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GENERATION THEOREMS FOR φ HILLE-YOSIDA OPERATORS

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ABSTRACT. This paper introduces the concept of φ Hille-Yosida operators and studies several generation theorems. We show that if a once-integrated semigroup $\{S(t)\}_{t\geq 0}$ satisfies $\Phi(t) := limsup_{h\to 0^+}\frac{1}{h}||S(t+h)-S(t)|| < \infty$ for all $t > 0$ *a.e.*, then $\Phi(\cdot)$ is locally bounded on $(0,\infty)$ and exponentially bounded. In addition, some other interesting results are presented.

1. INTRODUCTION

It is well known that a linear operator A on a Banach space X is called a *Hille-Yosida operator* of type (M,ω_0) (see [4]) if there exist $M > 0$ and $\omega_0 \in R$ such that $(\omega_0,\infty) \subseteq \rho(A)$ and

$$||R(\lambda, A)^n|| \leq \frac{M}{(\lambda-\omega_0)^n} \quad \forall \lambda > \omega_0,\ n \in N.$$

By the Hille-Yosida theorem, an operator A generates a C_0-semigroup if and only if it is a densely defined Hille-Yosida operator. But recently there has been a great deal of interest in operators that may not be densely defined. For example, in many population models, the operators under consideration are not densely defined.

This paper will study once-integrated semigroups and semigroups generated by φ Hille-Yosida operators (Definition 2.10) that may not be densely defined.

We say that a Banach space-valued function $f(\cdot)$ defined on $(0,\infty)$ (resp. on $[0,\infty)$) is *locally Lipschitz continuous* on $(0,\infty)$ (resp. on $[0,\infty)$) if for every bounded interval $[a,b] \subset (0,\infty)$ (resp. $[a,b] \subset [0,\infty)$), there exists $K_{a,b} > 0$ such that

$$limsup_{h\to 0^+}\frac{1}{h}||f(t+h)-f(t)|| \leq K_{a,b} \quad \forall\ t \in [a,b].$$

We say that $f(\cdot)$ is *exponentially bounded* if there exists $\omega_0 \in R$, the set of all real numbers, such that

$$||f(t)|| = O(e^{\omega_0 t}), \quad \text{as } t\to\infty. \tag{1.1}$$

Throughout, X is a Banach space, and A is a closed linear operator (not necessarily densely defined) on X. $\rho(A)$, $\sigma(A)$ are, respectively, the resolvent set and the spectrum of A.

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In Section 2 we will prove several Hille-Yosida theorems for once-integrated semigroups. It is surprising that if a once-integrated semigroup $\{S(t)\}_{t\geq 0}$ satisfies

$$\Phi(t) := limsup_{h\to 0}\frac{1}{h}||S(t+h)-S(t)|| < \infty \quad \forall t>0 \ a.e.,$$

then $\Phi(\cdot)$ will be locally bounded as well as exponentially bounded on $(0,\infty)$. This and other related results seem to be significant improvements of some well known ones (see Theorems 2.4, 2.11, 2.13 and Corollary 2.12 of this paper).

As a supplement of Section 2, in Section 3 we study the strong continuity of a semigroup generated by a φ Hille-Yosida operator on an extended space that contains X as a subset. As a special case, we examine the continuity of conjugate semigroups in a certain sense, which is similar to a result in [3, Section 5].

To illustrate the theorems established in Sections 2 and 3, in Section 4 we offer three examples.

As regards the theory of integrated semigroups, regularized semigroups and their applications to partial differential equations, quite a few important achievements in this area were summarized in the monograph [4] by R. deLaubenfels. The interested reader will find the book of importance.

Definition 1.1 ([1, 4, 5])**.** Suppose W is a Banach space and $\omega_0 \in R$. We say that $g:(\omega_0,\infty)\mapsto W$ is the *once-integrated Laplace transform* of G if $G:[0,\infty)\mapsto W$ is continuous, $G(0)=0$, and

$$g(\lambda)=\lambda\int_0^\infty e^{-\lambda t}G(t)\,dt \quad \forall\ \lambda>\omega_0.$$

We recall that $L^1_{loc}(R_+)$ is the space of all functions that are integrable on every bounded interval $[a,b]\subset R_+ := [0,\infty)$.

The following lemma is similar to but different than [5, Theorem 2.2], so we offer a proof.

Lemma 1.2. *Suppose $\varphi \in L^1_{loc}(R_+)$ is positive and satisfies* (1.1), *and $g:(\omega_0,\infty)\mapsto C$. Then the following are equivalent*:

(i) *g is the once-integrated Laplace transform of a continuous function $G:[0,\infty)\mapsto C,\ G(0)=0$ and there exists a constant $K_1>0$ so that*

$$|G(t+h)-G(t)| \leq K_1\int_t^{t+h}\varphi(s)\,ds \quad \forall t\geq 0,\ h>0. \tag{1.2}$$

(ii) *There exists a constant $K_2>0$ such that g is infinitely differentiable and*

$$|g^{(n)}(\lambda)| \leq K_2\int_0^\infty e^{-\lambda t}t^n\varphi(t)\,dt \quad \forall\lambda>\omega_0,\ n\in N\cup\{0\}.$$

K_1 and K_2 may be chosen so that $K_2\leq K_1\leq 2K_2$.

