nabla u|^{p-2} \nabla u, \nabla \varphi) \geq 0
\]

for all nonnegative \( \varphi \in C_0^\infty(G) \). Furthermore, we say that \( u \) is a \( p \)-subsolution if \( -u \) is a \( p \)-supersolution.

In this section we show that the direct approach to solve the Dirichlet problem taken by Anderson and Schoen in [4] also works in the nonlinear setting of \( p \)-harmonic functions.

Theorem 2.1. Let \( M \) be a Cartan-Hadamard manifold whose sectional curvature \( K \) satisfies

\[
-b^2 \leq K \leq -a^2
\]
for some constants \( b \geq a > 0 \). Let \( h \) be a continuous function on \( S(\infty) \). Then there exists a unique function \( u \in C(M) \) which is \( p \)-harmonic in \( M \) and \( u = h \) on \( S(\infty) \).

The proof requires some preliminaries. Let \( h \in C(S(\infty)) \) be given. Fix a point \( o \in M \) and write \( r(x) = d(x,o) \). We identify \( S(\infty) \) with the unit sphere \( S^{n-1} \subset T_o M \). Therefore, we may consider \( h \) as a continuous function on \( S^{n-1} \). Assume that \( h \colon S^{n-1} \to \mathbb{R} \) is Lipschitz. We extend \( h \) radially to a continuous function \( \tilde{h} \) on \( M \setminus \{o\} \). More precisely, we define \( \tilde{h} \) in polar coordinates about \( o \) by

\[
\tilde{h}(r, \vartheta) = h(\vartheta)
\]

for every \( r > 0 \) and \( \vartheta \in S^{n-1} \). The Lipschitz continuity of \( h \) and the curvature upper bound \( K \leq -a^2 \) imply that

\[
\text{osc}(\tilde{h}, B(x, 3)) := \sup_{B(x, 3)} \tilde{h} - \inf_{B(x, 3)} \tilde{h} \leq cLe^{-ar(x)},
\]

where \( L \) is the Lipschitz constant of \( h \); see \[4\]. Next we define a smooth function \( h \) on \( M \) such that

\[
\lim_{x \to \xi} h(x) = h(\xi)
\]

for every \( \xi \in S(\infty) \) and that first and second order derivatives of \( h \) are effectively controlled. For this purpose, we fix a maximal 1-separated set \( Q = \{q_1, q_2, \ldots \} \subset M \), that is,

\[
d(q_i, q_j) \geq 1
\]

whenever \( i \neq j \) and no more points can be added to \( Q \) without breaking the condition \[2.5\]. We may assume that \( o \notin Q \). In particular, the balls \( B(q_i, 1/2) \) are mutually disjoint and \( M = \bigcup_i B(q_i, 1) \). For each \( x \in M \), we write \( Q_x = Q \cap B(x, 3) \).

The curvature lower bound then implies that

\[
\text{card} Q_x \leq c,
\]

where \( c \) is independent of \( x \); see e.g. \[10\]. Then we define

\[
h(x) = \sum_{q_i \in Q} \tilde{h}(q_i) \varphi_i(x),
\]

where \( \{\varphi_i\} \) is a partition of unity subordinate to \( \{B(q_i, 3)\} \) defined as follows. First choose a \( C^\infty \) function \( f \colon [0, \infty] \to [0, 1] \) such that \( f[0, 1] = 1 \), \( f[2, \infty] = 0 \), and

\[
\max\{|f'(t)|, |f''(t)|\} \leq c\chi_{[1,2]}(t)
\]

for some constant \( c \), where \( \chi_{[1,2]} \) is the characteristic function of the interval \([1, 2]\). For \( q_i \in Q \) and \( x \in M \), let \( \eta_i(x) = f(r_i(x)) \), where \( r_i(x) = d(x, q_i) \). Finally we set

\[
\varphi_i(x) = \frac{\eta_i(x)}{\sum_j \eta_j(x)}.
\]

