LOCAL DUAL SPACES OF BANACH SPACES OF VECTOR-VALUED FUNCTIONS

MANUEL GONZÁLEZ AND ANTONIO MARTÍNEZ-ABEJÓN

(Communicated by Jonathan M. Borwein)

Abstract. We show that $L_∞(µ, X^*)$ is a local dual of $L_1(µ, X)$, and $L_1(µ, X^*)$ is a local dual of $L_∞(µ, X)$, where $X$ is a Banach space. A local dual space of a Banach space $Y$ is a subspace $Z$ of $Y^*$ so that we have a local representation of $Y^*$ in $Z$ satisfying the properties of the representation of $X^{**}$ in $X$ provided by the principle of local reflexivity.

1. Introduction

The principle of local reflexivity shows that there is a close relation between a Banach space $X$ and its second dual $X^{**}$ from a finite dimensional point of view. This means that $X$ can be considered “locally” as a dual of $X^*$.

In [6] we introduced the local dual spaces of $X$. These subspaces satisfy the thesis of the principle of local reflexivity in full force: $X^*$ is finitely dual representable in any of its local dual spaces by means of $ε$-isometries that fix points. Moreover, in [5] we studied the polar property for subspaces $Z$ of $X^*$ as a test to check if $X^*$ is finitely dual representable in $Z$. We observe that being a local dual is strictly stronger than satisfying the polar property.

It was proved in [6] that $ℓ_1(X^*)$ is a local dual of $ℓ_∞(X)$, and $ℓ_∞(X)$ is a local dual of $ℓ_1(X^*)$. Moreover, assuming the continuum hypothesis $2^ω = ω_1$, $C[0, 1]$ is a local dual of $L_1[0, 1]$ and $L_1[0, 1]$ is a local dual of $C[0, 1]$. Also every separable space $X$ with the metric approximation property has a local dual with the metric approximation property.

Here we describe local dual spaces for some spaces of vector-valued functions. Let $µ$ be a finite measure. We show that $L_1(µ, X^*)$ is a local dual of $L_∞(µ, X)$, and $L_∞(µ, X^*)$ is a local dual of $L_1(µ, X)$. Note that $L_1(µ, X)^*$ can be described as a space of $X^*$-valued, weak*-measurable functions [2], and $L_∞(µ, X)^*$ can be described as a direct sum of $L_1(µ, X^*)$ and a certain subspace of singular elements [1]. But these descriptions are not always manageable, especially in the latter case. Previously, Díaz [3] studied the duality between $L_∞(µ, X)$ and $L_1(µ, X^*)$, but our results are stronger and our proofs are more natural.

Received by the editors June 5, 2001.

2000 Mathematics Subject Classification. Primary 46B10, 46B20; Secondary 46B04, 46B08.

Key words and phrases. Local dual space, local reflexivity, norming subspace, Banach spaces of vector-valued functions.

This work was supported in part by DGICYT Grant PB 97–0349.

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In the paper $X$ and $Y$ are Banach spaces, $B_X$ the closed unit ball of $X$, $S_X$ the unit sphere of $X$, and $X^*$ the dual of $X$. We identify $X$ with a subspace of $X^{**}$. By a subspace we always mean a closed subspace. We denote by $B(X,Y)$ the space of all (bounded linear) operators from $X$ into $Y$. Given $T \in B(X,Y)$, $T(x)$ is essentially valued in a separable subspace of $X$. 

Given a number $0 < \varepsilon < 1$, an operator $T \in B(X,Y)$ is an $\varepsilon$-isometry if it satisfies $(1+\varepsilon)^{-1} < \|Tx\| < 1+\varepsilon$ for all $x \in S_X$. A space $X$ is said to be finitely representable in $Y$ if for each $\varepsilon > 0$ and each finite dimensional subspace $M$ of $X$ there is an $\varepsilon$-isometry $T : M \longrightarrow Y$.