Proof. (i) $\to$ (ii). From

$$g(\lambda)=\lambda\int_0^\infty e^{-\lambda t}G(t)\,dt=\int_0^\infty e^{-\lambda t}\,dG(t),$$

we have

$$\begin{aligned}|g^{(n)}(\lambda)| &= |\int_0^\infty e^{-\lambda t}t^n\,dG(t)| \\ &\leq K_1\int_0^\infty e^{-\lambda t}t^n\varphi(t)\,dt \quad \forall\ \lambda>\omega_0,\ n\in N\cup\{0\}.\end{aligned}$$

(ii) → (i). We first assume that g is real-valued. (ii) implies that

$$h_{\pm}(\lambda) := K_2 \int_0^{\infty} e^{-\lambda t}\varphi(t)\,dt \pm g(\lambda)$$

is completely monotonic ([14, p. 154]), that is,

$$(-1)^n h_{\pm}^{(n)}(\lambda) = K_2 \int_0^{\infty} e^{-\lambda t} t^n \varphi(t)\,dt \pm (-1)^n g^{(n)}(\lambda) \geq 0.$$

By Bernstein's theorem ([14, p. 156]), there exist nondecreasing functions $\alpha_{\pm}$ such that

$$h_{\pm}(\lambda) = \int_0^{\infty} e^{-\lambda t}\,d\alpha_{\pm}(t) = \lambda \int_0^{\infty} e^{-\lambda t}\alpha_{\pm}(t)\,dt.$$

Therefore

$$\begin{aligned} g(\lambda) =& \lambda \int_0^{\infty} e^{-\lambda t}[\alpha_+(t) - K_2 \int_0^t \varphi(s)\,ds]\,dt \\ &= \lambda \int_0^{\infty} e^{-\lambda t}[K_2 \int_0^t \varphi(s)\,ds - \alpha_-(t)]\,dt. \end{aligned}$$

By the uniqueness of the Laplace transform, we have

$$\alpha_+(t) - K_2 \int_0^t \varphi(s)\,ds = K_2 \int_0^t \varphi(s)\,ds - \alpha_-(t).$$

Denote them by $G(t)$. Then g is the once-integrated Laplace transform of G and

$$|G(t+h) - G(t)| \leq K_2 \int_t^{t+h} \varphi(s)\,ds \quad \forall\ t \geq 0.$$

(1.2) follows with $K_1 = K_2$.

If g is complex-valued, write $g(\lambda) = g_1(\lambda) + ig_2(\lambda)$, where g_1, g_2 are real-valued. From the foregoing, g_1, g_2 are, respectively, the once-integrated Laplace transforms of G_1, G_2 satisfying (1.2). Define $G := G_1 + iG_2$. Then g is the once-integrated Laplace transform of G satisfying (1.2) with $K_1 = 2K_2$. □

The following lemma is a vector-valued version of Lemma 1.2 and its proof is similar to [5, Theorem 2.3]; we omit the details.

Lemma 1.3. *Suppose that φ satisfies the conditions in Lemma* 1.2, *W is a Banach space and $g : (\omega_0, \infty) \mapsto W$. Then the following are equivalent*:

(i) *g is the once-integrated Laplace transform of a continuous function $G : [0,\infty) \mapsto W$ and there is a constant $K_1 > 0$ such that*

$$||G(t+h) - G(t)|| \leq K_1 \int_t^{t+h} \varphi(s)\,ds \quad \forall\ t \geq 0,\ h > 0.$$

(ii) *g is infinitely differentiable and there is a constant $K_2 > 0$ such that*

$$||g^{(n)}(\lambda)|| \leq K_2 \int_0^{\infty} e^{-\lambda t} t^n \varphi(t)\,dt \quad \forall\ \lambda > \omega_0,\ n \in N \cup \{0\}.$$

K_1 and K_2 may be chosen so that $K_2 \leq K_1 \leq 2K_2$.

2. Once-integrated semigroups generated by φ Hille-Yosida operators

Definition 2.1 ([1, 4])**.** A strongly continuous family $\{S(t)\}_{t\geq 0}$ of bounded linear operators on X is a once-integrated semigroup if the following are true:

(i) $S(0) = 0$.

(ii) $S(t)S(s)x = [\int_s^{t+s} - \int_0^t] S(r)x\, dr$ for all $t \geq 0,\ s \geq 0$ and $x \in X$.

$\{S(t)\}_{t\geq 0}$ is nondegenerate if $S(t)x = 0$, for all $t \geq 0$, implies $x = 0$.

Throughout, all once-integrated semigroups we are dealing with are nondegenerate.

Let $\{S(t)\}_{t\geq 0}$ be a once-integrated semigroup and let $x \in X$. If $y \in X$ is a solution of the equation

$$\int_0^t S(s)y\, ds = S(t)x - tx \quad \forall t \geq 0, \tag{2.1}$$

then y is uniquely determined by x by the nondegeneration.

Definition 2.2 ([10, 11, 12])**.** We define the *generator* A of $\{S(t)\}_{t\geq 0}$ to be the following operator:

$$D(A) := \{x : \exists\ y \text{ satisfying (2.1)}\} \text{ and } Ax := y.$$

The proof of the following lemma can be found in [1, 11, 12].