To estimate first and second order derivatives of \( h \), we first observe that

\[
\nabla \eta_i(x) = f'(r_i(x)) \nabla r_i(x)
\]

and

\[
\Delta \eta_i(x) = f'(r_i(x)) \Delta r_i(x) + \langle \nabla f'(r_i(x)), \nabla r_i(x) \rangle = f'(r_i(x)) \Delta r_i(x) + f''(r_i(x))
\]
since \((\nabla r_i(x), \nabla r_i(x)) = |\nabla r_i(x)|^2 \equiv 1\). Thus (2.8) and (2.10) imply that
\[
|\nabla \eta_i(x)| \leq c\mathcal{X}_{A(q_i:1,2)}(x),
\]
where \(A(y; s, t) = \bar{B}(y, t)\setminus B(y, s)\). By the Hessian comparison theorem ([8, Theorem A]),
\[
(n - 1)a \coth(ar_i(x)) \leq \Delta r_i(x) \leq (n - 1)b \coth(br_i(x)).
\]
Combining this with (2.8) and (2.11) yields
\[
|\Delta \eta_i(x)| \leq c\mathcal{X}_{A(q_i:1,2)}(x).
\]
Since \(\sum_j \eta_j(x) \geq 1, 0 \leq \eta_i(x) \leq 1\), and \(\text{card } Q \cap B(x, 3) \leq c\) for every \(x \in M\), we get by a simple computation that
\[
|\nabla \varphi_i(x)| \leq c\mathcal{X}_{B(q_i,4)}(x)
\]
and
\[
|\Delta \varphi_i(x)| \leq c\mathcal{X}_{B(q_i,4)}(x).
\]
In the next lemma we collect those properties of \(H\) that are crucial in the sequel.

**Lemma 2.2.** Let \(r: M \to \mathbb{R}\) be the distance function \(r(x) = d(x, o)\) and let \(h: M \to \mathbb{R}\) be the function given by (2.7). Furthermore, let \(v: M \setminus \{o\} \to \mathbb{R}\) be defined by
\[
v(x) = e^{-\delta r(x)},
\]
with \(\delta > 0\). Then there exists a constant \(c_0\) independent of \(h\) and \(\delta\) such that
\[
|\nabla h(x)| \leq c_0Le^{-ar(x)},
\]
\[
|\Delta h(x)| \leq c_0Le^{-ar(x)},
\]
\[
|\nabla (\nabla h, \nabla h)(x)| \leq (c_0L)^2e^{-2ar(x)},
\]
\[
|\nabla (\nabla h, \nabla v)(x)| \leq c_0L(1 + \delta)e^{-(a + \delta)r(x)}
\]
for \(r(x) \geq 1\). Moreover,
\[
\lim_{x \to \xi} h(x) = h(\xi)
\]
for every \(\xi \in S(\infty)\).