2. Main results

Local dual spaces of Banach spaces were introduced in [6, Definition 2.1] as follows:

**Definition 1.** A subspace $Z$ of $X^*$ is said to be a local dual space of $X$ if for every couple of finite dimensional subspaces $F$ of $X^*$ and $G$ of $X$, and every number $0 < \varepsilon < 1$, there is an $\varepsilon$-isometry $L : F \longrightarrow Z$ satisfying the following conditions:

(a) $\langle Lf, x \rangle = \langle f, x \rangle$ for all $x \in G$ and all $f \in F$, and

(b) $\|Lf - f\| \leq \varepsilon \|f\|$ for all $f \in F \cap Z$.

Obviously, $X^*$ is a local dual of $X$. Moreover, the principle of local reflexivity establishes that every isometric predual is a local dual of $X^*$, and the principle of local reflexivity for ultrapowers [7, Theorem 7.3] establishes that $(X^*)_{\mathcal{U}}$ is a local dual of $X_{\mathcal{U}}$. Observe that we do not have a description of $(X_{\mathcal{U}})^*$.

Given a couple of subspaces $Z$ of $X^*$ and $G$ of $Z^*$, an operator $L : G \longrightarrow X^{**}$ is said to be an extension operator if $Lf|_Z = f$, for every $f \in G$. The following characterization of the local dual spaces of a Banach space will be the key to proving our results.

**Theorem 2 ([6, Theorem 2.5]).** A subspace $Z$ of $X^*$ is a local dual of $X$ if (and only if) for every couple of finite dimensional subspaces $F$ of $X^*$ and $G$ of $X$, and every $0 < \varepsilon < 1$, there is an $\varepsilon$-isometry $L : F \longrightarrow Z$ such that

(a) $\|\langle Lf, x \rangle - \langle f, x \rangle\| < \varepsilon \|f\| \|x\|$, for all $x \in G$ and all $f \in F$, and

(b) $\|Lf - f\| \leq \varepsilon \|f\|$ for all $f \in F \cap Z$.

The relation “being a local dual” is symmetric. Let $\hat{x}$ denote the vector $x \in X$ as an element of $X^{**}$. We define a map $\Upsilon : X \longrightarrow Z^*$ by $\Upsilon(x) = \hat{x}|_Z$.

**Proposition 3 ([6, Proposition 2.10(b)]).** Let $Z$ be a local dual of $X$. Then $\Upsilon(X)$ is a local dual of $Z$ which is isometric to $X$.

Here is our main result.

**Theorem 4.** Let $\mu$ be a finite measure. Then:

(a) $L_1(\mu, X^*)$ is a local dual of $L_\infty(\mu, X)$.

(b) $L_\infty(\mu, X^*)$ is a local dual of $L_1(\mu, X)$.

**Proof.** The set $S_{\infty}(\mu, X^*)$ of all functions $g = \sum_{n=1}^{\infty} \chi_{A_n} \otimes x_n^*$, where $(A_n)$ is a disjoint sequence of measurable sets and $(x_n^*)$ is a sequence in $X^*$ so that $\sum_n \mu(A_n) \|x_n^*\| < \infty$, is dense in $L_1(\mu, X^*)$. Analogously, since each $f \in L_\infty(\mu, X)$ is essentially valued in a separable subspace of $X$, the set $S_{\infty}(\mu, X)$ of all functions
\( f = \sum_{n=1}^{\infty} \chi_{A_n} \otimes x_n \), where \((A_n)\) is a disjoint sequence of measurable sets and \((x_n)\) is a bounded sequence in \(X\), is dense in \(L_\infty(\mu, X)\).