Lemma 2.3. *For a once-integrated semigroup $\{S(t)\}_{t\geq 0}$, the following statements are true:*

(i) *If A is the generator of $\{S(t)\}_{t\geq 0}$, then $\int_0^t S(s)x\, ds \in D(A)$ for all $x \in X,\ t \geq 0$ and*

$$A\int_0^t S(s)x\, ds = S(t)x - tx. \tag{2.2}$$

(ii) *If $x \in D(A)$, then $S(t)x$ is continuously differentiable and*

$$AS(t)x = S(t)Ax = \frac{d}{dt}S(t)x - x \quad \forall t \geq 0.$$

(iii) *If A is closed and satisfies (2.2), then A is the generator of $\{S(t)\}_{t\geq 0}$.*

Theorem 2.4. *Let $\{S(t)\}_{t\geq 0}$ be a once-integrated semigroup. For every $t \geq 0$, denote*

$$\Phi(t) := limsup_{h\to 0^+}\frac{1}{h}||S(t+h) - S(t)||.$$

If $\Phi(t) < \infty$ for $t \geq 0$ a.e., then $\Phi(t)$ is bounded in every bounded interval $[a, b] \subset (0, \infty)$. Consequently, $\{S(t)\}_{t\geq 0}$ is locally Lipschitz continuous on $(0, \infty)$, and exponentially bounded, that is, there exists $\omega_0 \in R$ such that

$$limsup_{h\to 0^+}\frac{1}{h}||S(t+h) - S(t)|| = 0(e^{\omega_0 t}), \quad \text{as } t \to \infty.$$

To prove the theorem, we need the following lemmas.

Lemma 2.5. $\Phi(\cdot)$ *is measurable on* $[0, \infty)$.

Proof. Since $S(t+h) - S(t)$ is strongly continuous on $[0, \infty)$ for given $h > 0$, $||S(t+h) - S(t)||$ is lower semi-continuous. Hence it is measurable, thus $\Phi(t) = limsup_{h\to 0^+}\frac{1}{h}||S(t+h) - S(t)||$ is also. ☐

Lemma 2.6. *$\Phi(\cdot)$ satisfies $\Phi(t+s) \leq \Phi(t)\Phi(s)$ for $t, s \geq 0$ a.e.*

Proof. From

$$\begin{aligned}
&[S(t+h)-S(t)][S(s+k)-S(s)]x \\
&\quad = [(\int_{s+k}^{t+h+s+k} - \int_0^{t+h}) - (\int_s^{t+h+s} - \int_0^{t+h})]S(r)x\,dr \\
&\quad - [(\int_{s+k}^{t+s+k} - \int_0^{t}) - (\int_s^{t+s} - \int_0^{t})]S(r)x\,dr \\
&\quad = [\int_{t+s+k}^{t+s+k+h} - \int_{t+s}^{t+s+h}]S(r)x\,dr,
\end{aligned}$$

we have

$$\begin{aligned}
&||[\int_{t+s+k}^{t+s+k+h} - \int_{t+s}^{t+s+h}]S(r)x\,dr|| \\
&\quad \leq ||S(t+h)-S(t)||\;||S(s+k)-S(s)||\;||x||.
\end{aligned}$$

This implies that

$$\begin{aligned}
&||[S(t+s+k)-S(t+s)]x|| \\
&\quad = lim_{h\to 0}||\frac{1}{h}[\int_{t+s+k}^{t+s+k+h} - \int_{t+s}^{t+s+h}]S(r)x\,dr|| \\
&\quad \leq ||S(s+k)-S(s)||\;limsup_{h\to 0}\frac{1}{h}||S(t+h)-S(t)||\;||x||.
\end{aligned}$$

Hence $\Phi(t+s) \leq \Phi(t)\ \Phi(s)$ for $t, s \geq 0$ *a.e.* □

Lemma 2.7. *$\Phi(\cdot)$ is bounded on every bounded interval $[a,b] \subset (0,\infty)$ and hence exponentially bounded.*

Proof. We use the method employed in [6, 8] to prove the first conclusion. Assume $\Phi(\cdot)$ is unbounded in an interval $[a,b] \subset (0,\infty)$. Then there exist a $t_0 \in [a,b]$ and a sequence $\{t_n\} \subset [a,b]$ satisfying

$$t_n \to t_0 \text{ and } \Phi(t_n) \geq n.$$

Since $\Phi(\cdot)$ is measurable by Lemma 2.5 and finite almost everywhere, there exist a constant $M > 0$ and a measurable set $E_0 \subset (0, t_0]$ such that

$$m(E_0) > \frac{t_0}{2} \text{ and } \Phi(t) \leq M \quad \forall\ t \in E_0.$$

Denote $t_n - \{E_0 \cap (0, t_n]\}$ by E_n. Then $m(E_n) > t_0/2$ for n sufficiently large. Let $t \in E_n$ and $s_n \in E_0 \cap (0, t_n]$ be such that $t = t_n - s_n$. From Lemma 2.6

$$n \leq \Phi(t_n) \leq \Phi(t)\Phi(s_n) \leq M\Phi(t).$$

This is true for almost every $t \in E_n$. Hence $\Phi(t) \geq n/M\ \forall t \in E_n$ *a.e.* Denote $E := limsup_n E_n$. Then $m(E) \geq t_0/2$ and $\Phi(t) = \infty$ for almost every $t \in E$. This contradicts the fact that $\Phi(\cdot)$ is finite *a.e.*

It remains to show that $\Phi(\cdot)$ is exponentially bounded. Denote $K := sup\{\Phi(t) : t \in [1,2]\}$. If $t > 2$, write $t = n + q$ with $n \in N,\ 1 \leq q < 2$. Then the exponential boundedness follows from

$$\Phi(t) \leq \Phi(n)\Phi(q) \leq K[\Phi(1)]^n = Ke^{n[ln(\Phi(1))]}$$
$$= Ke^{-q[ln(\Phi(1))]}e^{t[ln(\Phi(1))]} = O(e^{\omega_0 t}),$$

where $\omega_0 := ln(\Phi(1))$. □

Theorem 2.4 is an immediate consequence of Lemma 2.7.