**Proof.** Fix \(x \in M \setminus B(a,1)\) and choose \(q \in Q\) such that \(x \in B(q,1)\). Then
\[
\nabla h(x) = \sum_{q_i \in Q} \hat{h}(q_i)\nabla \varphi_i(x) = \sum_{q_i \in Q_x} \hat{h}(q_i)\nabla \varphi_i(x)
\]
\[
= \sum_{q_i \in Q_x} (\hat{h}(q_i) - \tilde{h}(q))\nabla \varphi_i(x)
\]
since \(\sum_{q_i \in Q_x} \varphi_i = 1\) in a neighborhood of \(x\) and therefore
\[
\sum_{q_i \in Q_x} \hat{h}(q)\nabla \varphi_i(x) = \hat{h}(q)\nabla \left(\sum_{q_i \in Q_x} \varphi_i\right)(x) = 0.
\]
By (2.4), (2.6), and (2.12),
\[
|\nabla h(x)| \leq c(\text{card } Q_x) \text{osc}(h, B(x, 3)) \leq cLe^{-ar(x)}
\]
By (2.4) and (2.6) it suffices to prove that
\[ \langle \nabla h, \nabla h \rangle(x) = \sum_{q_i \in Q_x} (\tilde{h}(q_i) - \hat{h}(q_i)) \nabla \varphi_i, \sum_{q_j \in Q_x} (\tilde{h}(q_j) - \hat{h}(q_j)) \nabla \varphi_j \rangle(x) \]
\[ = \sum_{q_i, q_j \in Q_x} (\tilde{h}(q_i) - \hat{h}(q_i)) (\tilde{h}(q_j) - \hat{h}(q_j)) \langle \nabla \varphi_i, \nabla \varphi_j \rangle(x), \]
and so
\[ \nabla \langle \nabla h, \nabla h \rangle(x) = \sum_{q_i, q_j \in Q_x} (\tilde{h}(q_i) - \hat{h}(q_i)) (\tilde{h}(q_j) - \hat{h}(q_j)) \nabla \langle \nabla \varphi_i, \nabla \varphi_j \rangle(x). \]
By (2.4) and (2.6) it suffices to prove that
\[ |\nabla \langle \nabla \varphi_i, \nabla \varphi_j \rangle(x)| \leq c \]
for all \( q_i, q_j \in Q_x \) which reduces to establishing that
\[ |\nabla \langle \nabla r_i, \nabla r_j \rangle(x)| \leq c \]
whenever \( x \in A(q_i; 1, 2) \cap A(q_j; 1, 2) \). Let \( X_1, \ldots, X_n \) be an orthonormal frame in a neighborhood of \( x \). Then
\[ \nabla \langle \nabla r_i, \nabla r_j \rangle = \sum_k (X_k \langle \nabla r_i, \nabla r_j \rangle) X_k \]
\[ = \sum_k (\langle \nabla X_k \nabla r_i, \nabla r_j \rangle + \langle \nabla r_i, \nabla X_k \nabla r_j \rangle) X_k. \]
On the other hand,
\[ \langle \nabla X_k \nabla r_i, \nabla r_j \rangle = \nabla^2 r_i (X_k, \nabla r_j), \]
where \( \nabla^2 r_i \) is the Hessian of \( r_i \). By the Hessian comparison theorem all eigenvalues of \( \nabla^2 r_i \) are nonnegative and bounded from above by \( b \coth(br_i) \). Hence
\[ |\langle \nabla X_k \nabla r_i, \nabla r_j \rangle(x)| \leq b \coth(br_i(x)) |X_k(x)||\nabla r_j(x)| = b \coth(br_i(x)) \leq c \]
if \( r_i(x) \geq 1 \). Similarly, \( |\langle \nabla X_k \nabla r_j, \nabla r_i \rangle(x)| \leq c \) if \( r_j(x) \geq 1 \), and so (2.20) follows.
This proves (2.17). The estimate (2.18) can be established similarly since
\[ |\nabla \langle \nabla h, \nabla v \rangle(x)| \leq \delta e^{-\delta r(x)} \sum_{q_i \in Q_x} |\tilde{h}(q_i) - \hat{h}(q_i)||\nabla \langle \nabla \varphi_i, \nabla r \rangle(x)| \]
\[ + \delta^2 e^{-\delta r(x)} |\nabla r(x)| \sum_{q_i \in Q_x} |\tilde{h}(q_i) - \hat{h}(q_i)||\nabla \varphi_i, \nabla r \rangle(x)|. \]
Now \( |\nabla \langle \nabla \varphi_i, \nabla r \rangle(x)| \leq c \) if \( r(x) \geq 1 \) by a similar argument as above, and thus (2.18) follows. Finally, (2.19) follows easily from the definition (2.7) and from the continuity of \( h|S(\infty) \).

**Lemma 2.3.** Suppose that \( h: S^{n-1} \rightarrow \mathbb{R} \) is \( L \)-Lipschitz, where \( S^{n-1} \) is the unit sphere in \( T_xM \). Define \( h: M \rightarrow \mathbb{R} \) by (2.4) and let \( v = e^{-\delta r} \). Then there exist \( \delta_0 \in [0, a] \) such that, for every \( \delta \in [0, \delta_0] \), \( h + v \) is a \( p \)-supersolution and \( h - v \) is a \( p \)-subsolution in \( M \setminus \bar{B}(o, R_\delta) \), where \( R_\delta = R_\delta(a, \delta, c_0, L) \).
Proof. In what follows $R_1, \ldots, R_5$ are constants depending only on $a, \delta, c_0$, and $L$. Since $h$ and $v$ are smooth in $M \setminus \{o\}$, we can prove the claims by direct computation using the properties of $h$ for the first term in the right-hand side of (2.21). Write $u = h + v$ and note that

