PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 11, Pages 3255–3258
S 0002-9939(02)06626-1
Article electronically published on April 22, 2002

LOCAL DUAL SPACES OF BANACH SPACES OF VECTOR-VALUED FUNCTIONS

MANUEL GONZÁLEZ AND ANTONIO MARTÍNEZ-ABEJÓN

(Communicated by Jonathan M. Borwein)

ABSTRACT. We show that $L_\infty(\mu, X^*)$ is a local dual of $L_1(\mu,X)$, and $L_1(\mu, X^*)$ is a local dual of $L_\infty(\mu, X)$, where X is a Banach space. A local dual space of a Banach space Y is a subspace Z of Y^* so that we have a local representation of Y^* in Z satisfying the properties of the representation of X^{**} in X provided by the principle of local reflexivity.

1. Introduction

The principle of local reflexivity shows that there is a close relation between a Banach space X and its second dual X^{**} from a finite dimensional point of view. This means that X can be considered "locally" as a dual of X^*.

In [6] we introduced the local dual spaces of X. These subspaces satisfy the thesis of the principle of local reflexivity in full force: X^* is finitely dual representable in any of its local dual spaces by means of ε-isometries that fix points. Moreover, in [5] we studied the polar property for subspaces Z of X^* as a test to check if X^* is finitely dual representable in Z. We observe that being a local dual is strictly stronger than satisfying the polar property.

It was proved in [6] that $\ell_1(X^*)$ is a local dual of $\ell_\infty(X)$, and $\ell_\infty(X)$ is a local dual of $\ell_1(X^*)$. Moreover, assuming the continuum hypothesis $2^\omega = \omega_1$, $C[0,1]$ is a local dual of $L_1[0,1]$ and $L_1[0,1]$ is a local dual of $C[0,1]$. Also every separable space X with the metric approximation property has a local dual with the metric approximation property.

Here we describe local dual spaces for some spaces of vector-valued functions. Let μ be a finite measure. We show that $L_1(\mu, X^*)$ is a local dual of $L_\infty(\mu, X)$, and $L_\infty(\mu, X^*)$ is a local dual of $L_1(\mu, X)$. Note that $L_1(\mu, X)^*$ can be described as a space of X^*-valued, weak*-measurable functions [2], and $L_\infty(\mu, X)^*$ can be described as a direct sum of $L_1(\mu, X^*)$ and a certain subspace of singular elements [1]. But these descriptions are not always manageable, especially in the latter case. Previously, Díaz [3] studied the duality between $L_\infty(\mu, X)$ and $L_1(\mu, X^*)$, but our results are stronger and our proofs are more natural.

Received by the editors June 5, 2001.
2000 *Mathematics Subject Classification.* Primary 46B10, 46B20; Secondary 46B04, 46B08.
Key words and phrases. Local dual space, local reflexivity, norming subspace, Banach spaces of vector-valued functions.
This work was supported in part by DGICYT Grant PB 97–0349.

In the paper X and Y are Banach spaces, B_X the closed unit ball of X, S_X the unit sphere of X, and X^* the dual of X. We identify X with a subspace of X^{**}. By a *subspace* we always mean a closed subspace. We denote by $\mathcal{B}(X,Y)$ the space of all (bounded linear) operators from X into Y. Given $T \in \mathcal{B}(X,Y)$, $R(T)$ is the range of T, and T^* is the conjugate operator of T.

Given a number $0 < \varepsilon < 1$, an operator $T \in \mathcal{B}(X,Y)$ is an ε*-isometry* if it satisfies $(1+\varepsilon)^{-1} < \|Tx\| < 1+\varepsilon$ for all $x \in S_X$. A space X is said to be *finitely representable in* Y if for each $\varepsilon > 0$ and each finite dimensional subspace M of X there is an ε–isometry $T : M \longrightarrow Y$.