Remark 2.8. Theorem 2.4 dealt with locally Lipschitz continuous once-integrated semigroups; from [4, Example 4.10] and [7, Example 1.2] there exist once-integrated semigroups that are not locally Lipschitz continuous and hence $\Phi(t) = \infty$ for almost all $t \in (0, \infty)$.

The following lemma is well known and its proof was given in [2, Theorem 6.8] for increasing once-integrated semigroups. But the proof there is valid for our case.

Lemma 2.9. *If $\{S(t)\}_{t\geq 0}$ satisfies the conditions in Theorem* 2.4*, then A is the generator of $\{S(t)\}_{t\geq 0}$ if and only if the following hold*:

(i) *There exists $\omega_0 \in R$ such that $(\omega_0, \infty) \subset \rho(A)$.*

(ii) *For $\lambda > \omega_0$ and $x \in X$,*

$$(\lambda - A)^{-1}x = \lambda \int_0^\infty e^{-\lambda t} S(t)x\, dt.$$

Definition 2.10. Suppose $\varphi \in L^1_{loc}(R_+)$ is positive and exponentially bounded, that is, there exists $\omega_0 \in R$ such that φ satisfies (1.1). A closed operator A (not necessarily densely defined) is a φ *Hille-Yosida operator* if the following hold:

(i) $(\omega_0, \infty) \subset \rho(A)$.

(ii) A satisfies

$$||(\lambda - A)^{-n}|| \leq \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} \varphi(t)\, dt \quad \forall\ \lambda > \omega_0,\ n \in N.$$

The following theorem offers a Hille-Yosida type theorem for once-integrated semigroups that are automatically locally Lipschitz continuous and exponentially bounded.

Theorem 2.11. *Suppose A is closed. Then the following are equivalent*:

(i) *There exists φ satisfying the conditions in Definition* 2.10 *such that A is a φ Hille-Yosida operator.*

(ii) *A is the generator of a once-integrated semigroup $\{S(t)\}_{t\geq 0}$ and there exists a constant $K_1 > 0$ such that*

$$limsup_{h\to 0^+} \frac{1}{h} ||S(t+h) - S(t)|| \leq K_1 \varphi(t) \quad \forall\ t \geq 0\ a.e.$$

with φ given in (i)*.*

(iii) *A is the generator of a once-integrated semigroup $\{S(t)\}_{t\geq 0}$ satisfying*

$$limsup_{h\to 0^+} \frac{1}{h} ||S(t+h) - S(t)|| < \infty \quad \forall\ t \geq 0\ a.e.$$

The left-hand side is integrable on the interval $[0, 1]$*.*

(iv) *A is the generator of a once-integrated semigroup that is locally Lipschitz continuous on* $(0, \infty)$*, and the left-hand side of the expression in* (iii) *is integrable on* $[0, 1]$*.*

Proof. (i) → (ii) is an immediate consequence of Lemmas 1.3 and 2.9.

(ii) → (iii). Obvious.

(iii) → (iv) follows from Theorem 2.4.

(iv) → (i). Define

$$\varphi(t) := limsup_{h\to 0}\frac{1}{h}||S(t+h)-S(t)|| \quad \forall\, t \geq 0.$$

From Theorem 2.4, $\varphi(\cdot) \in L^1_{loc}(R_+)$ and there exists $\omega_0 \in R$ such that $|\varphi(t)| = O(e^{\omega_0 t}),\ as\ t \to \infty$. $\{S(t)\}_{t\geq 0}$ is therefore exponentially bounded so that $(\omega_0, \infty) \subset \rho(A)$, and A is thus a φ Hille-Yosida operator by Lemmas 1.3 and 2.9. □

Example 4.1 in Section 4 shows that the integrable condition in (iii) and (iv) of Theorem 2.11 cannot be removed.

The following corollary is an improvement of [7, Proposition 2.2].

Corollary 2.12. *Suppose A is closed. Then the following are equivalent*:

(i) *A is a Hille-Yosida operator.*

(ii) *A generates a once-integrated semigroup that is locally Lipschitz continuous on* $[0, \infty)$.

(iii) *A generates a once-integrated semigroup $\{S(t)\}_{t\geq 0}$ satisfying*

$$limsup_{h\to 0^+}\frac{1}{h}||S(t+h)-S(t)|| < \infty,\ \forall\, t \geq 0\ a.e.$$

and the left-hand side is bounded in a neighborhood of $t = 0$.