$$\nabla u = \nabla h - \delta e^{-\delta r} \nabla v \neq 0$$

if $\delta < a$ and $r > R_1$ by (2.14). Hence

$$\text{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2} \Delta u + \frac{p-2}{2} |\nabla u|^{p-4} \langle |\nabla u|^2, \nabla u \rangle$$

in $M \setminus \bar{B}(o, R_1)$. Next we deduce from (2.11) that

$$\Delta v = -\delta e^{-\delta r} \Delta r + \delta^2 e^{-\delta r} \leq \delta e^{-\delta r} (\delta - (n-1)a) \leq -c_1 \delta e^{-\delta r} < 0,$$

with $c_1 = (n-1)a/2$ whenever $\delta \leq (n-1)a/2$; cf. [4]. Given $\delta < a$ there exists $R_2$ such that

$$\delta^2 e^{-2\delta r} \leq |\nabla h + v|^2 = |\nabla h|^2 + 2 \langle \nabla h, \nabla v \rangle + |\nabla v|^2 \leq (c_0 L)^2 e^{-2a r} + 2c_0 L \delta e^{-(a+\delta)r} + \delta^2 e^{-2\delta r}$$

as soon as $r \geq R_2$. Hence

$$d_p^{-1} \delta^p e^{-\delta(p-2)r} \leq |\nabla h + v|^{p-2} \leq d_p \delta^{p-2} e^{-\delta(p-2)r},$$

where $d_p = 2^{p-2}/2$. If $\delta < a \land c_1$, we get an estimate

$$|\nabla h + v|^{p-2} (\Delta h + \Delta v) \leq d_p^{-1} \delta^p e^{-\delta(p-2)r} (d_p^2 c_0 L e^{-ar} - c_1 \delta e^{-\delta r})$$

for the first term in the right-hand side of (2.21). To estimate the second term in (2.21) we write

$$\langle |\nabla u|^2, \nabla u \rangle = \langle |\nabla (|\nabla h|^2), \nabla h + v \rangle + \langle |\nabla (|\nabla v|^2), \nabla u \rangle + 2 \langle \nabla \langle \nabla h, \nabla v \rangle, \nabla u \rangle$$

$$= A + B + C.$$

By (2.15) and (2.17),

$$A = \langle |\nabla (|\nabla h|^2), \nabla h + v \rangle \leq |\nabla \langle \nabla h, \nabla h \rangle| |\nabla h + v| \leq (c_0 L)^2 e^{-2ar} (c_0 L e^{-ar} + \delta e^{-\delta r}) \leq \delta^4 e^{-3\delta r}$$

if $r \geq R_3$. Similarly,

$$B = \langle |\nabla (|\nabla v|^2), \nabla h + v \rangle \leq 2c_0 L \delta^3 e^{-2\delta r} e^{-ar} + 2\delta^4 e^{-3\delta r} \leq 3\delta^4 e^{-3\delta r}$$

if $r \geq R_4$. Finally, (2.18) and (2.15) imply that

$$C = 2 \langle |\nabla (\nabla h, \nabla v), \nabla h + v \rangle \leq 2 |\nabla \langle \nabla h, \nabla v \rangle| |\nabla h + v| \leq 2c_0 L (1 + \delta) \delta e^{-(a+\delta)r} (c_0 L e^{-ar} + \delta e^{-\delta r}) \leq \delta^4 e^{-3\delta r}$$
whenever \( r \geq R_5 \). Putting these estimates together yields
\[
\text{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2} \Delta u + \frac{p-2}{2} |\nabla u|^{p-4} \langle \nabla (|\nabla u|^2), \nabla u \rangle \\
\leq a_p^{-1} \delta^{p-2} e^{-\delta(p-2)r} [a_p^2 c_0 L e^{-ar} - (c_1 - \delta C_p) \delta e^{-ar}],
\]
where
\[
C_p = 3|p - 2|^{\frac{p-2}{2}}.
\]
Choosing \( \delta_0 < \min\{a, c_1/(2C_p)\} \), with an obvious interpretation \( c_1/(2C_p) = \infty \) if \( p = 2 \), finally gives an estimate
\[
\text{div}(|\nabla u|^{p-2}\nabla u) \leq -c_2 \delta^{p-1} e^{-\delta(p-1)r} < 0
\]
if \( \delta \leq \delta_0 \) and \( r \geq R_5 \). Similarly, we obtain an estimate
\[
\text{div}(|\nabla u|^{p-2}\nabla u) \geq c_2 \delta^{p-1} e^{-\delta(p-1)r} > 0
\]
if \( \delta \leq \delta_0 \) and \( r \geq R_5 \).