(a) In order to apply Theorem 2, we fix \(0 < \varepsilon < 1\) and finite dimensional subspaces \(F\) of \(L_\infty(\mu, X)^*\) and \(G\) of \(L_\infty(\mu, X)\). Without loss of generality we can assume that \(G\) is large enough so that, for every \(\phi \in F\),

\[
\|\phi\| \leq (1 + \varepsilon) \sup\{|(\phi, f)| : f \in SG\}.
\]

Let \(\{\phi_1, \ldots, \phi_l\}\) be a basis for \(F \cap L_1(\mu, X^*)\). We take \(\{\phi_1^*, \ldots, \phi_l^*\}\) in \(L_\infty(\mu, X)^*\) so that \(\langle \phi_i^*, \phi_j \rangle = \delta_{ij}\), for \(i, j = 1, \ldots, l\). Let \(M = \sup \|\phi_i^*\|\). We select \(f_1, \ldots, f_l\) in \(S_1(\mu, X^*)\) so that \(\|\phi_i - f_i\| < \varepsilon/(2M)\), for \(j = 1, \ldots, l\).

We define an operator \(T\) on \(L_\infty(\mu, X)^*\) by

\[
T(\phi) = \phi - \sum_{j=1}^{l} \langle \phi_j^*, \phi \rangle (\phi_j - f_j).
\]

Then \(\|I - T\| < \varepsilon/2\) and \(T(\phi_j) = f_j\), for \(j = 1, \ldots, l\).

Analogously, let \(\{h_1, \ldots, h_k\}\) be a basis for \(G\). We can select \(g_1, \ldots, g_k\) in \(S_\infty(\mu, X)\), and define an operator \(S\) on \(L_\infty(\mu, X)\) such that \(\|I - S\| < \varepsilon/2\) and \(S(h_i) = g_i\), for \(i = 1, \ldots, k\).

We take a disjoint sequence \((C_n)\) of measurable sets and sequences \((x_{i,n})_{n=1}^{\infty}\) in \(X\) and \((x^*_{j,n})_{n=1}^{\infty}\) in \(X^*\) so that, for \(i = 1, \ldots, k\) and \(j = 1, \ldots, l\),

\[
g_i = \sum_{n=1}^{\infty} \chi_{C_n} \otimes x_{i,n}\quad \text{and}\quad f_j = \sum_{n=1}^{\infty} \chi_{C_n} \otimes x^*_{j,n}.
\]

We define a projection \(P\) on \(L_\infty(\mu, X)\) by

\[
P(f) = \sum_{n=1}^{\infty} \mu(C_n)^{-1} \chi_{C_n} \otimes \int_{C_n} f \, d\mu.
\]

Since \(\|P(f)\| = \sup_n \|\mu(C_n)^{-1} \chi_{C_n} \otimes \int_{C_n} f \, d\mu\| = \sup_n \|\mu(C_n)^{-1} \int_{C_n} f \, d\mu\| \leq \|f\|_\infty\), we get \(\|P\| = 1\). Moreover \(P(S(h)) = S(h)\), for every \(h \in G\). Let \(P^*\) be the conjugate projection acting on \(L_\infty(\mu, X)^*\). For each \(\phi \in L_\infty(\mu, X)^*\), we define \((P^*\phi)_n \in X^*\) by

\[
\langle (P^*\phi)_n, x \rangle = \langle \phi, \mu(C_n)^{-1} \chi_{C_n} \otimes x \rangle.
\]

Then \(P^* = \sum_{n=1}^{\infty} \chi_{C_n} \otimes (P^*\phi)_n\). Indeed, for every \(h \in L_\infty(\mu, X)\),

\[
\langle P^*(\phi), h \rangle = \langle \phi, P(h) \rangle
\]

\[
= \sum_{n=1}^{\infty} \mu(C_n)^{-1} \langle \phi, \chi_{C_n} \otimes \int_{C_n} h \, d\mu \rangle
\]

\[
= \sum_{n=1}^{\infty} \chi_{C_n} \otimes (P^*\phi)_n, h \rangle.
\]

Consequently, \(R(P^*) \subset L_1(\mu, X^*)\). Moreover, \(P^*(f_j) = f_j\), for every \(j = 1, \ldots, l\).