Assume $\{S(t)\}_{t\geq 0}$ is a once-integrated semigroup generated by A satisfying the equivalent conditions in Theorem 2.11. Then $\{S(t)\}_{t\geq 0}$ is locally Lipschitz continuous on $(0, \infty)$ and

$$limsup_{h\to 0^+}\frac{1}{h}||S(t+h)-S(t)|| \leq K_1\varphi(t) \quad \forall\, t > 0\ a.e. \tag{2.3}$$

where K_1 is the constant in (ii) of Theorem 2.11. Let X_0 be the space of x such that $S(t)x$ is continuously differentiable on $(0, \infty)$. Then $D(A) \subseteq X_0$ by (ii) of Lemma 2.3. From (2.3) and an argument similar to [11, p. 262], we can show that X_0 is closed in X. Define

$$T_0(t)x := \frac{d}{dt}S(t)x \quad \forall\, t > 0,\ x \in X_0. \tag{2.4}$$

From the proof of [7, Proposition 2.2] $\{T_0(t)\}_{t>0}$ is a semigroup on X_0 that is strongly continuous and satisfies

$$||T_0(t)|| \leq K_1\varphi(t) \quad \forall\, t > 0\ a.e.$$

Let A_0 be the part of A in X_0. From (2.4) and Definition 2.2, $x \in D(A_0)$ if and only if there exists $y \in X_0$ such that

$$\int_0^t T_0(s)y\, ds = T_0(t)x - x \quad \forall\, t > 0,$$
$$A_0 x = y.$$

We say that A_0 is the generator of $\{T_0(t)\}_{t>0}$. We do not know if A_0 is densely defined in X_0, yet $(\omega_0, \infty) \subseteq \rho(A_0)$ and $(\lambda - A_0)^{-1} = (\lambda - A)^{-1}|_{X_0}$. From Lemma 2.9 we have

$$(\lambda - A_0)^{-1}x = \int_0^\infty e^{-\lambda t}T_0(t)x\, dt \quad \forall\, x \in X_0,\ \lambda > \omega_0.$$

This, combined with [6, relation (11.5.1) and Theorem 11.5.2], concludes that our definition of generators here is consistent with the definition on [6, p. 302]. Thus we have proved

Theorem 2.13. *Suppose A is a φ Hille-Yosida operator. Then the following statements are true*:

(i) *The part A_0 of A in X_0 generates a semigroup $\{T_0(t)\}_{t>0}$ on X_0 that is strongly continuous and satisfies $||T_0(t)|| \leq K_1\varphi(t)$ for $t > 0$ a.e.*

(ii) *If $D(A)$ is dense in X, then $\{T_0(t)\}_{t>0}$ is a semigroup on X generated by A and satisfies the properties in* (i).

(iii) *If A is strongly Abel-ergodic, that is, for $x \in X$,*

$$\lambda(\lambda - A)^{-1}x \to x, \text{ as } \lambda \to \infty, \tag{2.5}$$

then $\{T_0(t)\}_{t>0}$ is of class $(1, A)$ (see [6]*).*

(i) and (ii) of Theorem 2.13 produce a new class of strongly continuous semigroups which are more general than the semigroups of class $(1, A)$, since (2.5) may not be true in general (see Example 4.2).

3. Semigroups on extended spaces

Let A be a closed linear operator on X with $\rho(A)$ nonempty. Given $\mu \in \rho(A)$, similar to [3, Section 5], [9, Section 3.1] and [10, Section 5], we define the following norm on X:

$$||x||_\mu := ||(\mu - A)^{-1}x|| \quad \forall\, x \in X, \tag{3.1}$$

and let X_μ be the completion of X under the norm $||\cdot||_\mu$.

The following lemma can be found in the above-listed references.

Lemma 3.1. *Let A be a closed linear operator on X with $\rho(A)$ nonempty. Then the following are true*:

(i) *The norms $||\cdot||_\mu$ $(\mu \in \rho(A))$ on X are mutually equivlent and hence X_μ are mutually isomorphic.*

(ii) *$D(A)$ is dense in X_μ under the norm $||\cdot||_\mu$.*

(iii) *A is closable in X_μ. Let A_μ be the closure of A in X. Then A is the part of A_μ in X and $\rho(A) \subseteq \rho(A_\mu)$.*

From (i) of Lemma 3.1, we may regard X_μ as the same space endowed with the same topology defined in (3.1). Fix $\mu \in \rho(A)$ and denote $X_1 := X_\mu$, $||x||_1 := ||x||_\mu$ for every $x \in X_1$. A_1 is thus the closure of A in X_1.

Let $\{S(t)\}_{t\geq 0}$ be the once-integrated semigroup satisfying the equivalent conditions in Theorem 2.11. Since $||S(t)x||_1 \leq ||S(t)||\ ||x||_1$ for all $x \in X$, $S(t)$ is bounded under the norm $||\cdot||_1$. Let $S_1(t)$ be the extension of $S(t)$ on X_1. Then $\{S_1(t)\}_{t\geq 0}$ is a once-integrated semigroup on X_1 that is locally Lipschitz continuous on $(0, \infty)$. A_1 is clearly a φ Hille-Yosida operator on X_1.