Lemma 2.4. Identify \( S(\infty) \) with the unit sphere \( S^{n-1} \subset T_o M \). Assume that \( h: \quad S^{n-1} \to \mathbb{R} \) is \( L \)-Lipschitz. Then there exists a \( p \)-harmonic function \( u \) in \( M \) satisfying
\[
\lim_{x \to \xi} u(x) = h(\xi)
\]
for every \( \xi \in S(\infty) \).

Proof. Define \( h: M \to \mathbb{R} \) by \( 2.7 \) and let \( \delta \in [0, \delta_0] \) and \( R_5 \) be given by Lemma 2.8.

First we note that \( h \) is bounded, and therefore we can choose a constant \( \lambda \in [0,1] \) such that
\[
\lambda \text{osc}(h,M) \leq e^{-\delta R_5}.
\]
Since \( \lambda h \mid S^{n-1} \) is also \( L \)-Lipschitz, \( \lambda h + v \) is a \( p \)-supersolution and \( \lambda h - v \) is a \( p \)-subsolution in \( M \setminus \overline{B}(a, R_5) \). For \( i = 1, 2, \ldots \), let \( u_i \in C(M) \) be the unique function such that \( u_i \) is \( p \)-harmonic in \( B(a, 2^i R_5) \) and \( u_i \equiv \lambda h \) in \( M \setminus B(a, 2^i R_5) \). Now \( \lambda h - v \leq u_i \leq \lambda h + v \) on \( \partial(B(a, 2^i R_5) \setminus \overline{B}(a, R_5)) \), and hence the same holds in \( B(a, 2^i R_5) \setminus \overline{B}(a, R_5) \) by the comparison principle; see \( 3.18 \) and \( 7.6 \). Hence there exists a subsequence, denoted again by \( (u_i) \) and a function \( u \in C(M) \) such that \( \lambda^{-1} u_i \to u \) locally uniformly in \( M \). Furthermore, the function \( u \) is \( p \)-harmonic in \( M \) and satisfies \( 2.22 \) for every \( \xi \in S(\infty) \).

Proof of Theorem 2.7. Fix \( o \in M \) and identify \( S(\infty) \) with \( S^{n-1} \subset T_o M \). Let \( (h_i) \) be a sequence of Lipschitz functions on \( S^{n-1} \) such that \( h_i \to h \) uniformly on \( S^{n-1} \). By Lemma 2.4, there are \( p \)-harmonic functions \( u_i \in C(M) \) with \( u_i = h_i \) in \( S(\infty) \). The sequence \( (u_i) \) converges uniformly in \( M \) to a function \( u \in C(M) \) which is \( p \)-harmonic in \( M \) and \( u = h \) in \( S(\infty) \). To prove the uniqueness, suppose that \( u \) and \( w \) are both \( p \)-harmonic in \( M \), continuous in \( M \), with \( u = w \) in \( S(\infty) \), and \( u(y) > w(y) \) for some \( y \in M \). Let \( \varepsilon = (u(y) - w(y))/2 \). Since \( u \) and \( w \) are continuous in \( M \) and they coincide on the compact set \( S(\infty) \), there exists \( R > 0 \) such that \(|u(x) - w(x)| < \varepsilon \) for every \( x \in M \setminus \overline{B}(o, R) \). Let \( D \) be the \( y \)-component of \( \{x \in M : u(x) > w(x) + \varepsilon\} \). It follows that \( D \) is a relatively compact domain in \( M \) and \( u = w + \varepsilon \) on \( \partial D \). Hence \( u = w + \varepsilon \) in \( D \) which leads to a contradiction since \( y \in D \). This proves the uniqueness and thus the whole theorem is proved.
References


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