We define \(L\) as the restriction of \(P^*\) to the subspace \(F\). Let \(\phi \in F\) and \(h \in G\). Since \(\|L\| \leq 1\) and \(\langle L\phi, S(h) \rangle = \langle \phi, P(S(h)) \rangle = \langle \phi, S(h) \rangle\),

\[
|\langle L\phi, h \rangle - \langle \phi, h \rangle| \leq \|\langle L\phi, (I - S)h \rangle\| + \|\langle \phi, (I - S)h \rangle\| \leq \varepsilon \|\phi\| \|h\|,
\]

where \(\varepsilon\) is the a priori constant.
which is (a) in Theorem 2. Moreover, formulas (1) and (2) imply that, for every \( \phi \in F \),
\[
\|L\phi\| \geq \|L(\phi)\|_C \geq \|\phi\|_C - \varepsilon \|\phi\| \geq (1 + \varepsilon)^{-1} \|\phi\| - \varepsilon \|\phi\| \geq (1 - 2\varepsilon)\|\phi\|.
\]
Hence \( L \) is a \( 2\varepsilon \)-isometry. Moreover, since \( \|I - T\| < \varepsilon/2 \) and \( L(T(\phi)) = T(\phi) \), for every \( \phi \in F \cap L_1(\mu, X^*) \),
\[
\|L(\phi) - \phi\| \leq \|L(\phi - T(\phi))\| + \|T(\phi) - \phi\| < \varepsilon \|\phi\|.
\]
Thus, an application of Theorem 2 finishes the proof of this part.

(b) The proof is similar. In this case \( F \) and \( G \) are finite dimensional subspaces of \( L_1(\mu, X^*) \) and \( L_1(\mu, X) \), respectively. The projection \( P \) on \( L_1(\mu, X) \), defined by
\[
P(f) = \sum_{n=1}^{\infty} \mu(C_n)^{-1} \chi_{C_n} \otimes \int_{C_n} fd\mu,
\]
satisfies \( \|P(f)\| \leq \sum_{n=1}^{\infty} \|\int_{C_n} fd\mu\| \leq \|f\|_1 \), hence \( \|P\| = 1 \). The remainder of the proof is similar to that of part (a).

Remark 5. We have \( L_\infty(\mu, X^*) = L_1(\mu, X)^* \) if and only if every \( T \in B(L_1(\mu), X^*) \) is representable; equivalently, \( X^* \) has the Radon-Nikodym property [4].

It was proved in [3] that \( L_\infty(\mu) \hat{\otimes}_\varepsilon X^* \) is a local dual of \( L_1(\mu, X) \equiv L_1(\mu) \hat{\otimes}_\varepsilon X \). In this case, \( L_\infty(\mu) \hat{\otimes}_\varepsilon X^* = L_1(\mu, X)^* \) if and only if every \( T \in B(L_1(\mu), X^*) \) is compact; equivalently, \( X \) is finite dimensional.

Let \( K \) denote a compact space. It was proved in [3] that \( C(K)^* \hat{\otimes}_\varepsilon X^* \) is a local dual of \( C(K, X) \equiv C(K) \hat{\otimes}_\varepsilon X \). Note that \( C(K)^* \hat{\otimes}_\varepsilon X^* = C(K, X)^* \) if and only if \( X^* \) has the Radon-Nikodym property [4].

The following result is a direct consequence of Theorem 4 and Proposition 3.

Corollary 6. (a) \( L_\infty(\mu, X) \) is a local dual of \( L_1(\mu, X^*) \).

(b) \( L_1(\mu, X) \) is a local dual of \( L_\infty(\mu, X^*) \).

Remark 7. Corollary 4 is a generalization of the principle of local reflexivity.

References


DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DE CANTABRIA, E-39071 SANTANDER, SPAIN
E-mail address: gonzalem@unican.es

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DE OVIEDO, E-33007 OVIEDO, SPAIN
E-mail address: ama@pinon.ccu.uniovi.es