From Lemma 2.9 and (iii) of Lemma 3.1, one can show that

$$(\lambda - A_1)^{-1}x = \lambda \int_0^\infty e^{-\lambda t} S_1(t)x\, dt \quad \forall\, \lambda > \omega_0,\ x \in X_1.$$

$\{S_1(t)\}_{t\geq 0}$ is thus nondegenerate and A_1 is the generator of $\{S_1(t)\}_{t\geq 0}$. From (ii) of Theorem 2.13 and (ii) of Lemma 3.1 A_1 generates a semigroup $\{T_1(t)\}_{t>0}$ on X_1

that is strongly continuous and

$$||T_1(t)||_1 \le K_1\varphi(t) \quad \forall\ t > 0\ a.e.$$

Clearly, $T_1(t)x = (\frac{d}{dt})_1 S_1(t)x$ for all $x \in X_1$, where $(\frac{d}{dt})_1$ means the derivative taken in the norm topology of X_1. The following relations are clear:

$$T_1(t)x = (\frac{d}{dt})_1 S(t)x \quad \forall\ x \in X,$$
$$T_1(t)x = (\frac{d}{dt})_1 S(t)x = \frac{d}{dt}S(t)x = T_0(t)x \quad \forall\ x \in D(A),$$

where $T_0(\cdot)$ is defined in Section 2. Moreover,

$$T_1(t)x = (\lambda - A)T_0(t)(\lambda - A)^{-1}x \quad \forall\ x \in X,\ \lambda \in \rho(A).$$

In [3, 9, 10], $\{T_1(t)\}_{t>0}$ is called the *intertwined semigroup* generated by A.

The following proposition offers several equivalent conditions such that X is invariant under $T_1(t)$ for all $t > 0$ (see also [3, 9, 10]).

Proposition 3.2. *The following assertions are equivalent*:

(i) $\{T_1(t)\}_{t>0}$ *can be restricted to a semigroup of bounded linear operators on* X *that is strongly continuous.*

(ii) $D(A)$ *is invariant under* $T_0(t)$ *for all* $t \in (0, \infty)$.

(iii) $(\frac{d}{dt})_1 S(t)x \in X$ *for every* $x \in X$.

(iv) $S(t)$ *maps* X *into* $D(A)$.

We now consider the conjugate semigroups.

Let Y be a Banach space and let $X := Y^*$ be the conjugate of Y and let B be a densely defined φ Hille-Yosida operator on Y. Then B generates a semigroup $\{\tilde{T}(t)\}_{t>0}$ on Y that enjoys the properties in (ii) of Theorem 2.13. The conjugate A of B is also a φ Hille-Yosida operator. We show that for this A, the equivalent conditions in Proposition 3.2 are satisfied. Let $\tilde{S}(t)y = \int_0^t \tilde{T}(s)y\, ds (y \in Y,\ t \ge 0)$ and let $S(t) := [\tilde{S}(t)]^*$. Then $\{S(t)\}_{t\ge 0}$ is the once-integrated semigroup generated by A.

Theorem 3.3. *For all* $x \in X$ *we have* $S(t)x \in D(A)$. *Therefore* $\{\tilde{T}^*(t)\}_{t>0}$ *is strongly continuous in the topology induced by the norm* $||\cdot||_1$ *on* X.

Proof. Let $y \in D(B)$ and let $x \in X$. From

$$\langle By, S(t)x\rangle = \langle By, [\tilde{S}(t)]^* x\rangle = \langle \tilde{S}(t)By, x\rangle$$
$$= \langle \tilde{T}(t)y - y, x\rangle = \langle y, \tilde{T}^*(t)x - x\rangle,$$

it follows that $\langle By, S(t)x\rangle$ is a bounded linear functional of $y \in D(B)$, therefore $S(t)x \in D(A)$ and hence X is invariant under $T_1(t)$ for all $t > 0$ by Proposition 3.2. Recall that $||x||_1 := ||(\mu - A)^{-1}x||$ for all $x \in X$. Let $x_n, x \in X$ be such that $||x_n - x||_1 \to 0$. Then for $y \in D(B)$ and $\lambda \in \rho(A)$,

$$lim_{n\to\infty}\langle y, x_n\rangle = lim_{n\to\infty}\langle(\lambda - B)y, (\lambda - A)^{-1}x_n\rangle$$
$$= \langle(\lambda - B)y, (\lambda - A)^{-1}x\rangle = \langle y, x\rangle.$$

This implies that the following calculation makes sense:

$$\begin{aligned}\langle y, T_1(t)x\rangle &= \langle y, (\frac{d}{dt})_1 S(t)x\rangle = \frac{d}{dt}\langle y, S(t)x\rangle \\ &= \frac{d}{dt}\langle \tilde{S}(t)y, x\rangle = \langle \tilde{T}(t)y, x\rangle \\ &= \langle y, \tilde{T}^*(t)x\rangle \quad \forall\ y \in D(B),\ x \in X.\end{aligned}$$

Hence $\tilde{T}^*(t) = T_1(t)$. The theorem follows from Proposition 3.2. □

4. Examples

The following example shows that the integrable condition in (iii) and (iv) of Theorem 2.11 may not be satisfied.