2. Main results

Local dual spaces of Banach spaces were introduced in [6, Definition 2.1] as follows:

Definition 1. A subspace Z of X^* is said to be a *local dual space of* X if for every couple of finite dimensional subspaces F of X^* and G of X, and every number $0 < \varepsilon < 1$, there is an ε-isometry $L : F \longrightarrow Z$ satisfying the following conditions:

(a) $\langle Lf, x\rangle = \langle f, x\rangle$ for all $x \in G$ and all $f \in F$, and

(b) $L(f) = f$ for all $f \in F \cap Z$.

Obviously, X^* is a local dual of X. Moreover, the principle of local reflexivity establishes that every isometric predual is a local dual of X^*, and the principle of local reflexivity for ultrapowers [7, Theorem 7.3] establishes that $(X^*)_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$. Observe that we do not have a description of $(X_{\mathfrak{U}})^*$.

Given a couple of subspaces Z of X^* and G of Z^*, an operator $L : G \longrightarrow X^{**}$ is said to be an *extension operator* if $Lf|_Z = f$, for every $f \in G$. The following characterization of the local dual spaces of a Banach space will be the key to proving our results.

Theorem 2 ([6, Theorem 2.5])**.** *A subspace Z of X^* is a local dual of X if (and only if) for every couple of finite dimensional subspaces F of X^* and G of X, and every $0 < \varepsilon < 1$, there is an ε-isometry $L : F \longrightarrow Z$ such that*

(a) $|\langle Lf, x\rangle - \langle f, x\rangle| < \varepsilon\|f\|\|x\|$, *for all $x \in G$ and all $f \in F$, and*

(b) $\|L(f) - f\| \leq \varepsilon\|f\|$ *for all $f \in F \cap Z$.*

The relation "being a local dual" is symmetric. Let $\hat{x}$ denote the vector $x \in X$ as an element of X^{**}. We define a map $\Upsilon : X \longrightarrow Z^*$ by $\Upsilon(x) = \hat{x}|_Z$.

Proposition 3 ([6, Proposition 2.10(b)])**.** *Let Z be a local dual of X. Then $\Upsilon(X)$ is a local dual of Z which is isometric to X.*

Here is our main result.

Theorem 4. *Let μ be a finite measure. Then:*
(a) $L_1(\mu, X^*)$ *is a local dual of* $L_\infty(\mu, X)$.
(b) $L_\infty(\mu, X^*)$ *is a local dual of* $L_1(\mu, X)$.

Proof. The set $\mathcal{S}_1(\mu, X^*)$ of all functions $g = \sum_{n=1}^{\infty} \chi_{A_n} \otimes x_n^*$, where (A_n) is a disjoint sequence of measurable sets and (x_n^*) is a sequence in X^* so that $\sum_n \mu(A_n)\|x_n^*\| < \infty$, is dense in $L_1(\mu, X^*)$. Analogously, since each $f \in L_\infty(\mu, X)$ is essentially valued in a separable subspace of X, the set $\mathcal{S}_\infty(\mu, X)$ of all functions

$f = \sum_{n=1}^{\infty} \chi_{A_n} \otimes x_n$, where (A_n) is a disjoint sequence of measurable sets and (x_n) is a bounded sequence in X, is dense in $L_\infty(\mu, X)$.

(a) In order to apply Theorem 2, we fix $0 < \varepsilon < 1$ and finite dimensional subspaces F of $L_\infty(\mu, X)^*$ and G of $L_\infty(\mu, X)$. Without loss of generality we can assume that G is large enough so that, for every $\phi \in F$,

$$\|\phi\| \leq (1+\varepsilon) \sup\{|\langle \phi, f\rangle| : f \in S_G\}. \tag{1}$$

Let $\{\phi_1, \dots, \phi_l\}$ be a basis for $F \cap L_1(\mu, X^*)$. We take $\{\phi_1^*, \dots, \phi_l^*\}$ in $L_\infty(\mu, X)^{**}$ so that $\langle \phi_i^*, \phi_j\rangle = \delta_{ij}$, for $i, j = 1, \dots, l$. Let $M = \sup \|\phi_i^*\|$. We select $f_1, \dots, f_l$ in $\mathcal{S}_1(\mu, X^*)$ so that $\|\phi_j - f_j\| < \varepsilon/(2lM)$, for $j = 1, \dots, l$.