Example 4.1. Let X be the weighted Hilbert space of complex sequence pairs $\{(\xi_n, \eta_n)\}$, endowed with the norm

$$||\{(\xi_n, \eta_n)\}|| := \sum_{n=1}^{\infty}[|\xi_n|^2 + n^2|\eta_n|^2]^{1/2} < \infty.$$

Define $T(t)$ on X by the formula

$$T(t)(\{(\xi_n, \eta_n)\}) := \{(\xi'_n(t), \eta'_n(t))\} \quad \forall\ t > 0,$$

where

$$\begin{aligned}\xi'_n(t) &:= exp[-(n+1)t](\xi_n cos\ nt - \eta_n sin\ nt), \\ \eta'_n(t) &:= exp[-(n+1)t](\xi_n sin\ nt + \eta_n cos\ nt).\end{aligned}$$

It is routine to show that

$$||T(t)|| \le sup_n\ ne^{-(n+1)t} \le \frac{1}{te^{1+t}}$$

and hence $\{T(t)\}_{t>0}$ is a semigroup of bounded linear operators on X. Clearly, $\{T(t)\}_{t>0}$ is strongly continuous. Let $t \in (0, \pi/2]$ and let $m \in N$ be such that $t \in [\pi/2^{m+1}, \pi/2^m]$. Choose $\chi_m := \{(\xi_n, \eta_n)\} \in X$ with all entries zero except for $\xi_{2^m} = 1$. Then

$$||T(t)\chi_m|| = e^{-(2^m+1)t}(cos^2\ 2^m t + 2^{2m} sin^2\ 2^m t)^{1/2}$$

is decreasing for $t \in [\pi/2^{m+1}, \pi/2^m]$ by the following calculation:

$$\begin{aligned}\frac{d}{dt}||T(t)\chi_m||^2 &= e^{-2(2^m+1)t}[-2(2^m+1)(cos^2\ 2^m t + 2^{2m} sin\ 2^m t) \\ &\quad + (2^{3m} - 2^m) sin\ 2^{m+1}t] < 0 \quad \forall\ t \in (\pi/2^{m+1}, \pi/2^m).\end{aligned}$$

This implies that

$$\begin{aligned}||T(t)|| &\ge ||T(t)\chi_m|| \ge ||T(\frac{\pi}{2^{m+1}})\chi_m|| \\ &= 2^m e^{-(2^m+1)\pi/2^{m+1}} \ge \frac{\pi}{2e^{\pi}t}.\end{aligned}$$

We now define

$$S(t)\chi := \int_0^t T(s)\chi\, ds$$
$$= \{\int_0^t e^{-(n+1)s}(\xi_n cos\ ns - \eta_n sin\ ns,\ \xi_n sin\ ns + \eta_n cos\ ns)\, ds\}$$
$$\forall\ \chi = \{(\xi_n, \eta_n)\} \in X,\ t \in [0, \infty).$$

Then

$$||S(t)|| \le sup_n\ n \int_0^t e^{-(n+1)s}\, ds \le 1.$$

$\{S(t)\}_{t\ge 0}$ is thus a once-integrated semigroup and satisfies

$$limsup_{h\to 0^+} \frac{1}{h}||S(t+h) - S(t)|| = ||T(t)|| \ge \frac{\pi}{2e^{\pi}t},$$

which is not integrable on $[0, 1]$.

The following example shows that there exists a φ Hille-Yosida operator A such that φ is unbounded in any neightborhood of zero and the semigroup generated by A is not of class $(1, A)$.

Example 4.2 ([6, p. 371]). Let X be the weighted Hilbert space of complex sequence pairs $\{(\xi_n, \eta_n)\}$, endowed with the norm

$$||\{(\xi_n, \eta_n)\}||^2 := \sum_{n=1}^{\infty}[|\xi_n|^2 + n|\eta_n|^2] < \infty.$$

Define $T(t)$ on X by the formula

$$T(t)(\{(\xi_n, \eta_n)\}) := \{(\xi_n'(t), \eta_n'(t))\} \quad \forall\ t > 0,$$

where

$$\xi_n'(t) := [exp(-t(n+1))](\xi_n cos\ nt - \eta_n sin\ nt),$$
$$\eta_n'(t) := [exp(-t(n+1))](\xi_n sin\ nt + \eta_n cos\ nt).$$

It was shown in [6, p. 371] that $\{T(t)\}_{t>0}$ is a semigroup strongly continuous for $t > 0$. Its generator, denoted by A, is densely defined, and

$$||T(t)|| \le (2et)^{-\frac{1}{2}}e^{-t} \quad \forall\ t > 0,$$
$$||R(\lambda, A)|| \ge \frac{1}{10(\lambda+1)^{\frac{1}{2}}} \quad \forall\ \lambda > 0. \tag{4.1}$$

We now show that the semigroup is unbounded in any neighborhood of zero. Let $\chi_m := \{(\xi_n, \eta_n)\}$ be in X with all entries zero except for $\xi_m = 1$. An easy calculation shows that

$$||T(\frac{\pi}{2m}) \ge ||T(\frac{\pi}{2m})|| \ge ||T(\frac{\pi}{2m})\chi_m|| \ge \frac{m^{1/2}}{e^{3\pi/2}}.$$

If we define $\varphi(t) := ||T(t)||$, then A is a φ Hille-Yosida operator. But (4.1) shows that $\{T(t)\}_{t>0}$ is not a semigroup of class $(1, A)$.

The following example shows that there exist operators satisfying the equivalent conditions of Proposition 3.2.

Example 4.3 ([13]). Let $0 < \alpha < 1$ be fixed and let X be the space of complex-valued $Lip\alpha$ continuous functions over $[0, 1]$ vanishing at $s = 1$. X, endowed with the norm

$$||x||_\alpha := sup\{\frac{|x(s') - x(s'')|}{|s' - s''|^\alpha} : \ s', s'' \in [0, 1], s' \neq s''\},$$

is a nonseparable Banach space. Define $A := \frac{d}{ds}$ on X with maximal domain. It was proved in [13] that A is not densely defined. For $t \geq 0$, define

$$[T(t)x](s) := \begin{cases} x(s+t), & \forall\ 0 \leq s \leq 1 \text{ and } 0 \leq s+t \leq 1, \\ 0, & \forall\ 0 \leq s \leq 1 \text{ and } 1 < s+t. \end{cases}$$

Then $\{T(t)\}_{t\geq 0}$ is a semigroup on X not strongly continuous for $t \geq 0$ (see [13]). We will show that it is strongly continuous in the topology induced by the norm $||x||_1 = ||(\mu - A)^{-1}x||_\alpha$, where $\mu \in \rho(A)$ (see the first paragraph of Section 3). Define

$$[S(t)x](s) := \int_0^t T(r)x(s)\,dr = \int_0^t x(s+r)\,dr \quad \forall\ x \in X,\ t \geq 0. \tag{4.2}$$

Then $\{S(t)\}_{t\geq 0}$ is a once-integrated semigroup. Since $||T(t)|| \leq 1$, we have

$$||[S(t+h) - S(t)]x||_\alpha \leq h||x||_\alpha \quad \forall\ t, h \geq 0.$$

Hence the generator A of $\{S(t)\}_{t\geq 0}$ is a Hille-Yosida operator. Similar to Section 2, let X_0 be the space of those x such that $S(t)x$ is continuously differentiable on $[0, \infty)$. From (4.2) we have

$$[T_0(t)x](s) := [\frac{d}{dt}S(t)x](s) = x(s+t) \quad \forall\ x \in X_0.$$

It is easily seen that $D(A)$ is invariant under the translation $x_t(s) = x(s+t)$, hence invariant under $T_0(t)$. This implies that the following calculation makes sense:

$$\begin{aligned} [AT_0(t)(1-A)^{-1}x](s) &= A\int_0^\infty e^{-r}[\int_0^r x(s+t+v)\,dv]\,dr \\ &= A\int_0^\infty e^{-r}x(s+t+r)\,dr = \frac{d}{ds}[e^s\int_s^\infty e^{-r}x(t+r)\,dr] \\ &= e^s\int_s^\infty e^{-r}x(t+r)\,dr - x(t+s) \\ &= [T_0(t)(1-A)^{-1}x](s) - [T(t)x](s) \quad \forall\ x \in X. \end{aligned}$$

Hence $(1-A)T_0(t)(1-A)^{-1} = T(t)$.

From Proposition 3.2, $\{T(t)\}_{t\geq 0}$ is strongly continuous in the topology induced by the norm $||x||_1 = ||(1-A)^{-1}x||_\alpha$ on X.

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References

1. W. Arendt, *Vector-valued Laplace transforms and Cauchy problems*, Israel J. Math., V. 59 (1987), 327-353. MR **89a:**47064
2. W. Arendt *et. al*, *Resolvent Positive Operators*, Proc. London Math., V. 54 (1987), 321-349. MR **88c:**47074
3. Ph. Clement *et. al*, *A Hille-Yosida theorem for a class of weak* continuous semigroups*, Semigroup Forum, V. 38 (1989), 157-178. MR **90c:**47066
4. R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations*, Lect. Notes in Math., Springer-Verlag, V. 1570, 1994. MR **96b:**47047
5. R. deLaubenfels, Q. P. Vũ and S. W. Wang, *Laplace transforms of vector-valued functions with growth ω and semigroups of operators*, Semiproup Furom, to appear.
6. E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Pub., Vol. V. 31, Providence, R. I., 1957. MR **19:**664d; reprinting of revised edit. MR **54:**11077
7. H. Kellerman and M. Hieber, *Integrated Semigroups*, J. Funct. Anal., V. 84 (1989), 160-180. MR **90h:**47072
8. I. Miyadera, *On one-parameter semi-groups of operators*, J. Math. Tokyo, V. 1 (1951), 23-26. MR **14:**564d
9. J. van Neerven, *The Adjoint of a Semigroup of Linear Operators*, Lect. Notes in Math., Springer-Verlag, V. 1529, 1992. MR **94j:**47059
10. H. R. Thieme, *Integrated semigroups and integrated solutions to abstract Cauchy problems*, J. Math. Anal. Appl., V. 152 (1990), 416-447. MR **91k:**47093
11. S. W. Wang, *Mild integrated C-existence families*, Studia Math., V. 112 (1995), 251-266. MR **95m:**47067
12. S. W. Wang, *Quasi-distribution semigroups and integrated semigroups*, J. Funct. Anal., V. 146 (1997), 352-381. MR **98d:**47088
13. S. W. Wang and I. Erdelyi, *Abel-Ergodic properties of Pseudo-resolvents and Applications to Semigroups*, Tôkohu Math. J., V. 45 (1993), 539-554. MR **95g:**47011
14. D. V. Widder, *An Introduction to Transform Theory*, Acad. Press, New York, 1971.

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