We define an operator T on $L_\infty(\mu, X)^*$ by

$$T(\phi) = \phi - \sum_{j=1}^{l} \langle \phi_j^*, \phi\rangle (\phi_j - f_j).$$

Then $\|I - T\| < \varepsilon/2$ and $T(\phi_j) = f_j$, for $j = 1, \dots, l$.

Analogously, let $\{h_1, \dots, h_k\}$ be a basis for G. We can select $g_1, \dots, g_k$ in $\mathcal{S}_\infty(\mu, X)$, and define an operator S on $L_\infty(\mu, X)$ such that $\|I - S\| < \varepsilon/2$ and $S(h_i) = g_i$, for $i = 1, \dots, k$.

We take a disjoint sequence (C_n) of measurable sets and sequences $(x_{i,n})_{n=1}^\infty$ in X and $(x_{j,n}^*)_{n=1}^\infty$ in X^* so that, for $i = 1, \dots, k$ and $j = 1, \dots, l$,

$$g_i = \sum_{n=1}^{\infty} \chi_{C_n} \otimes x_{i,n} \quad \text{and} \quad f_j = \sum_{n=1}^{\infty} \chi_{C_n} \otimes x_{j,n}^*.$$

We define a projection P on $L_\infty(\mu, X)$ by

$$P(f) = \sum_{n=1}^{\infty} \mu(C_n)^{-1} \chi_{C_n} \otimes \int_{C_n} f d\mu.$$

Since $\|P(f)\| = \sup_n \|\mu(C_n)^{-1}\chi_{C_n} \otimes \int_{C_n} f d\mu\| = \sup_n \|\mu(C_n)^{-1} \int_{C_n} f d\mu\| \leq \|f\|_\infty$, we get $\|P\| = 1$. Moreover $P(S(h)) = S(h)$, for every $h \in G$. Let P^* be the conjugate projection acting on $L_\infty(\mu, X)^*$. For each $\phi \in L_\infty(\mu, X)^*$, we define $(P^*\phi)_n \in X^*$ by

$$\langle (P^*\phi)_n \, , \, x\rangle = \langle \phi \, , \, \mu(C_n)^{-1} \chi_{C_n} \otimes x\rangle.$$

Then $P^*(\phi) = \sum_{n=1}^{\infty} \chi_{C_n} \otimes (P^*\phi)_n$. Indeed, for every $h \in L_\infty(\mu, X)$,

$$\begin{aligned} \langle P^*(\phi) \, , \, h\rangle &= \langle \phi \, , \, P(h)\rangle \\ &= \sum_{n=1}^{\infty} \mu(C_n)^{-1} \langle \phi \, , \, \chi_{C_n} \otimes \int_{C_n} h d\mu\rangle \\ &= \langle \sum_{n=1}^{\infty} \chi_{C_n} \otimes (P^*\phi)_n \, , \, h\rangle. \end{aligned}$$

Consequently, $R(P^*) \subset L_1(\mu, X^*)$. Moreover, $P^*(f_j) = f_j$, for every $j = 1, \dots, l$.

We define L as the restriction of P^* to the subspace F. Let $\phi \in F$ and $h \in G$. Since $\|L\| \leq 1$ and $\langle L\phi, S(h)\rangle = \langle \phi, P(S(h))\rangle = \langle \phi, S(h)\rangle$,

$$|\langle L\phi, h\rangle - \langle \phi, h\rangle| \leq |\langle L\phi, (I-S)h\rangle| + |\langle \phi, (I-S)h\rangle| \leq \varepsilon \|\phi\| \|h\|, \tag{2}$$

which is (a) in Theorem 2. Moreover, formulas (1) and (2) imply that, for every $\phi \in F$,

$$\|L\phi\| \geq \|L(\phi)|_G\| \geq \|\phi|_G\| - \varepsilon\|\phi\| \geq (1+\varepsilon)^{-1}\|\phi\| - \varepsilon\|\phi\| \geq (1-2\varepsilon)\|\phi\|.$$

Hence L is a 2ε-isometry. Moreover, since $\|I - T\| < \varepsilon/2$ and $L(T(\phi)) = T(\phi)$, for every $\phi \in F \cap L_1(\mu, X^*)$,

$$\|L(\phi) - \phi\| \leq \|L(\phi - T(\phi))\| + \|T(\phi) - \phi\| < \varepsilon\|\phi\|.$$

Thus, an application of Theorem 2 finishes the proof of this part.

(b) The proof is similar. In this case F and G are finite dimensional subspaces of $L_1(\mu, X)^*$ and $L_1(\mu, X)$, respectively. The projection P on $L_1(\mu, X)$, defined by

$$P(f) = \sum_{n=1}^{\infty} \mu(C_n)^{-1}\chi_{C_n} \otimes \int_{C_n} f d\mu,$$

satisfies $\|P(f)\| \leq \sum_{n=1}^{\infty} \| \int_{C_n} f d\mu\| \leq \|f\|_1$, hence $\|P\| = 1$. The remainder of the proof is similar to that of part (a). □

Remark 5. We have $L_\infty(\mu, X^*) = L_1(\mu, X)^*$ if and only if every $T \in \mathcal{B}(L_1(\mu), X^*)$ is representable; equivalently, X^* has the Radon-Nikodym property [4].

It was proved in [6] that $L_\infty(\mu)\hat{\otimes}_\epsilon X^*$ is a local dual of $L_1(\mu, X) \equiv L_1(\mu)\hat{\otimes}_\pi X$. In this case, $L_\infty(\mu)\hat{\otimes}_\epsilon X^* = L_1(\mu, X)^*$ if and only if every $T \in \mathcal{B}(L_1(\mu), X^*)$ is compact; equivalently, X is finite dimensional.

Let K denote a compact space. It was proved in [6] that $C(K)^*\hat{\otimes}_\pi X^*$ is a local dual of $C(K, X) \equiv C(K)\hat{\otimes}_\epsilon X$. Note that $C(K)^*\hat{\otimes}_\pi X^* = C(K, X)^*$ if and only if X^* has the Radon-Nikodym property [4].

The following result is a direct consequence of Theorem 4 and Proposition 3.

Corollary 6. (a) $L_\infty(\mu, X)$ *is a local dual of* $L_1(\mu, X^*)$.
(b) $L_1(\mu, X)$ *is a local dual of* $L_\infty(\mu, X^*)$.

Remark 7. Corollary 6 is a generalization of the principle of local reflexivity.

References

[1] C. Castaing and A. Valadier. *Convex analysis and measurable multifunctions.* Lecture Notes in Math. **580**. Springer-Verlag, Berlin, 1977. MR **57:**7169

[2] P. Cembranos and J. Mendoza. *Banach spaces of vector-valued functions.* Lecture Notes in Math. **1676**. Springer-Verlag, Berlin, 1997. MR **99f:**46049

[3] S. Díaz. A local approach to functionals on $L^\infty(\mu, X)$, *Proc. Amer. Math. Soc.* **128** (2000), 101-109. MR **2000c:**46066

[4] J. Diestel and J.J. Uhl, Jr. Vector measures. *Math. Surveys* **15**. Amer. Math. Soc., Providence, 1977. MR **56:**12216

[5] M. González and A. Martínez-Abejón. *Local reflexivity of dual Banach spaces,* Pacific J. Math. **189** (1999), 263-278. MR **2000g:**46010

[6] M. González and A. Martínez-Abejón. *Local dual spaces of a Banach space,* Studia Math. **147** (2001), 155–168.

[7] S. Heinrich. *Ultraproducts in Banach space theory,* J. Reine Angew. Math. **313** (1980), 72-104. MR **82b:**46013

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, E-39071 Santander, Spain

E-mail address: gonzalem@unican.es

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, E-33007 Oviedo, Spain

E-mail address: ama@pinon.ccu.uniovi.es